In this paper, we develop a hybrid control framework for addressing multiagent formation control protocols for general nonlinear dynamical systems using hybrid stabilization of sets. The proposed framework develops a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations, which can include flocking, cyclic pursuit, rendezvous, and consensus control of multiagent systems. These dynamic controllers combine a logical switching architecture with the continuous system dynamics to guarantee that a system generalized energy function whose zero level set characterizes a specified system formation is strictly decreasing across switchings. The proposed approach addresses general nonlinear dynamical systems and is not limited to systems involving single and double integrator dynamics for consensus and formation control or unicycle models for cyclic pursuit. Finally, several numerical examples involving flocking, rendezvous, consensus, and circular formation protocols for standard system formation models are provided to demonstrate the efficacy of the proposed approach. [DOI: 10.1115/1.4027501]

Keywords: energy-based set stabilization, formation control, hybrid control, impulsive dynamical systems, consensus protocols, multiagent systems

1 Introduction

Using system-theoretic thermodynamic concepts, an energy- and entropy-based hybrid control architecture was proposed in Refs. [1] and [2] as a means for achieving enhanced energy dissipation in lossless and dissipative dynamical systems. These dynamic controllers combined a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework developed in Ref. [1] leads to closed-loop systems described by impulsive differential equations [2]. In particular, the authors in Refs. [1] and [2] construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. Specifically, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy- and entropy-based hybrid controllers involving state-dependent switching were also developed to show the efficacy of the approach.

Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks [3]. Network systems involve distributed decision making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, and congestion control in communication networks. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. For example, in a group of autonomous vehicles, this property might be a common heading angle or a shared communication frequency. Moreover, it is important to develop information consensus protocols for networks of dynamic agents, wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or consensus [4–6]. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, semistability [7–9], and not asymptotic stability, is the relevant notion of stability. In addition, system-theoretic thermodynamic concepts [4–6,10] have proved invaluable in addressing Lyapunov stability and convergence for nonlinear dynamical networks.

Convergence and state equipartitioning also arise in numerous complex large-scale dynamical networks that demonstrate a degree of synchronization. System synchronization typically involves coordination of events that allows a dynamical system to operate in unison resulting in system self-organization. The onset of synchronization in populations of coupled dynamical networks have been studied for various complex networks including network models for mathematical biology, statistical physics, kinetic theory, bifurcation theory, as well as plasma physics [11]. Synchronization of firing neural oscillator populations also appears in the neuroscience literature [12,13].

Alternatively, in other applications of multiagent systems, groups of agents are required to achieve and maintain a prescribed geometric shape. This formation control problem includes flocking [14,15] and cyclic pursuit [16], wherein parallel and circular formations of vehicles are sought. For formation control of multiple vehicles, cohesion, separation, and alignment constraints are typically required for individual agent steering which describe how a given vehicle maneuvers based on the positions and velocities of nearby agents. Specifically, cohesion refers to a steering rule, wherein a given vehicle attempts to move toward the average position of local vehicles, separation refers to collision avoidance with nearby vehicles, whereas alignment refers to velocity matching with nearby vehicles.

Since a specified formation of multiagent systems, which can include flocking, cyclic pursuit, rendezvous, or consensus, can be characterized by a hyperplane or manifold in the state space; in this paper, we extend the results of Refs. [1] and [2] to develop a state-dependent hybrid control framework for addressing...
multiajgent formation control protocols for general nonlinear dynamical systems using hybrid stabilization of sets. The proposed framework involves a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations. These dynamic controllers combine a logical switching architecture with continuous dynamics to guarantee that a system generalized energy function, whose zero level set characterizes a specified system formation, is strictly decreasing across switchings. The general framework leads to hybrid closed-loop systems described by impulsive differential equations and addresses general nonlinear dynamical systems without limiting consensus and formation control protocols to single and double integrator models.

The contents of the paper are as follows: In Sec. 2, we establish definitions, notation, and review some basic results on impulsive differential equations which provide the mathematical foundation for designing formation control protocols for nonlinear dynamical systems using logic-based hybrid controllers. In Sec. 3, we present a general state-dependent hybrid control framework for stabilization of states. The main result in this section extends the results of Ref. [1] to hybrid stabilization of sets and is not limited to lossless systems. The main result in this section extends the results of and for designing formation control protocols for nonlinear dynamical differential equations which provide the mathematical foundation systems of the form

\[ \dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z} \]  

where

\[ \Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{Z} \]  

and \( \mathcal{Z} = \{ x \in \mathcal{D} : (x, h_p(x)) \in \mathcal{Z}_c \} \), with \( n \triangleq n_p + n_c \) and \( \mathcal{D} \triangleq \mathcal{D}_c \times \mathcal{D}_p \). We refer to the differential equation (6) as the continuous-time dynamics, and we refer to the difference equation (7) as the resetting law. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from Eq. (8) that only those states associated with the controller are reset. To ensure well posedness of the solutions to Eqs. (6) and (7), we make the following additional assumptions [2].

**Assumption 1.** If \( x \in \mathcal{Z} \), then there exists \( \epsilon > 0 \) such that, for all \( 0 < \delta < \epsilon \), \( \psi(\delta, x) \notin \mathcal{Z} \), where \( \psi(\cdot, \cdot) \) denotes the solution to the continuous-time dynamics (6).

**Assumption 2.** If \( x \in \mathcal{Z} \), then \( x + f_d(x) \notin \mathcal{Z} \).

Assumption 1 ensures that if a trajectory reaches the closure of \( \mathcal{Z} \) at a point that does not belong to \( \mathcal{Z} \), then the trajectory must be directed away from \( \mathcal{Z} \); that is, a trajectory cannot enter \( \mathcal{Z} \) through a point that belongs to the closure of \( \mathcal{Z} \) but not to \( \mathcal{Z} \). Furthermore, Assumption 2 ensures that when a trajectory intersects the resetting set, \( \mathcal{Z} \), it instantaneously exits \( \mathcal{Z} \). Finally, we note that if \( x_0 \in \mathcal{Z} \), then the system initially resets to \( x_0^* = x_0 + f_d(x_0) \notin \mathcal{Z} \), which serves as the initial condition for the continuous-time dynamics (6).

A function \( x : \mathcal{I}_{x_0} \to \mathcal{D} \) is a solution to the impulsive dynamical systems (6) and (7) on the interval \( \mathcal{I}_{x_0} \subseteq \mathbb{R} \) with initial condition \( x(0) = x_0 \), where \( \mathcal{I}_{x_0} \) denotes the maximal interval of existence of a solution to Eqs. (6) and (7), if \( x(t) \) is left continuous and \( x(t) \) satisfies Eqs. (6) and (7) for all \( t \in \mathcal{I}_{x_0} \). For further discussion on solutions to impulsive differential equations, see Refs. [2] and [17–22]. For convenience, we use the notation \( x(t, x_0) \) to denote the solution \( x(t) \) of Eqs. (6) and (7) at time \( t \geq 0 \) with initial condition \( x(0) = x_0 \).

For a particular closed-loop trajectory \( x(t) \), we let \( t_k = t_k^{x_0} \) denote the \( k \)-th instant of time at which \( x(t) \) intersects \( \mathcal{Z} \), and we call the times \( t_k \) the resetting times. Thus, the trajectory of the closed-loop systems (6) and (7) from the initial condition \( x(0) = x_0 \) is given by \( y(t, x_0) \) for \( 0 \leq t \leq t_k \). If and when the trajectory reaches a state \( x_1 \) satisfying \( x_1 \in \mathcal{Z} \), then the state is instantaneously transferred to \( x_1^* = x_1 + f_d(x_1) \) according to the resetting law (7). The trajectory \( x(t) \), \( t_1 < t \leq t_2 \), is then given by \( \psi(t - t_1, x_1^*) \), and so on. Our convention here is that the solution \( x(t) \) of Eqs. (6) and (7) is left continuous, that is, it is continuous everywhere except at the resetting times \( t_k \), and \( x_k \triangleq x(t_k) = \lim_{t \to t_k^-} x(t) \), \( x_k^* \triangleq x(t_k^*) = \lim_{t \to t_k^+} x(t) \), \( t_1, t_2, \ldots \), for \( k = 1, 2, \ldots \).

