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Is the Frobenius Matrix Norm Induced?

Vijaya-Sekhar Chellaboina and Wassim M. Haddad

***Abstract*—In this note we answer the question of whether (or not) there exist *normed input–output vector spaces that induce the Frobenius* matrix norm. Specifically, using the notion of dual norms we show that, up to a scalar multiple, the maximum singular value is the only unitarily invariant induced norm. As a special case of this result, it follows that the Frobenius matrix norm is not induced.**

I. Introduction

A central issue in feedback control is the performance analysis of a control system for its ability to reject disturbances. One standard mathematical framework for addressing this problem is to consider the dynamical system

$$\dot{x}(t) = Ax(t) + Bw(t), \qquad t \geq 0, \tag{1}$$

$$z(t) = Cx(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^d$, $z(t) \in \mathbb{R}^q$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$, and $C \in \mathbb{R}^{q\times n}$, where $w(\cdot)$ is the exogenous disturbance signal belonging to a class of disturbances $\mathcal{D}$, and where $z(\cdot)$ is the error signal belonging to a class of error signals $\mathcal{E}$. Here, (1) and (2) denote a control system in closed-loop configuration. Hence, the standard feedback control performance analysis problem is to determine the "size" in some specified sense of $z(\cdot)$ given that $w(\cdot) \in \mathcal{D}$.

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The authors are with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150 USA.

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Several settings are generally considered for this standard feedback control performance analysis problem. Specifically, in the H_2 (stochastic) case, $w(\cdot)$ denotes a white noise disturbance and the performance variable $z(\cdot)$ is measured by the steady-state quadratic performance criterion [1]

$$J_{H_2} \triangleq \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t z^T(s)z(s)ds = \operatorname{tr} CQC^T \tag{3}$$

where $\mathbb{E}$ denotes expectation, tr is the trace operator, and $Q \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)x^T(t)]$ is the steady-state covariance satisfying the Lyapunov equation

$$0 = AQ + QA^T + BB^T \tag{4}$$

or, equivalently

$$Q = \int_0^\infty e^{At}BB^Te^{A^Tt}dt. \tag{5}$$

Hence, using (3), (5), and Plancherel's theorem, it follows that

$$\begin{aligned} J_{H_2} &= \operatorname{tr}\int_0^\infty Ce^{At}BB^Te^{A^Tt}C^T\,dt \\ &= \frac{1}{2\pi}\operatorname{tr}\int_{-\infty}^\infty C(\jmath\omega I_n - A)^{-1} \\ &\quad \cdot BB^T(-\jmath\omega I_n - A)^{-T}C^T\,d\omega \end{aligned} \tag{6}$$

or, equivalently [2]

$$\begin{aligned} J_{H_2} &= \int_0^\infty \|Ce^{At}B\|_F^2 dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \|G(\jmath\omega)\|_F^2 d\omega = \|G(s)\|_2^2 \end{aligned} \tag{7}$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm and $G(s)$ is the transfer function from disturbances w to performance variables z. Hence, the performance criterion (3) can be written in terms of the L_2 norm of the impulse response or, equivalently, the H_2 norm of the dynamic system (1) and (2).

In the deterministic setting, if the input–output signals are constrained to finite energy signals so that $\mathcal{D}$ and $\mathcal{E}$ are L_2 spaces on $[0,\infty)$ and define the equi-induced signal norm

$$J_{H_\infty} \triangleq \sup_{w(\cdot)\in L_2} \frac{\|z\|_2}{\|w\|_2} \tag{8}$$

then it follows that ([2])

$$J_{H_\infty} = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[G(\jmath\omega)] = \|G(s)\|_\infty \tag{9}$$

were $\sigma_{\max}(\cdot)$ denotes the maximum singular value and hence J_{H_∞} is the H_∞ norm of $G(s)$.

Alternatively, if mixed-induced signal norms are assigned to the input–output spaces, then mixed input–output signals can be considered. For example, if $\mathcal{D}$ is an L_2 space with Euclidean spatial norm and $\mathcal{E}$ is an L_∞ space with Euclidean spatial norm it follows that the resulting induced operator norm is ([3])

$$J \triangleq \sup_{w(.)\in L_2} \frac{\|z\|_\infty}{\|w\|_2} = \lambda_{\max}(CQC^T) \tag{10}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue and hence J provides a worst-case measure of amplitude errors due to finite energy signals. Within the context of the standard feedback control performance analysis problem, several other mixed-induced signal norms are considered in [4].

