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Approximate Consensus of Multiagent Systems With Inaccurate Sensor Measurements

One of the main challenges in robotics applications is dealing with inaccurate sensor data. Specifically, for a group of mobile robots, the measurement of the exact location of the other robots relative to a particular robot is often inaccurate due to sensor measurement uncertainty or detrimental environmental conditions. In this paper, we address the consensus problem for a group of agent robots with a connected, undirected, and time-invariant communication graph topology in the face of uncertain interagent measurement data. Using agent location uncertainty characterized by norm bounds centered at the neighboring agent's exact locations, we show that the agents reach an approximate consensus state and converge to a set centered at the centroid of the agents' initial locations. The diameter of the set is shown to be dependent on the graph Laplacian and the magnitude of the uncertainty norm bound. Furthermore, we show that if the network is all-to-all connected and the measurement uncertainty is characterized by a ball of radius r, then the diameter of the set to which the agents converge is 2r. Finally, we also formulate our problem using set-valued analysis and develop a set-valued invariance principle to obtain set-valued consensus protocols. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approximate consensus protocol framework.
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Keywords: approximate consensus, sensor uncertainty, multiagent systems, mobile robots, set-valued protocols

1 Introduction

Modern military and national command and control infrastructure capabilities involve large-scale, multilayered network systems placing stringent demands on controller design and implementation of increasing complexity. In numerous large-scale network system applications, agents can detect the location of the neighboring agents only approximately. This problem can arise in network defense systems involving low sensor quality, sensor failure, or detrimental environmental conditions resulting from a large-scale catastrophic event. This problem also arises in many robotics applications with inaccurate sensor data as well as low-cost, small-sized unmanned vehicles with relatively cheap sensors. In such a setting, it is desirable that the agents reach consensus approximately.

In this paper, we consider a multiagent consensus problem in which agents possess sensors with limited accuracy. Specifically, we consider a group of agents with a connected, undirected, and time-invariant communication graph topology and develop consensus control protocols for continuous- and discrete-time network systems that guarantee that the agents reach an approximate consensus state and converge to a set centered at the centroid of the agents' initial locations. This set is shown to be time-varying, in the sense that only the differences between agent positions are, in the limit, small.

For discrete-time network systems, we also use difference inclusions and set-valued analysis to describe the inaccurate sensor measurement problem formulation. Set-valued analysis has been previously used for consensus control. In Ref. [1], the author uses set-valued Lyapunov functions to study convergence of multiagent dynamical systems. The approach involves constructing set-valued Lyapunov functions from convex sets that depend on the agent states. In Refs. [1–3], the authors address stability of each equilibrium point in the sense that the system solutions approach an equilibrium from a neighborhood of equilibria. Lorenz and Lorenz [3] consider barycentric coordinate maps, whereas Moreau [1] and Angeli and Bliman [2] consider difference equations and difference inclusions, respectively.

Necessary and sufficient conditions for semistability for multiagent consensus problems using set-valued Lyapunov analysis are presented in Ref. [4]. More recently, Xiao and Wang [5] consider an asynchronous rendezvous problem using set-valued consensus theory. Specifically, a design strategy for multiagent consensus is developed by requiring two consecutive way-points to be included within a minimum convex region covering the two associated anticipated-way-point sets.

The proposed set-valued consensus protocol builds on the framework of Refs. [1], [4], and [6] to develop approximate consensus protocols for multiagent systems with uncertain interagent measurements. Specifically, the proposed protocol algorithm modifies the set-valued consensus update maps of the agents by assuming that the locations of all the agents, including the agents calculating the update map, are within a ball of radius r. However, since the update sets of our design protocol do not satisfy a strict convexity assumption, our results go beyond the results of Ref. [1] by employing a set-valued invariance principle.

This paper can be viewed as a contribution to the literature addressing multiagent systems in the presence of adversarial attacks. Specifically, the authors in Refs. [7–15] utilize stochastic tools and methods to study the behavior of multiagent systems in the presence of communication noise, transmission delays, and packet losses, whereas the authors in Refs. [16–22] utilize nonlinear and adaptive system theory to study the behavior of multiagent systems in the presence of agent and graph topology uncertainties. The results in this paper addressing multiagent systems with inaccurate sensor measurements add to this literature by utilizing complementary analysis methods including Lyapunov theory, difference inclusions, and set-valued analysis.