It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times \( t_k = t_k(x_0) \) are distinct and well defined [2]. Since the resetting set \( \mathcal{Z} \) is a subset of the state space and is independent of time, impulsive dynamical systems of the form Eqs. (6) and (7) are time-invariant systems. These systems are called state-dependent impulsive dynamical systems [2]. Since

\[ x_k(t) = f_c(x_k(t), y(t)), \quad x_k(0) = x_0^*, \quad x_k(t) \notin \mathcal{Z} \]

\[ \Delta x_k(t) = f_d(x_k(t)), \quad x_k(t) \in \mathcal{Z} \]

\[ u(t) = h_w(x(t), y(t)) \]

where \( t \geq 0 \), \( x_k(t) \in \mathcal{D}_c \subseteq \mathbb{R}^n_c, \mathcal{D}_c \) is an open set, \( \Delta x_k(t) \triangleq x_k(t^+) - x_k(t) \), where \( x_k(t^+) \triangleq x_k(t) + f_d(x_k(t), y(t)) = \lim_{t \to t_k^+} x(t) \) for \( k = 1, 2, \ldots \).
the resetting times are well defined and distinct and since the solution to Eq. (6) exists and is unique, it follows that the solution of the impulsive dynamical systems (6) and (7) also exists and is unique over a forward time interval. For details on the existence and uniqueness of solutions of impulsive dynamical systems in forward time, see Refs. [17–20].

Remark 2.1. Let \( x^* \in \mathcal{D} \) satisfy \( f_d(x^*) = 0 \). Then \( x^* \notin \mathcal{Z} \). To see this, suppose \( x^* \in \mathcal{Z} \). Then \( x^* + f_d(x^*) = x^* \in \mathcal{Z} \), which contradicts the assumption that if \( x \in \mathcal{Z} \), then \( x + f_d(x) \notin \mathcal{Z} \). Hence, if \( x = x^* \) is an equilibrium point of Eqs. (6) and (7), then \( x^* \notin \mathcal{Z} \).

For the statement of the next result, the following key assumption is needed.

**Assumption 3.** Consider the impulsive dynamical systems (6) and (7), and let \( s(t, x_0), t \geq 0 \), denote the solution to Eqs. (6) and (7) with initial condition \( x_0 \). Then, for every \( x_0 \notin \mathcal{Z} \) and every \( \varepsilon > 0 \) and \( t \neq t_k \), there exists \( \delta(t, x_0, t) > 0 \) such that if \( \| x_0 - z \| < \delta(t, x_0, t) \), \( x \in \mathcal{D} \), then \( |s(t, x_0) - s(t, z)| < \varepsilon \). Assumption 3 is a weakened version of the quasi-continuous dependence assumption given in Refs. [2] and [21] and is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Specifically, by letting \( t \in [0, \infty) \), Assumption 3 specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system’s initial conditions \( x_0 \in \mathcal{D} \) for every time instant. It should be noted that the standard continuous dependence property for dynamical systems with continuous flows is defined uniformly in time on compact intervals. Since solutions of impulsive dynamical systems are not continuous in time and solutions are not continuous functions of the system initial conditions, Assumption 3 involving pointwise continuous dependence is needed to apply the hybrid invariance principle developed in Refs. [2] and [21] to hybrid closed-loop systems. Sufficient conditions that guarantee that the impulsive dynamical systems (6) and (7) satisfy a stronger version of Assumption 3 are given in Refs. [2] and [21] (see also Ref. [23]). The following proposition provides a generalization of Proposition 4.1 in Ref. [21] for establishing sufficient conditions for guaranteeing that the impulsive dynamical systems (6) and (7) satisfy Assumption 3.

**Proposition 2.1.** Consider the impulsive dynamical system \( \mathcal{G} \) given by Eqs. (6) and (7). Assume that Assumptions 1 and 2 hold. Then \( \tau_d(\cdot) \) is continuous at every \( x \notin \mathcal{Z} \) such that \( 0 < \tau_d(x) < \infty \), and if \( x \in \mathcal{Z} \), then \( x + f_d(x) \in \mathcal{Z} \). Furthermore, for every \( x \in \mathcal{Z} \) such that \( 0 < \tau_d(x) < \infty \), assume that the following statements hold:

(i) If a sequence \( \{x_i\}_{i=1}^{\infty} \in \mathcal{D} \) is such that \( \lim_{i \to \infty} x_i = x \) and \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \), then either both \( f_d(x_i) = 0 \) and \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \).

(ii) If a sequence \( \{x_i\}_{i=1}^{\infty} \in \mathcal{Z} \) is such that \( \lim_{i \to \infty} x_i = x \) and \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \), then either both \( f_d(x_i) = 0 \) and \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \),

then \( \mathcal{G} \) satisfies Assumption 3.

The following result provides sufficient conditions for establishing continuity of \( \tau_d(\cdot) \) at \( x \notin \mathcal{Z} \) and sequential continuity of \( \tau_d(\cdot) \) at \( x \in \mathcal{Z} \), that is, \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \) for \( \{x_i\}_{i=1}^{\infty} \notin \mathcal{Z} \) and \( \lim_{i \to \infty} \tau_d(x_i) = \tau_d(x) \). For this result, the following definition is needed. First, however, recall that the Lie derivative of a smooth function \( \chi : \mathcal{D} \to \mathbb{R} \) along the vector field of the continuous-time dynamics \( f_d(x) \) is given by \( L_d^\chi \chi(x) = (\partial \chi(x)/\partial t) \chi(\psi(t,x),x) = (\partial \chi(x)/\partial t) f_d(x) \), and the zeroth and higher order Lie derivatives are, respectively, defined by \( L_0^\chi \chi(x) = \chi(x) \) and

\[
L_k^\chi \chi(x) = L_k^\chi (L_{k-1}^\chi \chi(x)), \quad k \geq 1.
\]

**Definition 2.1.** Let \( Q \neq \{x \in \mathcal{D} : \chi(x) = 0\} \), where \( \chi : \mathcal{D} \to \mathbb{R} \) is an infinitely differentiable function. A point \( x \in Q \), for which \( f_d(x) \neq 0 \), is k- transversal to Eq. (6) if \( f_d(x) \neq 0 \) and \( \chi(x) \neq 0 \) for every \( k \in \{1, 2, \ldots\} \) such that

\[
L_k^\chi \chi(x) = 0, \quad r = 0, 1, \ldots, 2k-2, \quad L_{2k-1}^\chi \chi(x) \neq 0.
\]
respect to compact sets of initial conditions. For further details on this subtle point, see Ref. [26].

**Theorem 2.1.** Consider the impulsive dynamical systems (6) and (7) and assume Assumptions I–3 hold. Assume \( D_{\mathcal{A}} \subset D \) is a positively invariant set with respect to Eqs. (6) and (7), assume that if \( x_0 \in Z \), then \( x_0 + f_a(x_0) \in \mathbb{Z} \), and assume that there exists a continuously differentiable function \( V : D_{\mathcal{A}} \to \mathbb{R} \) such that

\[
V(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise}. \end{cases}
\]

and Eq. (10) is satisfied. Then the set \( D_{\mathcal{A}} \subset D_{\mathcal{A}} \) is asymptotically stable with respect to Eqs. (6) and (7) and \( D_{\mathcal{A}} \) is a subset of the domain of attraction.