A natural question that arises from the above observations is whether (or not) the H_2 norm is an induced norm. That is, do there exist normed (or seminormed) input–output function spaces $\mathcal{D}$ and $\mathcal{E}$ that induce the H_2 norm? As mentioned in a survey paper by Bernstein [5] and as can readily be seen from (7), the answer to this question is related to whether (or not) the Frobenius matrix norm is induced which, in itself, is another fundamental open problem. Since $\|I_n\|_F = \sqrt{n}$ the Frobenius matrix norm is not equi-induced, that is, the domain and range spaces of I_n cannot be assigned the same spatial norm to induce the Frobenius norm. Of course, this does not preclude the possibility of the Frobenius matrix norm being mixed induced, that is, assigning different spatial norms on the domain and range spaces for inducing the Frobenius norm.

In this note we answer one of these fundamental questions, namely, the Frobenius matrix norm is not induced, and provide further insight to the question of whether (or not) the H_2 norm is induced. Specifically, using the notion of dual norms we show that a class of unitarily invariant norms and consequently all singular value norms (σp-norms) with the exception of the spectral norm (maximum singular value) are not induced. Since $\|\cdot\|_{\sigma 2} = \|\cdot\|_F$, it follows that the Frobenius matrix norm is not induced. Hence, if a necessary condition for inducing the H_2 norm is that the Frobenius matrix norm is an induced norm then it would follow that the H_2 norm is not induced.

Notation

$\mathbb{R}, \mathbb{C}$	Real numbers, complex numbers.
$\mathbb{R}^{m\times n}$	$m \times n$ real matrices.
$\mathbb{C}^{m\times n}$	$m \times n$ complex matrices.
A^*	Complex conjugate transpose of A.
$\det A, \operatorname{tr} A$	Determinant of A, trace of A.
$\|x\|_2$	Euclidian norm of vector $x (= \sqrt{x^*x})$.
$\sigma_i(A)$	ith singular value of A.
$\sigma_{\max}(A)$	Maximum singular value of A.
$\|A\|_F$	Frobenius norm of A $(= (\operatorname{tr} AA^*)^{1/2})$.
$A_{(i,j)}$	(i,j)th element of A.
E_{11}	$m \times n$ elementary matrix with unity in the $(1,1)$ position and zeros elsewhere.
$\|A\|_{\sigma p}$	$[\sum_{i=1}^{r} \sigma_i^p(A)]^{1/p}$, $1 \le p < \infty$, $r = \operatorname{rank} A$.

II. Mathematical Preliminaries

In this section we give certain definitions and lemmas concerning matrix norms. A matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is unitarily invariant if $\|UAV\| = \|A\|$ for all $A \in \mathbb{C}^{m\times n}$ and for all unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$. Furthermore, a unitarily invariant matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is normalized if $\|A\| = \sigma_{\max}(A)$ for all rank one matrices $A \in \mathbb{C}^{m\times n}$.

Next, let $\|\cdot\|'$ and $\|\cdot\|''$ denote vector norms on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, where $m, n > 1$. Then $\|\cdot\|: \mathbb{C}^{m\times n} \to \mathbb{R}$ defined by

$$\|A\| \triangleq \max_{\|x\|'=1} \|Ax\|''$$

is the matrix norm induced by $\|\cdot\|'$ and $\|\cdot\|''$. A matrix norm $\|\cdot\|$ is not induced on $\mathbb{C}^{m\times n}$ if there do not exist vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $\|\cdot\|$ is the matrix norm induced by $\|\cdot\|'$ and $\|\cdot\|''$.

Finally, let $\|\cdot\|$ denote a vector norm on $\mathbb{C}^m$. Then we define the dual norm $\|\cdot\|_D$ of $\|\cdot\|$ by

$$\|y\|_D \triangleq \max_{\|x\|=1} |y^*x|$$

where $y \in \mathbb{C}^m$ [6]. The following key lemmas are needed for the main results of this paper.

Lemma 2.1: Let $\|\cdot\|$ denote the matrix norm on $\mathbb{C}^{m\times n}$ induced by vector norms $\|\cdot\|'$ and $\|\cdot\|''$ and let $x \in \mathbb{C}^m$, $y \in \mathbb{C}^n$. Then $\|xy^*\| = \|x\|''\|y\|'_D$.