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2 Notation, Definitions, and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ column vectors, $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices, $\overline{\mathbb{Z}}_+$ denotes the set of non-negative integers, and $(\cdot)^{\mathrm{T}}$ denotes transpose. We write $\lambda_{\min}(M)$ (respectively, $\lambda_{\max}(M)$) for the minimum (respectively, maximum) eigenvalue of the Hermitian matrix M, $\sigma_{\max}(M)$ for the maximum singular value of the matrix M, $\rho(M)$ for the spectral radius of the square matrix M, spec(M) for the spectrum of the square matrix M including multiplicity, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the *open ball centered* at α with *radius* ε, $\mathrm{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(k) \to \mathcal{M}$ as $k \to \infty$, where $k \in \overline{\mathbb{Z}}_+$, to denote that the trajectory $x(k)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $N_0 > 0$, such that $\mathrm{dist}(x(k), \mathcal{M}) < \varepsilon$ for all $k > N_0$.

Moreover, in this paper we distinguish between the set inclusions $\subset$ and $\subseteq$, namely, $\subset$ denotes a strict inclusion, whereas $\subseteq$ denotes a nonstrict inclusion. In addition, we use the Minkowski sum for summation of sets with an analogous definition for set subtraction. Namely, for the sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ denote, respectively, the set of all the vectors $z \in \mathbb{R}^n$ such that $z = x + y$ and $z = x - y$, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Finally, the notions of openness, convergence, continuity, and compactness that we use throughout the paper refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

The consensus problem we address in this paper appears frequently in coordination of multiagent network systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with undirected and directed information flow [23–25]. In this paper, we use *undirected graphs* to represent a static network. The graph-theoretic notation and terminology we use in the paper are standard [26]. Specifically, $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denotes a weighted *directed graph* (or *digraph*) denoting the static network (or static graph) with the set of *nodes* (or *vertices*) $\mathcal{V} = \{1, \ldots, N\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow between agents, and a *weighted adjacency matrix* $\mathcal{A} \in \mathbb{R}^{N\times N}$ such that $\mathcal{A}_{(i,j)} = a_{ij} > 0$, $i, j = 1, \ldots, N$, if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. The edge $(j, i) \in \mathcal{E}$ denotes that agent i can obtain information from agent j, but not necessarily vice versa. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$.

Note that if the weights a_{ij}, $i, j = 1,\ldots, N$, are not relevant, then a_{ij} is set to 1 for all $(j, i) \in \mathcal{E}$. In this case, $\mathcal{A}$ is called a *normalized adjacency matrix*. Every edge $\ell \in \mathcal{E}$ corresponds to an ordered pair of vertices $(i, j) \in \mathcal{V} \times \mathcal{V}$, where i and j are the *initial* and *terminal* vertices of the edge ℓ. In this case, ℓ is *incident into* j and *incident out of* i. Finally, we say that $\mathfrak{G}$ is *strongly* (respectively, *weakly*) *connected* if for every ordered pair of vertices (i, j), $i \neq j$, there exists a *directed* (respectively, *undirected*) *path*, that is, a directed (respectively, undirected) sequence of arcs, leading from i to j.

The *in-neighbors* and *out-neighbors* of node i are, respectively, defined as $\mathcal{N}_{\mathrm{in}}(i) \triangleq \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_{\mathrm{out}}(i) \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The *in-degree* $\deg_{\mathrm{in}}(i)$ of node i is the number of edges incident into i, and the *out-degree* $\deg_{\mathrm{out}}(i)$ of node i is the number of edges incident out of i, that is, $\deg_{\mathrm{in}}(i) \triangleq \sum_{j=1}^N a_{ji}$ and $\deg_{\mathrm{out}}(i) \triangleq \sum_{j=1}^N a_{ij}$. We say that the node i of a digraph $\mathfrak{G}$ is *balanced* if $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i)$, and a graph $\mathfrak{G}$ is called *balanced* if all of its nodes are balanced, that is, $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$, $i = 1, \ldots, N$. Furthermore, we define the *graph Laplacian* and *Perron matrix* of $\mathfrak{G}$ as $\mathcal{L} \triangleq \Delta - \mathcal{A}$ and $\mathcal{P} \triangleq I - \varepsilon\mathcal{L}$, respectively, where $\varepsilon > 0$ and $\Delta \triangleq \mathrm{diag}[\deg_{\mathrm{in}}(1), \ldots, \deg_{\mathrm{in}}(N)]$.