**Proof.** It follows from Eq. (12) that \( \mathcal{R} = \{ x \in D_{\mathcal{A}} : x \notin Z, V(x) = 0 \} \). Since for every \( x_0 \in D_{\mathcal{A}} \), there exists \( \tau \geq 0 \) such that \( x(\tau) \in Z \), it follows that the largest invariant set contained in \( \mathcal{R} \) is \( D_{\mathcal{A}} \). Now, the result is a direct consequence of Theorem 2.1.

**3 Hybrid Stabilization of Sets**

In this section, we present a hybrid controller design framework for stabilization of sets. Specifically, we consider nonlinear dynamical systems \( \mathcal{G}_0 \) of the form given by Eqs. (1) and (2). Furthermore, we consider hybrid resetting dynamic controllers \( \mathcal{G}_o \) of the form

\[
\dot{x}_c(t) = f_c(x_c(t), y(t)), \quad x_c(0) = x_{0c}, \quad (x_c(t), y(t)) \notin Z_c
\]

\[
\Delta x(t) = \eta(y(t)) - x_c(t), \quad (x_c(t), y(t)) \in Z_c
\]

\[
y_c(t) = h_c(x_c(t), y(t))
\]

where \( x_c(t) \in D_c \subseteq \mathbb{R}^n \), \( D_c \) is an open set, \( y(t) \in \mathbb{R}^l \), \( y(t) \in \mathbb{R}^n \), \( f_c : D_c \times \mathbb{R}^l \to \mathbb{R}^n \) is smooth on \( D_c \times \mathbb{R}^l \), \( \eta : \mathbb{R}^l \to D_c \) is continuous, and \( h_c : D_c \times \mathbb{R}^n \to \mathbb{R}^n \) is smooth.

Consider the negative feedback interconnection of \( \mathcal{G}_0 \) and \( \mathcal{G}_o \) given by \( y = u_c \) and \( u = -u_c \). In this case, the closed-loop system \( \mathcal{G} \) is given by

\[
\dot{x}(t) = f_c(x(t)), \quad (x(0) = x_0, x(t) \notin Z, \quad t \geq 0
\]

where \( t \geq 0 \), \( x(t) = x_c^T(t) \in \mathbb{R}^n \), \( Z \subseteq \{ x \in D : (x_c, h_c(x_0)) \}

The objective is to design the hybrid resetting controller (13)–(15) in such a way that the set \( D_0 = \{ (x_p, x_0) \in D_P \times D_c : x_0 \notin Z \} \), where \( D_P \subset D_c \), is asymptotically stable with respect to the closed-loop systems (16) and (17). In order to do this, we associate with the plant a generalized energy function \( V : D_c \to \mathbb{R}^n \), such that \( V(x_0, x_0) = 0 \), \( x_0 \in D_P \), and \( V_0(x_0) > 0 \), \( x_0 \in D_P \). Furthermore, we associate with the controller a generalized energy function \( V_c : D_c \times \mathbb{R}^l \to \mathbb{R}^n \) such that \( V_c(x_c, y) \geq 0 \), \( x_c \in D_c \), \( y \in \mathbb{R}^l \), and \( V_c(x_c, y) = 0 \) if and only if \( x_c = \eta(y) \). Finally, we associate with the closed-loop system the generalized energy function \( V(x) = V_c(x_c, y) + V_0(x_0) \).

Next, we construct the resetting set for the closed-loop system \( \mathcal{G} \) in the following way:

\[
Z = \{ (x_p, x_0) \in D_P \times D_c : L_2 V_c(x_c, h_c(x_0)) = 0 \text{ and } V_c(x_c, h_c(x_0)) > 0 \}
\]

The resetting set \( Z \) is thus defined to be the set of all points in the closed-loop state space that correspond to the instant when the controller is at the verge of decreasing its generalized energy function \( V_c(x_0) \). By resetting the controller states, the generalized energy function \( V_c(x_0) \) can never increase after the first resetting event. Furthermore, if the closed-loop system generalized energy function \( V_0(x_0) \) is conserved between resetting events, then a decrease in \( V_c(x_0) \) is accompanied by a corresponding increase in \( V_c(x_0) \). Hence, this approach allows the generalized plant energy to flow to the controller, where it increases the emulated generalized controller energy but does not allow the emulated generalized controller energy to flow back to the plant after the first resetting event.

This energy dissipating hybrid controller effectively enforces a one-way generalized energy transfer between the plant and the controller after the first resetting event. For practical implementation, knowledge of \( x_c \) and \( y \) is sufficient to determine whether or not the closed-loop state vector is in the set \( Z \). That is, the full state \( x_0 \) need not be known in order to determine whether or not the closed-loop state vector is in the set \( Z \), neither is it needed for feedback control between resetting determined by Eq. (15).

The next theorem gives sufficient conditions for asymptotic stability of the set \( D_0 \subset D_P \times D_c \) with respect to the closed-loop system \( \mathcal{G} \) using state-dependent hybrid controllers.

**Theorem 3.1.** Consider the closed-loop impulsive dynamical system \( \mathcal{G} \) given by Eqs. (16) and (17) and assume that \( D_c \subset D \) is a positively invariant set with respect to \( \mathcal{G} \) such that \( D_0 \subset D_c \), where \( D_0 = \{ (x_p, x_0) \in D_P \times D_c : x_0 \notin Z \} \) and \( D_0 \subset D_0 \). Assume that there exists a continuously differentiable function \( V : D_c \to \mathbb{R}^n \) such that \( V(x_0, x_0) = 0 \), \( x_0 \in D_0 \), and \( V(x_0) > 0 \), \( x_0 \in D_0 \), and assume that there exists a smooth (i.e., infinitely differentiable) function \( V_c : D_c \times \mathbb{R}^l \to \mathbb{R}^n \) such that \( V_c(x_c, y) \geq 0 \), \( x_c \in D_c \), \( y \in \mathbb{R}^l \), and \( V_c(x_c, y) = 0 \) if and only if \( x_c = \eta(y) \). Furthermore, assume that every \( x_0 \in Z \) is k-transversal to Eq. (16) and

\[
V_0(x_0) + V_c(x_c, y) = 0, \quad (x_0) \notin Z
\]

where \( y = u_c \) and \( Z \) is given by Eq. (19). Then, the set \( D_0 \subset D_c \) is asymptotically stable with respect to the closed-loop system \( \mathcal{G} \). Finally, if \( D_0 = \mathbb{R}^n \), \( D_c = \mathbb{R}^n \) and \( V \) is radially

**051020-4 / Vol. 136, September 2014** Transactions of the ASME
unbounded, then the set $D_0 \subset D_0$ is globally asymptotically stable with respect to $G$.