Proof: It need only be noted that $\|xy^*\| = \max_{\|z\|'=1} \|xy^*z\|'' = \max_{\|z\|'=1} \|x\|''|y^*z| = \|x\|''\|y\|'_D$. □

Lemma 2.2: Let $\|\cdot\|$ denote a unitarily invariant matrix norm on $\mathbb{C}^{m\times n}$. Then there exists $c > 0$ such that $\|A\| = c\sigma_{\max}(A)$ for all rank one matrices $A \in \mathbb{C}^{m\times n}$.

Proof: Note that for all rank one matrices $A \in \mathbb{C}^{m\times n}$, it follows from the singular value decomposition that there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that $A = \sigma_{\max}(A)UE_{11}V$. Next, it follows from unitary invariance of $\|\cdot\|$ that $\|A\| = \sigma_{\max}(A)\|E_{11}\|$. The result is now immediate with $c = \|E_{11}\|$. □

Remark 2.1: Special cases of Lemmas 2.1 and 2.2 for equi-induced norms are given in [7].

III. The Frobenius Matrix Norm

In this section we show that the Frobenius matrix norm is not induced. Noting that the Frobenius matrix norm is a normalized unitarily invariant norm, we first present a result that gives necessary and sufficient conditions for a unitarily invariant norm to be induced. In what follows, we assume m and n are integers greater than one.

Theorem 3.1: Let $\|\cdot\|$ denote a unitarily invariant matrix norm on $\mathbb{C}^{m\times n}$. Then there exist vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $\|\cdot\|$ is the matrix norm induced by $\|\cdot\|'$ and $\|\cdot\|''$ if and only if there exists $k > 0$ such that $\|A\| = k\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$. Furthermore, if $\|A\| = k\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$ then $k = \|E_{11}\|$.

Proof: If there exists $k > 0$ such that $\|A\| = k\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$ then $\|\cdot\|$ is the matrix norm induced by the vector norms $\|\cdot\|' = \|\cdot\|_2$ and $\|\cdot\|'' = k\|\cdot\|_2$. Conversely, suppose there exist vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $\|\cdot\|$ is the matrix norm induced by $\|\cdot\|'$ and $\|\cdot\|''$. Then, for all $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, it follows from Lemma 2.2 that there exists $c > 0$ such that $c\|x\|_2\|y\|_2 = c\sigma_{\max}(xy^*) = \|xy^*\|$. Furthermore, it follows from Lemma 2.1 that $\|xy^*\| = \|x\|''\|y\|'_D$, and hence

$$c\|x\|_2\|y\|_2 = \|x\|''\|y\|'_D \tag{11}$$

for all $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. Now, letting $y = \bar{y}$ where $\bar{y} \in \mathbb{C}^n$ is a constant, (11) implies that there exists $k_1 > 0$ such that $\|x\|'' = k_1\|x\|_2$. Similarly, letting $x = \bar{x}$ where $\bar{x} \in \mathbb{C}^m$ is a constant, (11) implies that $\|y\|'_D/\|y\|_2$ is a constant for all $y \in \mathbb{C}^n$. Next noting that $\|y\|'_{DD} = \|y\|'$ and $\|y\|_{2D} = \|y\|_2$ for all $y \in \mathbb{C}^m$, it follows that there exists $k_2 > 0$ such that $\|y\|' = k_2\|y\|_2$. Hence, it follows that $\|A\| = (k_1/k_2)\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$ as required. Finally, if $\|A\| = k\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$ then it follows from Lemma 2.2 that $k = \|E_{11}\|$. □

Remark 3.1: In the case where $\|\cdot\|$ is assumed to be an equi-induced matrix norm, Theorem 3.1 specializes to Corollary 5.6.35 of [7].

The following corollary is now immediate.

Corollary 3.1: Let $\|\cdot\|$ denote a normalized unitarily invariant matrix norm on $\mathbb{C}^{m\times n}$ and let $1 \le p < \infty$. Then the following statements hold:

i) There exist vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, such that $\|\cdot\|$ is the matrix norm induced by $\|\cdot\|'$ and $\|\cdot\|''$ if and only if $\|A\| = \sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$.
ii) $\|\cdot\|_{\sigma p}$ is not induced.
iii) The Frobenius matrix norm is not induced.