A *graph* or *undirected graph* $\mathfrak{G}$ associated with the adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ is a directed graph for which the arc set is symmetric, that is, $\mathcal{A} = \mathcal{A}^{\mathrm{T}}$. In this case, $\mathcal{N}_{\mathrm{in}}(i) = \mathcal{N}_{\mathrm{out}}(i) \triangleq \mathcal{N}(i)$ and $\deg_{\mathrm{in}}(i) = \deg_{\mathrm{out}}(i) \triangleq \deg(i)$, $i = 1, \ldots, N$. Furthermore, in this case, we say that $\mathfrak{G}$ is *connected* if for every ordered pair of vertices (i, j), $i \neq j$, there exists a *path*, that is, a sequence of arcs, leading from i to j. A graph is *all-to-all connected* if every node of $\mathfrak{G}$ is connected to every other node of $\mathfrak{G}$. Finally, we denote the value of the node $i \in \{1,\ldots, N\}$ at time t (respectively, time step k) by $x_i(t) \in \mathbb{R}^n$ (respectively, $x_i(k) \in \mathbb{R}^n$).

In light of the above definitions, the consensus problem involves the design of a dynamic algorithm that guarantees system state equipartition [23,25], that is, $\lim_{t\to\infty} x_i(t) = q \in \mathbb{R}^n$ for $i = 1,\ldots, N$. This problem can be characterized as a network involving trajectories of a multiagent dynamical system $\mathcal{G}$ given by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N \tag{1}$$

$$u_i(t) = \sum_{j\in\mathcal{N}_{\mathrm{in}}(i)} (x_j(t) - x_i(t)), \quad i = 1, \ldots, N \tag{2}$$

Here, $x_i(t)$, $t \geq 0$, represents an *information state* and $u_i(t)$, $t \geq 0$, represents an *information control input* with a distributed consensus algorithm involving neighbor-to-neighbor interaction between agents.

In this paper, we consider the continuous-time distributed consensus algorithms (1) and (2) resulting in closed-loop systems of the form [23]

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_{\mathrm{in}}(i)} (x_j(t) - x_i(t)), \quad x_i(0) = x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N \tag{3}$$

as well as a discrete-time distributed consensus algorithm resulting in closed-loop systems of the form [25]

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j\in\mathcal{N}_{\mathrm{in}}(i)} (x_j(k) - x_i(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, N \tag{4}$$

where $\varepsilon > 0$. Even though in this paper we limit our attention to a network involving a chain of integrator multiagent dynamical systems $\mathcal{G}$, the proposed framework can be readily extended to designing low-level feedback consensus controllers for a network involving high-order, complex multiagent dynamical systems $\mathcal{G}$ (see Example 7.2 in Sec. 7).

3 Consensus Control Problem With Uncertain Interagent Location Measurements

In this paper, we consider a multiagent network in which N agents with a connected, undirected, and time-invariant communication graph topology reach an approximate consensus state and we use the terminology *agent state* and *agent location* interchangeably. Here, we do not consider time delays and communication losses between agents. In particular, each agent $i \in \{1,\ldots, N\}$ has a sensor with accuracy r, that is, each agent i can detect the location of the other agents with an accuracy of up to a ball of radius r centered at the actual location of the other agents. The approximate location of agent i as measured by agent j is given by the set

$$\mathcal{X}_i = \{p \in \mathbb{R}^n : \|p - x_i\|_2 \leq r\}, \quad i = 1, \ldots, N$$

The network consensus problem considered in this paper involves the design of a dynamic protocol that guarantees approximate system state equipartition, that is, the difference between any two agent states decreases to below a certain threshold that is dependent on the sensor accuracy r. Specifically, each agent i uses an update protocol resulting in a closed-loop system similar to

Eq. (3) or (4). However, since only approximate information of the location of the other agents is available at any given instant of time, the update protocol is constructed using approximate location information only.