Proof. First, note that since $V_c(x_c, y) \geq 0$, $x_c \in D_c, y \in \mathbb{R}^l$, it follows that

$$
\mathcal{Z} = \{(x, y) \in D_0 \times D_c : L_y V_c(x_c, h_p(x_c)) = 0 \text{ and } V_c(x_c, h_p(x_c)) \geq 0\} = \{(x, y) \in D_0 \times D_c : V(x) = 0\}
$$

(21)

where $V(x) = L_y V_c(x_c, h_p(x_c))$. Next, we show that if the $k$-transversality condition (9) holds, then Assumptions 1–3 hold and, for every $x_0 \in D_0$, there exists $t \geq 0$ such that $x(t) \in \mathcal{Z}$. Note that if $x_0 \not\in \mathcal{Z}$, then $V_c(x_0, h_p(x_0)) = 0$, and $L_y V_c(x_0, h_p(x_0)) = 0$, it follows from the $k$-transversality condition that there exists $\delta > 0$ such that for all $t \in (0, \delta)$, $L_y V_c(x(t), h_p(x(t))) \neq 0$. Hence, since $V_c(x(t), h_p(x(t))) = V_c(x_0, h_p(x_0)) + J_y V_c(x(t), h_p(x(t)))$ for some $t \in (0, \delta)$ and $V_c(x_0, h_p(x_0)) \geq 0$, $x_c \in D_c$, $y \in \mathbb{R}^l$, it follows that $V_c(x(t), h_p(x(t))) > 0$, $t \in (0, \delta)$, which implies that Assumption 1 is satisfied. Furthermore, if $x \in \mathcal{Z}$ then, since $V_c(x_c, y) = 0$ if and only if $x_c = \eta(y)$, it follows from Eq. (17) that $x + \delta(x) \not\in \mathcal{Z}$. Hence, Assumption 2 holds.

Next, consider the set $M_\delta \equiv \{x \in D_0 : V_c(x_c, h_p(x_c)) = \gamma\}$, where $\gamma \geq 0$. It follows from the $k$-transversality condition that for every $\gamma \geq 0$, $M_\delta$ does not contain any nontrivial trajectory of $\mathcal{G}$. To see this, suppose, ad absurdum, there is a nontrivial trajectory $x(t) \in M_\delta, t > 0$, for some $\gamma \geq 0$. In this case, it follows that $x(t) = x_0 + \int_0^t J_y V_c(x(t), h_p(x(t))) dt \equiv 0, k = 1, 2, \ldots$, which contradicts the $k$-transversality condition.

Next, we show that for every $x_0 \not\in \mathcal{Z}$, $x_0 \not\in D_0$, there exists $\tau > 0$ such that $x(t) \in \mathcal{Z}$. Thus, it follows that for every $x_0 \not\in \mathcal{Z}$, $0 < \tau(t_0) < \infty$. Now, it follows from Proposition 2.2 that $\tau(t)$ is continuous at $x_0 \not\in \mathcal{Z}$. Furthermore, for all $x_0 \in \overline{\mathcal{Z}}$, and for every unbounded sequence $\{x_i\}_{i=1}^\infty$ converging to $x_0 \in \overline{\mathcal{Z}}$, it follows from the $k$-transversality condition and Proposition 2.2 that $\lim_{t \to \infty} \tau_i(t) = \tau_i(x_0)$. Next, let $x_0 \in \overline{\mathcal{Z}}$, and let $\{x_i\}_{i=1}^\infty$ be such that $\lim_{i \to \infty} x_i = x_0$ and $\lim_{i \to \infty} \tau_i(t)$ exists. In this case, it follows from Proposition 2.2 that either $\lim_{t \to \infty} \tau_i(t) = 0$ or $\lim_{t \to \infty} \tau_i(t) = \tau_i(x_0)$. Furthermore, since $x_0 \in \overline{\mathcal{Z}}$, it follows to the case where $V_c(x_0, h_p(x_0)) = 0$, it follows that $\lim_{t \to \infty} \eta(h_p(x_0)) = 0$, and hence, $\lim_{t \to \infty} f_d(x_0) = 0$. Now, it follows from Proposition 2.1 that Assumption 3 holds.

Next, note that if $x_0 \in \mathcal{Z}$ and $x_0 + f_d(x_0) \not\in D_0$, then it follows from the above analysis that there exists $\tau > 0$ such that $x(t) \not\in \mathcal{Z}$. Alternatively, if $x_0 \in \mathcal{Z}$ and $x_0 + f_d(x_0) \not\in D_0$, then the solution of the closed-loop system reaches $D_0$ in finite time, which is a stronger condition than reaching $D_0$ as $t \to \infty$.

To show that the set $D_0 \subset D_0$ is asymptotically stable, consider the Lyapunov function candidate $V(x) = V_p(x_p) + V_c(x_c, h_p(x_c))$ corresponding to the total generalized energy function. It follows from Eq. (20) that

$$
V(x(t)) = 0, \ x(t) \not\in \mathcal{Z} \tag{24}
$$

Furthermore, it follows from Eqs. (18) and (19) that

$$
\Delta V(x(t)) = V_c(x_c(t^+), h_p(x_c(t^+))) - V_c(x_c(t^-), h_p(x_c(t^-)))
$$

$$
= V_c(x_c(t), h_p(x_c(t))) - V_c(x_c(t), h_p(x_c(t))) = -V_c(x_c(t), h_p(x_c(t)))
$$

$$
x(t) \not\in \mathcal{Z}, \ k \in \overline{\mathcal{Z}} \tag{25}
$$

Thus, it follows from Theorem 2.2 that the set $D_0 \subset D_0$ is asymptotically stable. Finally, if $D_0 = \mathbb{R}^{m_p}, D_c = \mathbb{R}^l$, and $V(t)$ is radially unbounded, then global asymptotic stability is immediate using standard arguments.

To demonstrate the utility of Theorem 3.1, let the set $D_0$ be given by the zero level set of the function $Q_p : D_p \to \mathbb{R}^p$ and let $V_p : D_p \to \mathbb{R}^p$ be given by

$$
V_p(x_p) = Q^T(x_p) P Q(x_p), \ \ x_p \in D_p \tag{26}
$$

where $P \in \mathbb{R}^{m_p \times m_p}$ and $P > 0$. Furthermore, let $V_c : D_c \times \mathbb{R}^l \to \mathbb{R}$ be given by

$$
V_c(x_c, h_p(x_c)) = (x_c - \eta(h_p(x_c)))^T P_c (x_c - \eta(h_p(x_c)))(x_c, x_c) \in D_p \times D_c \tag{27}
$$

where $P_c \in \mathbb{R}^{m_p \times m_p}$ and $P_c > 0$. In this case, the functions $f_d(\cdot), h_p(\cdot),$ and $\eta(\cdot)$ can be selected using Eq. (20) in Theorem 3.1. These constructions are shown for the specific problems of consensus and formation control for multiantisystems in Secs. 4–6.

4 Specialization to Linear Dynamical Systems

In this section, we specialize the results of Sec. 3 to the class of linear dynamical systems given by

$$
x_p(t) = A x_p(t) + B u(t), \ x_p(0) = x_p_0, \ t \geq 0 \tag{28}
$$

$$
y(t) = C x_p(t) \tag{29}
$$

where $x_p(t) \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m_p}$, and $C \in \mathbb{R}^{l \times n}$. Here, for simplicity of exposition, we assume $n_p = n, n_c = n_c$, and $C = I_n$. The case where $C \neq I_n$ can be addressed using an analogous analysis as shown below with $F_2, F_2, H_2, H_2$, and $M$ in Eqs. (32)–(34) replaced by $F_2, C, H_2, C_2$, and $F_2$, respectively. For the systems (28) and (29), we construct a hybrid feedback controller of the form (13)–(15) that asymptotically stabilizes the set $D_0$ given by

$$
D_0 = \{(x_p, x_c) \in D_p \times D_c : x_p \in D_0\} \tag{30}
$$

where

$$
D_0 = \{x_p \in D_p : T x_p = 0\} \tag{31}
$$

and $T \in \mathbb{R}_+^{n \times n}$. Specifically, we set

$$
f_d(x_c, x_p) = F_1 x_c + F_2 x_p \tag{32}
$$

$$
h_c(x_c, x_p) = -H_1 x_c - H_2 x_p \tag{33}
$$

$$
\eta(x_p) = M x_p \tag{34}
$$

where $F_1 \in \mathbb{R}_+^{n \times n}, F_2 \in \mathbb{R}_+^{n \times n}, H_1 \in \mathbb{R}_+^{n \times n}, H_2 \in \mathbb{R}_+^{n \times n}$, and $M \in \mathbb{R}_+^{n \times n}$. Thus, the closed-loop system (28), (29), and (13)–(15) with the negative feedback interconnection $u = -x_c$ is given by

$$
x_p(t) = (A + B H_2) x_p(t) + B H_1 x_c(t), \ (x_p(t), x_c(t)) \not\in \mathcal{Z} \tag{35}
$$
and only if there exist skew-symmetric matrices $A$ where

$$
\Delta x_c(t) = M x_p(t) - x_c(t), \quad (x_p(t), x_c(t)) \in Z
$$

(37)

where $Z$ is given by Eq. (19).