Proof: If $\|\cdot\|$ is a normalized unitarily invariant matrix norm then $\|A\| = \sigma_{\max}(A)$ for all rank one $A \in \mathbb{C}^{m\times n}$ and hence if there exists $k > 0$ such that $\|A\| = k\sigma_{\max}(A)$ for all $A \in \mathbb{C}^{m\times n}$ then $k = 1$. Now i) is a direct consequence of Theorem 3.1. Next note that $\|\cdot\|_{\sigma p}$ is a normalized unitarily invariant matrix norm for all $1 \le p < \infty$. Furthermore, since there exists $A \in \mathbb{C}^{m\times n}$ such that $\|A\|_{\sigma p} \neq \sigma_{\max}(A)$ ii) follows from i). Finally, noting that $\|\cdot\|_F = \|\cdot\|_{\sigma 2}$ iii) follows from ii). □

Alternatively, to show iii) in Corollary 3.1 for the case $m = n > 1$, suppose that the Frobenius matrix norm $\|\cdot\|_F$ is induced on $\mathbb{C}^{m\times m}$, that is, suppose there exist vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^m$ such that $\|A\|_F = \max_{\|x\|'=1}\|Ax\|''$, for all $A \in \mathbb{C}^{m\times m}$. Then, for all $x \in \mathbb{C}^m$ it follows that

$$x^*x = \|xx^*\|_F = \max_{\zeta\in\mathbb{C}^m}\frac{\|xx^*\zeta\|''}{\|\zeta\|'} \ge \frac{\|xx^*x\|''}{\|x\|'} = x^*x\frac{\|x\|''}{\|x\|'}$$

which implies that $\|x\|'' \le \|x\|'$ for all $x \in \mathbb{C}^m$. Hence, $\sqrt{m} = \|I\|_F = \max_{x\in\mathbb{C}^m}\|x\|''/\|x\|' \le 1$ is a contradiction.

Remark 3.2: Note that Corollary 3.1-iii) answers one of the questions posed in [5], specifically, whether (or not) the submultiplicative property ($\|AB\| \le \|A\|\,\|B\|$) of a matrix norm implies that the norm is induced. Since the Frobenius matrix norm is a submultiplicative matrix norm and as shown in Corollary 3.1-iii) is not induced, it follows that the submultiplicative property does not imply that the matrix norm is induced. More generally, the same can be said for matrix norms satisfying a mixed submultiplicative property ($\|AB\| \le \|A\|'\|B\|''$).

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Connections Between Local Stability in Lyapunov and Input/Output Senses

Jinhoon Choi

***Abstract*—The concept of gain over set was recently introduced as a tool for local input/output analysis of nonlinear systems. In this setting, the finiteness of the gain over set defines a local stability in the input/output sense. In this note, we show that the finite gain over ball stability is related to the local stability in the sense of Lyapunov.**

I. INTRODUCTION

There have been two paradigms in the analysis and design of control systems: the input/output and the state-space approaches. The classical control theory was mainly concerned with the input/output approaches to linear systems in frequency domain. The advent of state-space theory due to Kalman revolutionized the control theory and launched the age of so-called modern control. During the age of modern control, there have been great advances in state-space theory of linear and nonlinear systems as well as input/output approaches to linear and nonlinear systems in time domain. The connections between these two theories in linear systems are also well established during this period. These well-established connections between two theories in linear systems have been very powerful in addressing the linear system problems in the so-called post-modern control. For instance, the H_∞ optimization formulated in input/output theory has been solved very efficiently using the state-space theoretic tools [4]. This example shows that we can have much more powerful control theory when the state-space and input/output theories are combined. For nonlinear systems, however, the connections between the two theories are far beyond the completion, although both the input/output and the state-space theories are well developed in their own right.

One of the most important problems in control systems is stability, because the first concern in the design of control systems is to guarantee the stability of the closed-loop systems. Each of the two approaches to control systems has its own corresponding notion of stability. The state-space theory is based on the Lyapunov stability and the input/output approach on the finite gain or bounded-input/bounded-output (BIBO) stability. The connections between stability in these two approaches are well established for linear systems [2], although they are still under development for nonlinear systems.

There have been several attempts to establish the relationships between input/output and Lyapunov stability for nonlinear systems. The first attempt in this direction traces back to early 1970's when Willems [10] studied the problem of finding the Lyapunov functions for input/output stable systems and identified conditions under which finite gain or BIBO stability implies global asymptotic Lyapunov stability. Hill and Moylan [6] extended Willems' result by finding conditions under which finite gain implies local asymptotic stability and established a theorem in reverse direction for a class of nonlinear systems by proving that global exponential stability implies finite gain stability under some assumptions. The application of these global results is restrictive because large portions of nonlinear systems are only locally stable in Lyapunov or input/output sense. Thus, it is very

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The author is with the Department of Chemical Engineering, University of California, Los Angeles, CA 90095 USA.
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