In particular, for a discrete-time network system, the update protocol for a connected graph has the form

$$x_i(k+1) \in \mathcal{F}_i(x(k)), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, N \tag{5}$$

where

$$\mathcal{F}_i(x(k)) \triangleq x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_{\text{in}}(i)} (\mathcal{X}_j(k) - x_i(k))$$

$x \triangleq [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, and $\mathcal{X}_j - x_i$ denotes the set of all the vectors $z \in \mathbb{R}^n$ such that $z = y - x_i$ with $y \in \mathcal{X}_j$. Note that for the protocol given by Eq. (4), every agent has information of the exact location of the other agents, whereas for the protocol given by Eq. (5) only approximate location information of the other agents is available.

To further elucidate the protocol architecture given by Eq. (5), consider an all-to-all connected network consisting of three agents. In this case, the update protocol for agent 1 is given by

$$\begin{aligned} x_1(k+1) \in \mathcal{F}_1(x(k)) = x_1(k) + \varepsilon(\mathcal{X}_1(k) - x_1(k) + \mathcal{X}_2(k) \\ - x_1(k) + \mathcal{X}_3(k) - x_1(k)) \\ x_1(0) = x_{10}, \quad k \in \overline{\mathbb{Z}}_+ \end{aligned}$$

where the sets $\mathcal{X}_2 - x_1$ and $\mathcal{X}_3 - x_1$ are depicted in Fig. 1; that is, the measurement of the exact location of agents 2 and 3 are uncertain due to sensor measurement uncertainty or detrimental environmental conditions.

4 Continuous-Time Consensus With a Connected Graph Topology

In this section, we consider the continuous-time consensus problem over an undirected communication network with a connected graph topology. We assume that only approximate information of the location of neighboring agents is available at any given instant of time with the ith agent uncertainty satisfying $\|d_i(t)\|_2 \le r$, $t \ge 0$, for $i = 1, \ldots, N$. Here, we assume that the class of uncertainties we consider does not affect the graph topology (see, for example, Ref. [21] for a class of uncertainties affecting the communication graph topology). In particular, we consider the update protocol for agent i given by

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}(i)} (z_j(t) - z_i(t)), \quad x_i(0) = x_{i0}, \quad t \ge 0, \quad i = 1, \ldots, N \tag{6}$$

where

$$z_j(t) - z_i(t) \triangleq (x_j(t) - d_j(t)) - (x_i(t) - d_i(t))$$

Fig. 1 Visualization of sets $\mathcal{X}_2 - x_1$ and $\mathcal{X}_3 - x_1$ used in agent's 1 update map

In this case, it follows from Eq. (6) that

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}(i)} (x_j(t) - x_i(t)) + \sum_{j \in \mathcal{N}(i)} (d_i(t) - d_j(t)), \quad x_i(0) = x_{i0}, \quad t \ge 0, \quad i = 1, \ldots, N$$

or, equivalently, in compact form

$$\dot{x}(t) = -\tilde{\mathcal{L}}x(t) + \tilde{\mathcal{L}}d(t), \quad x(0) = x_0, \quad t \ge 0 \tag{7}$$

where $\tilde{\mathcal{L}} \triangleq I_n \otimes \mathcal{L} \in \mathbb{R}^{nN \times nN}$, $\mathcal{L} \in \mathbb{R}^{N \times N}$ denotes the graph Laplacian, $\otimes$ denotes Kronecker product, $x \triangleq [x_1^1, \ldots, x_N^1, \ldots, x_1^n, \ldots, x_N^n]^{\mathrm{T}}$, $d \triangleq [d_1^1, \ldots, d_N^1, \ldots, d_1^n, \ldots, d_N^n]^{\mathrm{T}}$, and x_i^j and d_i^j denote the jth component of x_i and d_i, respectively.