Next, define the generalized energy functions

$$
V_p(x_p) = \frac{1}{2} x_p^T T x_p, \quad x_p \in D_p
$$

(38)

$$
V_c(x_c, x_p) = \frac{1}{2} (x_c - M x_p)^T P_c (x_c - M x_p), \quad (x_p, x_c) \in D_p \times D_c
$$

(39)

where $P_c \in \mathbb{R}^{n_2 \times n_2}$ and $x_c \in \mathbb{R}^{n_2 \times m_2}$. Note that

$$
V_p(x_p) = 0, \quad x_p \in D_p;
$$

and $V_p(x_p) > 0, \quad x_p \notin D_p \setminus D_p$. Furthermore, note that $V_c(x_c, x_p) \geq 0$, $(x_c, x_p) \in D_p \times D_c$, and $V_c(x_c, x_p) = 0$ if and only if $x_c = \eta(x_p)$. For the closed-loop system (35)–(37), condition (20) in Theorem 3.1 gives

$$
\dot{V}_p(x_p(t)) + \dot{V}_c(x_c(t), x_p(t)) = x_p^T (T^T T x_p(t) + P_c (x_c(t), x_p(t)))
$$

(40)

Since $x_p$ and $x_c$ are independent state variables, Eq. (40) holds if and only if there exist skew-symmetric matrices $A_p \in \mathbb{R}^{n \times n}$ and $A_c \in \mathbb{R}^{n_2 \times n_2}$ such that

$$
T^T T B H_1 + F_1^T P_c - A^T T^T M^T P_c - H^2_2 B^T M^T P_c
$$

(41)

$$
+ M^T P_c F_1 + M^T P_c MBH_1 = 0
$$

$$
T^T T A + T^T T B H_2 - M^T P_c F_2 + M^T P_c MA + M^T P_c MBH_2 = A_p
$$

(42)

$$
P_c F_1 - P_c MBH_1 = A_c
$$

(43)

The skew-symmetric matrices $A_p \in \mathbb{R}^{n \times n}$ and $A_c \in \mathbb{R}^{n_2 \times n_2}$ are free design parameters. Furthermore, if the matrices $H_1 \in \mathbb{R}^{n \times n}$ and $H_2 \in \mathbb{R}^{n_2 \times n_2}$ are fixed, then it follows from Eqs. (41)–(43) that

$$
F_1 = P_c^{-1} A_c + MBH_1
$$

(44)

$$
F_2 = MA + MBH_2 - P_c^{-1} A_c M - P_c^{-1} H_2^2 B^T T^T T
$$

(45)

where $M \in \mathbb{R}^{n \times n}$ satisfies

$$
T^T T A + T^T T B H_2 + M^T A_c M + M^T H_2^2 B^T T^T T = A_p
$$

(46)

Note that if $A_c$ is skew symmetric, then $M^T A_c M$ is also skew symmetric. In this case, we can set $A_c = A_p + M^T A_c M$, where $A_p \in \mathbb{R}^{n \times n}$ is an arbitrary skew-symmetric matrix, so that

$$
N M = L
$$

(47)

where

$$
N \triangleq T^T T BH_1
$$

(48)

$$
L \triangleq -A_p - A^T T T - H^2_2 B^T T^T T
$$

(49)

Recall that a solution $M$ to the matrix equation (47) exists if and only if [27, Fact 6.4.43, p. 421]

$$
NN^T L = L
$$

(50)

where $N^T \in \mathbb{R}^{n \times n}$ is the Moore–Penrose generalized inverse of $N \in \mathbb{R}^{n \times n}$. If Eq. (50) is satisfied, then every solution to Eq. (47) is given by

$$
M = N^T L + Y - N^T N Y
$$

(51)

where $Y \in \mathbb{R}^{n \times n}$ is an arbitrary matrix; and if $Y = 0$, then $\text{tr} M^T M$ is minimized. Thus, the existence of a hybrid controller that asymptotically stabilizes the set $D_0$ given by Eqs. (30) and (31) is characterized by a matrix condition (50). Finally, if

$$
T^T T B H_1 \neq 0
$$

(52)

then the $k$-transversality condition (9) is satisfied. To see this, note that Eq. (52) implies that $V_p(x_p) \neq 0$, which, using Eq. (20), implies that $V_c(x_c, x_p) \neq 0$. This shows that $k$-transversality condition, with $k = 1$, holds for the closed-loop system (35)–(37). Note that Eq. (52) is guaranteed by Eq. (50).

5 Hybrid Control Design for Parallel and Rendezvous Formations

In this section, we apply the hybrid control framework developed in Sec. 4 to multiagent systems composed of double integrators executing various coordinated tasks. First, we consider the parallel formation problem for multiagent systems. Specifically, let $q$ denote the number of mobile agents so that $x_p = [x_{p1}, y_{p2}]^T \in \mathbb{R}^{2q}$, where $x_{p1} \in \mathbb{R}^{2q}$ represents a vector of positions, $y_{p2} \in \mathbb{R}^{q}$ represents a vector of velocities, $d$ represents the number of degrees-of-freedom of each agent, and $A$ and $B$ in Eq. (28) are given by

$$
A = \begin{bmatrix} 0 & I_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

(53)

The control aim is to design a hybrid feedback control law so that a parallel formation is achieved, wherein the agents are collectively required to maintain a prescribed geometric shape with constant velocities, and the relative position between any two mobile agents is asymptotically stabilized to a constant value. For this task, we set $d = 2$ and let $q = 5$ so that $x_{p1} = [x_1, y_1, \ldots, x_5, y_5]^T$, where $x_i, y_i, i = 1, \ldots, 5$, are, respectively, horizontal and vertical coordinates of the $i$th agent. The individual agent dynamics are thus given by

$$
\ddot{x}_i(t) = u_x(t), \quad x_i(0) = x_{i0}, \quad t \geq 0
$$

(54)

$$
\ddot{y}_i(t) = u_y(t), \quad y_i(0) = y_{i0}
$$

(55)

where $i = 1, \ldots, 5$ and $u_x$ and $u_y$ are individual control inputs in the horizontal and vertical directions, respectively.

For our hybrid controller design, we set $H_1 = [I_{10}, \; I_{10}], \quad H_2 = [I_{10 \times 10}, \; H_{22}], \quad P_c = 2I_{20}, \quad A_p = 0$, where

$$
\begin{bmatrix}
-2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(56)

and we choose $A_c \in \mathbb{R}^{20 \times 20}$ to be a random skew-symmetric matrix. The specifications of the parallel formation along the $x$
axis with equal distances along the $y$ axis and equal velocities can be characterized by Eq. (31) with

$$T = \begin{bmatrix} T_1 & 0_{1 \times 10} \\ 0_{9 \times 10} & T_2 \end{bmatrix}$$

where

$$T_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

In this case, condition (50) is verified and $M \in \mathbb{R}^{20 \times 20}$ is obtained from Eq. (51) with $y = 0$. Consequently, the matrices $F_1 \in \mathbb{R}^{20 \times 20}$ and $F_2 \in \mathbb{R}^{20 \times 20}$ are computed using Eqs. (44) and (45).