Although our results can be directly extended to the case of Eq. (7), for simplicity of exposition, we will focus on individual agent states evolving in $\mathbb{R}$ (i.e., $n = 1$). In this case, Eq. (7) becomes

$$\dot{x}(t) = -\mathcal{L}x(t) + \mathcal{L}d(t), \quad x(0) = x_0, \quad t \ge 0 \tag{8}$$

For the statement of the next result, let $\mathbf{e}_N \triangleq [1, \ldots, 1]^{\mathrm{T}}$ denote the ones vector of order N and $\overline{x} \triangleq (1/N)\mathbf{e}_N^{\mathrm{T}}x$. Furthermore, recall that the Laplacian of an undirected connected graph is a symmetric positive semidefinite matrix with a single zero eigenvalue [25]; specifically, the eigenvalues of the graph Laplacian are given by $0 = \lambda_{\min}(\mathcal{L}) \triangleq \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \le \lambda_3(\mathcal{L}) \le \cdots \le \lambda_N(\mathcal{L}) \triangleq \lambda_{\max}(\mathcal{L})$. Hence, the Schur decomposition of $-\mathcal{L}$ is given by $-\mathcal{L} = P_\Sigma \Sigma P_\Sigma^{\mathrm{T}}$, where $P_\Sigma \triangleq [p_1, \ldots, p_{N-1}, (1/\sqrt{N})\mathbf{e}_N]$, with $p_i \in \mathbb{R}^N$, $i = 1, \ldots, N-1$

$$\Sigma \triangleq \begin{bmatrix} \Sigma_0 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix}$$

and $\Sigma_0 \in \mathbb{R}^{(N-1)\times(N-1)}$ is Hurwitz.

THEOREM 4.1: *Consider an undirected network of N agents with a connected graph topology given by Eq. (8). Then,*

$$\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N\overline{x}(t)\|_2 \le \frac{\lambda_N(\mathcal{L})\sqrt{N}r}{\lambda_2(\mathcal{L})}$$

Proof. First, define $\delta(t) \triangleq x(t) - \mathbf{e}_N\overline{x}(t)$ and note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{N}\mathbf{e}_N^{\mathrm{T}}x(t)\right) = \frac{1}{N}\mathbf{e}_N^{\mathrm{T}}(-\mathcal{L}x(t) + \mathcal{L}d(t)) = 0_N$$

where we used the fact that $\mathcal{L}\mathbf{e}_N = 0_N$ and $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$. Hence, $\overline{x}(t) = (1/N)\mathbf{e}_N^{\mathrm{T}}x(t) = (1/N)\mathbf{e}_N^{\mathrm{T}}x(0) = \overline{x}$, $t \ge 0$, which shows that the centroid of the network does not change over time in the presence of time-varying interagent measurement uncertainties.

Next, differentiating $\delta(t)$ with respect to time yields

$$\begin{aligned} \dot{\delta}(t) &= \dot{x}(t) - \mathbf{e}_N\dot{\overline{x}}(t) \\ &= -\mathcal{L}x(t) + \mathcal{L}d(t) \\ &= -\mathcal{L}[\delta(t) + \mathbf{e}_N\overline{x}(t)] + \mathcal{L}d(t) \\ &= -\mathcal{L}\delta(t) + \mathcal{L}d(t), \quad \delta(0) = \delta_0, \quad t \ge 0 \end{aligned} \tag{9}$$

Introducing the transformation $q(t) \triangleq P_\Sigma^{\mathrm{T}}\delta(t)$, it follows from Eq. (9) that

$$\begin{aligned} \dot{q}(t) &= P_\Sigma^{\mathrm{T}}\dot{\delta}(t) \\ &= -P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma P_\Sigma^{\mathrm{T}}\delta(t) + P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma P_\Sigma^{\mathrm{T}}d(t) \\ &= -P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma q(t) + P_\Sigma^{\mathrm{T}}\mathcal{L}P_\Sigma\overline{d}(t), \quad q(0) = q_0, \quad t \ge 0 \end{aligned}$$

where $\overline{d}(t) \triangleq P_\Sigma^{\mathrm{T}} d(t)$, and hence

$$\dot{q}(t) = \begin{bmatrix} \Sigma_0 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix} [q(t) - \overline{d}(t)], \quad q(0) = q_0, \quad t \geq 0 \tag{10}$$