For our simulation, we set $x_p(0) = [0.4, -1, -0.3, 1.3, -1.3, -0.8, -0.1, -0.7, 1.1, 1.9, -0.34, -0.26, 0.2, -0.12, 0.47, 0.47, 0.15, 0.36, -0.1, 0.13]^T$ and $x_c(0) = Mx_p(0)$. Figure 1 shows the positions of the agents in the plane, whereas Figs. 2 and 3 show the control forces in $x$ and $y$ directions, respectively, acting on each agent. Figures 4 and 5 show the agent velocities in the $x$ and $y$ directions, respectively. Finally, Fig. 6 shows the time history of the generalized energy functions $V_p(x_p(t))$ and $V_c(x_c(t), x_p(t))$, $t \geq 0$.

For the next task, we design a hybrid controller (13)–(15) for the rendezvous problem of planar double integrator agents. Specifically, a cooperative rendezvous task requires that each agent determines the rendezvous time and location through team negotiation. In the following simulation, we consider four agents coming to a square formation with zero terminal velocities. In this case, $T = \begin{bmatrix} T_1 & 0_{6 \times 8} \\ 0_{8 \times 8} & T_2 \end{bmatrix}$, where

$$T_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
For our simulation, we set \( H_1 = [I_k, I_k] \), \( H_2 = [0_{k \times k}, -2I_k] \), \( A_c = A_p = 0 \), and \( P_c = 0.7I_{16} \). As in our previous example, condition (50) is verified and \( M \in \mathbb{R}^{10 \times 10} \) is obtained from Eq. (51) with \( Y = 0 \). For the initial conditions \( x_p(0) = [10, 10, 1, 5, 2, 4, 9, 1, 0, 0, 0, 0, 0, 0, 0] \) and \( x_c(0) = Mx_p(0) \), Fig. 7 shows the positions of the agents in the plane, whereas Figs. 8 and 9 show the control forces in \( x \) and \( y \) directions, respectively, acting on each agent. Finally, Fig. 10 shows the time history of the generalized energy functions \( V_p(x_p(t)) \) and \( V_c(x_c(t), x_p(t)) \), \( t \geq 0 \).

6 Hybrid Control Design for Consensus in Multiagent Networks

In this section, we specialize the results of Sec. 4 to design hybrid consensus controllers for multiagent networks of single integrator systems. Specifically, the consensus problem involves the design of a dynamic protocol algorithm that guarantees system state equipartition \([4, 6]\), that is, \( \lim_{t \to \infty} x_p(t) = x \in \mathbb{R} \) for each agent. Finally, Fig. 10 shows the time history of the generalized energy functions \( V_p(x_p(t)) \) and \( V_c(x_c(t), x_p(t)) \), \( t \geq 0 \).

\[
T_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
\]

\[
\begin{align*}
T_3 &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\end{bmatrix}
\end{align*}
\]

where, for each \( i \in \{1, \ldots, q\} \), \( x_p(i) \) denotes the \( i \)th component of the system state vector \( x_p(t) \). In particular, consider \( q \) continuous-time integrator agents with dynamics

\[
\dot{x}_p(i) = u_i(t), \quad x_p(0) = x_0, \quad t \geq 0, \quad i = 1, \ldots, q
\]

\[
y_i(t) = y_p(i)
\]
In order to stabilize $D_0$ given by Eq. (64), consider the hybrid feedback controller (13)–(15) with

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{(q-1)\times q} \quad (66)$$

Note that if $q$ is even, then we can always choose a skew-symmetric matrix $A_c \in \mathbb{R}^{q\times q}$ such that $A_c^{-1}$ exists. In this case, it follows from Eqs. (73) and (74) that

$$M = A_c^{-1}(A_c - H^T T^T T) \quad (75)$$

$$F = P_c^{-1} A_c + A_c^{-1}(A_c - H^T T^T T) H \quad (76)$$

For the following numerical example, we consider four agents with the dynamics given by Eqs. (62) and (63) and the objective being to stabilize the equipartitioned consensus state with $T \in \mathbb{R}^{4\times 4}$ given by Eq. (66). For our design, we set

$$A_c = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0.5 \\ 0 & 1 & -0.5 & 0 \end{bmatrix} \quad (77)$$

Note that with the above choice of $T \in \mathbb{R}^{4\times 4}$ and $H \in \mathbb{R}^{4\times 4}$, condition (52) is satisfied. For the initial conditions, $x_c(0)=[0.5, 1, 0.7, 1.2]^T$ and $x_p(0)=Mx_p(0)$. Fig. 11 shows the system states history versus time, whereas Fig. 12 shows the control input history versus time. Finally, Fig. 13 shows the time history of the generalized energy functions $V_c(x_c(t))$ and $V_p(x_p(t))$ versus time. It can be seen from Fig. 12 that the control inputs $u_i$, $i=1,\ldots,4$, are discontinuous functions of time.

### 7 Hybrid Control Design for Cyclic Pursuit

In this section, we use the results of Sec. 2 to develop a hybrid resetting controller of the form (3)–(5) to achieve circular formations involving cyclic pursuit [16,29,30]. The proposed controller has a leaderless, dynamic distributed architecture, which is
more robust and exhibits faster convergence than static, leader-based, or partially leaderless control designs [16,28–30]. Consider $q$ mobile autonomous agents in a plane described by the unicycle model given by

$$
\dot{x}_i(t) = v_i(t) \cos \theta_i(t), \quad x_i(0) = x_{0i}, \quad t \geq 0
$$

(78)

$$
\dot{y}_i(t) = v_i(t) \sin \theta_i(t), \quad y_i(0) = y_{0i}
$$

(79)

$$
\dot{\theta}_i(t) = \omega_i(t), \quad \theta_i(0) = \theta_{0i}
$$

(80)

where, for each $i \in \{1, \ldots, q\}$, $[x_i, y_i]^T \in \mathbb{R}^2$ denotes the position vector of the $i$th agent, $v_i \in \mathbb{R}$ denotes the orientation of the $i$th agent, $\omega_i \in \mathbb{R}$ denotes the velocity of the $i$th agent, and $u_i = [v_i, \omega_i]^T$ is the control input of the $i$th agent.

For our result, we assume that the graph $G$ of the communication topology for the mobile agents is undirected and strongly connected [3]. The control aim is to design $u_i$ by means of neighboring information so that a circular formation is achieved; that is, the system state asymptotically converges to an invariant manifold characterized by the following constraints [29]:

$$
v_i = c_1, \quad \omega_i = c_2, \quad \sum_{i=1}^{q} \sin \theta_i = 0, \quad \sum_{i=1}^{q} \cos \theta_i = 0
$$

(81)

$$
\sum_{j \in N_i} ([x_i - x_j] \cos \theta_i + [y_i - y_j] \sin \theta_i) = 0, \quad i = 1, \ldots, q
$$

(82)

where $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}$, and $N_i$ denote the set of all neighbors which can communicate with the $i$th agent. Note that if $\theta_i(t) - \theta_j(t) = (2i - j)/q$ for all $i, j = 1, \ldots, q$, then $\sum_{i=1}^{q} \sin \theta_i = 0$ and $\sum_{i=1}^{q} \cos \theta_i = 0$.

Define $h_i(x) = \sum_{j \in N_i} ([x_i - x_j] \cos \theta_i + [y_i - y_j] \sin \theta_i)$, $\hat{h}(x) = \sum_{j \in N_i} ([x_i - x_j] \cos \theta_i + [y_i - y_j] \sin \theta_i)$, where $x = [x_1, \ldots, x_q]^T, \hat{x} = [\hat{x}_1, \ldots, \hat{x}_q]^T, \hat{x} = [\hat{x}_1, \ldots, \hat{x}_q]^T$, and $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_q]^T$, and consider the hybrid resetting controller given by

$$
\dot{x}_{c1}(t) = -Lx_{c1}(t) + h(x_i(t)), \quad x_{c1}(0) = x_{c10}, \quad t \geq 0
$$

(83)

$$
\dot{x}_{c2}(t) = -Lx_{c2}(t) + L\dot{\theta}(t), \quad x_{c2}(0) = x_{c20}, \quad t \geq 0
$$

(84)

$$
\Delta x_{c1}(t) = -L^T \Delta x_{c1}(t), \quad \Delta x_{c1}(0) \in \mathbb{Z}
$$

(85)

$$
\Delta x_{c2}(t) = -L^T \Delta x_{c2}(t), \quad \Delta x_{c2}(0) \in \mathbb{Z}
$$

(86)

$$
\nu(t) = -x_{c1}(t) - h(x_i(t))
$$

(87)

where $L \in \mathbb{R}^{q \times q}$ denotes the Laplacian of the graph $G, x = [x_1, \ldots, x_q]^T, x_{c1} = [x_{c1}, \ldots, x_{c1}]^T$, and $x_{c2} \in \mathbb{R}^q$, and $x_{c2} \in \mathbb{R}^q$. Since, by assumption, $G$ is undirected and strongly connected, it follows that $L^T \geq 0$ and the rank of $L$ is $q - 1$.

Remark 7.1. It follows from Eqs. (87) and (88) that the system velocities and angular velocities are reset due to the resetting of $x_{c1}(t)$ and $x_{c2}(t)$ when $x(t) \in \mathbb{Z}$. This resetting involves an impulse velocity as discussed in Ref. [31]. Using Eq. (4.17) of Ref. [31], p. 125, it follows from Eqs. (85)–(88) that

$$
\nu(t) = \nu(t) + \int_t^{t+} L^T x_{c2} (s) \delta(s-t) \, ds
$$

(89)

$$
\omega(t) = \omega(t) + \int_t^{t+} L^T x_{c2} (s) \delta(s-t) \, ds
$$

(90)

where $L^T x_{c1}(t)$ and $L^T x_{c2}(t)$ can be viewed as input moments and $\delta(t)$ is the Dirac delta function. These equations are a restatement of Eq. (4.17) of Ref. [31], p. 125. Hence, in this case, the control input is an impulse. The notion of impulse velocities has been used in engineering dynamics to model dynamical processes when a moment acts over a very short time interval (see Ref. [31] for further details).

Next, let

$$
V_p(x_p) = \frac{1}{2} x^T L \dot{x} + \frac{1}{2} \dot{x}^T L \dot{x} + \frac{1}{2} \dot{\theta}^T L \dot{\theta}
$$

(91)

and

$$
V_c(x_c, x_p) = \frac{1}{2} x_c^T L x_c + \frac{1}{2} \dot{x}_c^T L \dot{x}_c
$$

(92)

and define the resetting set $\mathcal{Z}$ by

$$
\mathcal{Z} = \{ x \in \mathbb{R}^{2q} : x_{c1}^T [-Lx_{c1} + h(x_p)] + x_{c2}^T L(-Lx_{c2} + L\dot{\theta}) = 0, x_{c2}^T Lx_{c2} > 0 \}
$$

(93)

Furthermore, note that

$$
\mathcal{X} = \{ x \in \mathbb{R}^{2q} : x_{c1}^T [-Lx_{c1} + h(x_p)] + x_{c2}^T L(-Lx_{c2} + L\dot{\theta}) \}
$$

(94)

where “$\Delta$” denotes a diagonal matrix. In addition, note that

$$
\frac{\partial h}{\partial x} =
\begin{bmatrix}
L_{(1,1)} \cos \theta_1 & L_{(1,2)} \cos \theta_1 & \ldots & L_{(1,q)} \cos \theta_1 \\
L_{(2,1)} \cos \theta_2 & L_{(2,2)} \cos \theta_2 & \ldots & L_{(2,q)} \cos \theta_2 \\
\vdots & \vdots & \ddots & \vdots \\
L_{(q,1)} \cos \theta_q & L_{(q,2)} \cos \theta_q & \ldots & L_{(q,q)} \cos \theta_q
\end{bmatrix}
$$

(95)

$$
\frac{\partial \nu}{\partial \dot{y}} =
\begin{bmatrix}
L_{(1,1)} \sin \theta_1 & L_{(1,2)} \sin \theta_1 & \ldots & L_{(1,q)} \sin \theta_1 \\
L_{(2,1)} \sin \theta_2 & L_{(2,2)} \sin \theta_2 & \ldots & L_{(2,q)} \sin \theta_2 \\
\vdots & \vdots & \ddots & \vdots \\
L_{(q,1)} \sin \theta_q & L_{(q,2)} \sin \theta_q & \ldots & L_{(q,q)} \sin \theta_q
\end{bmatrix}
$$

(96)

$$
\frac{\partial \theta}{\partial \theta} =
\begin{bmatrix}
L_{(1,1)} \sin \theta_1 & L_{(1,2)} \sin \theta_1 & \ldots & L_{(1,q)} \sin \theta_1 \\
L_{(2,1)} \sin \theta_2 & L_{(2,2)} \sin \theta_2 & \ldots & L_{(2,q)} \sin \theta_2 \\
\vdots & \vdots & \ddots & \vdots \\
L_{(q,1)} \sin \theta_q & L_{(q,2)} \sin \theta_q & \ldots & L_{(q,q)} \sin \theta_q
\end{bmatrix}
$$

(97)

where $L_{(i,j)}$ denotes the $(i, j)$th entry of $L$, $i, j = 1, \ldots, q$, and

$$
\hat{h}_i(x_p) \overset{\Delta}{=} \sum_{j \in N_i} [-x_i - x_j] \sin \theta_i + (y_i - y_j) \cos \theta_i, \quad i = 1, \ldots, q.
$$

(98)
Then, it follows that

\[
L_{j} \mathcal{X}(x) = x_{1} \frac{\partial}{\partial x_{1}} \text{diag}[\cos \theta_{1}, \ldots, \cos \theta_{q}](x_{1} - h) \\
+ x_{1} \frac{\partial}{\partial x_{1}} \text{diag}[\sin \theta_{1}, \ldots, \sin \theta_{q}](x_{1} - h) - x_{1} \frac{\partial}{\partial x_{1}^{2}} x_{2} \\
- (L x_{2})^{T} L x_{2} + (x_{1} - h) (x_{1} - h)^{T} \\
+ (L x_{1})^{T} (L x_{1} - h) - 2(L x_{2})^{T} (L) \\
x_{2} \frac{\partial}{\partial x_{2}} \text{diag}[\sin \theta_{1}, \ldots, \sin \theta_{q}](x_{1} - h) \\
- x_{2} \frac{\partial}{\partial x_{2}^{2}} x_{2} \\
+ (L x_{1} + h)^{T} (L x_{1} + h) + (L x_{2} + h) \\
- 2(L x_{2} + h)^{T} (L x_{1} + h) \\
+ (L) \\
\] 

\[
= x_{1}^{2} C(\theta)(x_{1} - h) + x_{2}^{2} S(\theta)(x_{1} - h) \\
- (L x_{1} + h)^{T} (L x_{1} + h) + (L x_{2} + h) \\
- 2(L x_{2} + h)^{T} (L x_{1} + h) \\
+ (L) \\
\] 

(98)

where \( h = h(x_{p}), \tilde{h}_{i} = \tilde{h}_{i}(x_{p}), i = 1, \ldots, q, \) and

\[
C(\theta) = \begin{bmatrix}
L_{(1,1)} \cos \theta_{1} & L_{(1,2)} \cos \theta_{1} \cos \theta_{2} & \ldots & L_{(1,q)} \cos \theta_{1} \cos \theta_{q} \\
L_{(2,1)} \cos \theta_{2} \cos \theta_{1} & L_{(2,2)} \cos \theta_{2} & \ldots & L_{(2,q)} \cos \theta_{2} \cos \theta_{q} \\
\vdots & \vdots & \ddots & \vdots \\
L_{(q,1)} \cos \theta_{q} \cos \theta_{1} & L_{(q,2)} \cos \theta_{q} \cos \theta_{2} & \ldots & L_{(q,q)} \cos \theta_{q} \cos \theta_{q}
\end{bmatrix}
\]

(99)

\[
S(\theta) = \begin{bmatrix}
L_{(1,1)} \sin \theta_{1} & L_{(1,2)} \sin \theta_{1} \sin \theta_{2} & \ldots & L_{(1,q)} \sin \theta_{1} \sin \theta_{q} \\
L_{(2,1)} \sin \theta_{2} \sin \theta_{1} & L_{(2,2)} \sin \theta_{2} & \ldots & L_{(2,q)} \sin \theta_{2} \sin \theta_{q} \\
\vdots & \vdots & \ddots & \vdots \\
L_{(q,1)} \sin \theta_{q} \sin \theta_{1} & L_{(q,2)} \sin \theta_{q} \sin \theta_{2} & \ldots & L_{(q,q)} \sin \theta_{q} \sin \theta_{q}
\end{bmatrix}
\]

(100)

Hence, if \( L_{j} \mathcal{X}(x) \neq 0 \) for all \( x \in \mathbb{R}^{5} \) satisfying

\[
\begin{bmatrix}
x_{1} & L x_{2}
\end{bmatrix} \begin{bmatrix}
-L x_{1} + h \\
-L x_{2} + h
\end{bmatrix} = 0
\]

(101)

and

\[
\begin{bmatrix}
diag[\cos \theta_{1}, \ldots, \cos \theta_{q}](x_{1} - h) \\
\text{diag}[\sin \theta_{1}, \ldots, \sin \theta_{q}](x_{1} - h)
\end{bmatrix} \neq 0
\]

(102)

then the \( k \)-transversality condition (9) holds with \( k = 1 \).

Now, it follows from Eqs. (91), (92), (78)–(80), (83), (84), (87), and (88) that

\[
V_{p}(x_{p}) + \tilde{V}(x_{c}, x_{p}) = \frac{1}{2} x_{1}^{2} (L x_{1} - L^{T} L)(L x_{1} - L^{T} L) x_{1} \\
\leq \frac{1}{2} x_{1}^{2} x_{1} x_{c} - \frac{1}{2} x_{2}^{2} L x_{2} \\
< 0, \quad x \in \mathbb{Z}
\]

(103)

Hence, combining Eqs. (103) and (104) yields that

\[
V_{p}(x_{p}) + \tilde{V}(x_{c}, x_{p}) = -h^{2} h - x_{1}^{2} L x_{1} - x_{2}^{2} L x_{2} \leq 0, \quad x \in \mathbb{Z}
\]

(105)

Alternatively, noting that \( L x_{1} - L^{T} L \leq 0 \) and \( ||L x_{1} - L^{T} L||_{2} \leq 1 \), it follows from Eqs. (91), (92), (85), and (86) that

\[
\Delta V_{p}(x_{p}) + \Delta V_{c}(x_{c}, x_{p}) = \frac{1}{2} x_{1}^{2} (L x_{1} - L^{T} L)(L x_{1} - L^{T} L) x_{1} \\
\leq \frac{1}{2} x_{1}^{2} x_{1} x_{c} - \frac{1}{2} x_{2}^{2} L x_{2} \\
< 0, \quad x \in \mathbb{Z}
\]

(106)

Next, let \( R \triangleq \{ x \in D : \tilde{V}_{p}(x_{p}) + \tilde{V}(x_{c}, x_{p}) = 0 \} \), where \( D \subset \mathbb{R}^{5} \) is positively invariant with respect to Eqs. (78)–(80) and (83)–(88). Then it follows that \( R = \{ x \in D : L x_{1} = 0, h = 0 \} \). Let \( M \) denote the largest invariant set contained in \( R \) and note that \( L e = L^{T} e = 0 \) and rank \( L = q - 1 \), where \( e \triangleq [1, \ldots, 1]^{T} \in \mathbb{R}^{q} \). Then it follows from Eq. (84) that \( e^{T} L e = 0 \). Since on \( M, L x_{2} = 0 \), it follows that \( x_{2} = c_{1} e \), and hence, \( \omega = -c_{1} e \), where \( c_{1} \in \mathbb{R} \). Now, it follows from Eq. (84) that \( L \tilde{h} = 0 \). Thus, it follows that \( \tilde{h} = \tilde{h}_{j} \), and hence, \( \tilde{h}_{j} - \tilde{h}_{j} = (2(j - i)/q) \), which further implies that \( \sum_{j=1}^{q} \sin \theta_{j} = 0 \) and \( \sum_{j=1}^{q} \cos \theta_{j} = 0 \). Next, since \( L x_{1} = 0 \) and \( h = 0 \) on \( M \), it follows from Eq. (83) that \( x_{1} = 0 \), and together with \( L x_{1} = 0 \), we thus have \( x_{1} = c_{1} e \), where \( c_{1} \in \mathbb{R} \). Hence, it follows from Eq. (87) that on \( M, \nu = -c_{2} e \). Finally, it follows from Theorem 2.1 that there exists \( D_{0} \subset D \) such that for every system initial condition in \( D_{0}, (x(t), y(t), \theta(t), x_{1}(t), z_{1}(t)) \rightarrow M \) as \( t \rightarrow \infty \), which implies convergence to a circular formation characterized by the manifold (81) and (82).
To show the efficacy of our framework, let $q = 10$ and let

$$L = \begin{bmatrix}
  2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
  -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
  -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}$$

(107)

For this system, the transversality condition was verified numerically for $k = 1$. A group of 10 agents is initialized with random initial positions $(\tilde{x}, \tilde{y})$ in the range of $[-10, 10] \times [-10, 10]$. The initial states of the hybrid resetting controller (83)–(88) are chosen randomly within $[-10, 10]$. Figure 14 shows circular formation is achieved using the hybrid resetting controller (83)–(88). Figures 15–17 show the time histories of the velocities and orientations for circular formation design.

8 Conclusion

In this paper, we have developed a general energy-based hybrid control framework for formation control protocols of general dynamical systems using hybrid stabilization of sets. The proposed framework is used to develop a novel class of fixed-order, energy-based hybrid controllers as a means for achieving cooperative control formations which include flocking, cyclic pursuit, rendezvous, and consensus control of multiagent systems. Specifically, a specified formation is characterized by a hyperplane or manifold in the state space and a hybrid feedback architecture is designed that achieves set stabilization for the desired formation thereby addressing formation control protocols for general nonlinear dynamical models.

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References