Now, it follows from Eq. (10) that

$$\dot{q}_1(t) = \Sigma_0 q_1(t) - \Sigma_0 \overline{d}_1(t), \quad q_1(0) = q_{10}, \quad t \geq 0 \tag{11}$$

$$\dot{q}_2(t) = 0, \quad q_2(0) = q_{20} \tag{12}$$

where

$$q_1(t) \triangleq [I_{(N-1)\times(N-1)} \quad 0_{(N-1)\times 1}] q(t),$$
$$\overline{d}_1(t) \triangleq [I_{(N-1)\times(N-1)} \quad 0_{(N-1)\times 1}] \overline{d}(t)$$

and $q_2 \in \mathbb{R}$. Furthermore, note that $q_{20} = 0$ since $\mathbf{e}_N^{\mathrm{T}} \delta(t) = \mathbf{e}_N^{\mathrm{T}} x(t) - (1/N)\mathbf{e}_N^{\mathrm{T}} \mathbf{e}_N \mathbf{e}_N^{\mathrm{T}} x(t) = 0$.

Next, consider the Lyapunovlike function $V : \mathbb{R}^{(N-1)} \to \mathbb{R}$ given by $V(q_1) = q_1^{\mathrm{T}} S q_1$, where $S = S^{\mathrm{T}} > 0$, $S \in \mathbb{R}^{(N-1)\times(N-1)}$, satisfies

$$0 = \Sigma_0^{\mathrm{T}} S + S\Sigma_0 + Q \tag{13}$$

with $Q = Q^{\mathrm{T}} > 0$ and $Q \in \mathbb{R}^{(N-1)\times(N-1)}$. Now, note that the derivative of $V(q_1)$ along the trajectories of Eq. (11) is given by

$$\begin{aligned} \dot{V}(q_1(t)) &= -q_1^{\mathrm{T}}(t) Q q_1(t) - 2 q_1^{\mathrm{T}}(t) S \Sigma_0 \overline{d}_1(t) \\ &\leq -\lambda_{\min}(Q) \|q_1(t)\|_2^2 + 2\sigma_{\max}(S\Sigma_0) \\ &\quad \times \sigma_{\max}([I_{(N-1)\times(N-1)} \quad 0_{(N-1)\times 1}]) \\ &\quad \times \sigma_{\max}(P_\Sigma^{\mathrm{T}}) \|d(t)\|_2 \|q_1(t)\|_2 \\ &\leq -\lambda_{\min}(Q) \|q_1(t)\|_2^2 + 2\sigma_{\max}(S\Sigma_0)\sqrt{N} r \|q_1(t)\|_2 \\ &= -\|q_1(t)\|_2 [\lambda_{\min}(Q)\|q_1(t)\|_2 - 2\sigma_{\max}(S\Sigma_0)\sqrt{N} r], \quad t \geq 0 \end{aligned} \tag{14}$$

where we used the fact that $\sigma_{\max}([I_{(N-1)\times(N-1)} \quad 0_{(N-1)\times 1}]) = 1$, $\sigma_{\max}(P_\Sigma^{\mathrm{T}}) = 1$, and $\|d(t)\|_2 \leq \sqrt{N} r$, $t \geq 0$.

Next, it follows from Eq. (14) that $\dot{V}(q_1(t)) \leq 0$ for $\|q_1(t)\|_2 \geq ((2\sigma_{\max}(S\Sigma_0)\sqrt{N}r)/(\lambda_{\min}(Q))) \triangleq \beta$ and $t \geq 0$, and hence, $q_1(t)$, $t \geq 0$, is decreasing for $\|q_1(t)\|_2 > \beta$. Moreover, since $\dot{q}_2(t) = 0$, $t \geq 0$, and $q_2(0) = 0$, $q_2(t) = 0$ for all $t \geq 0$. Hence, it follows from the definition of $q(t)$ and Eq. (14) that

$$\|\delta(t)\|_2 = \left\| \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \right\|_2 = \|q_1(t)\|_2 \leq \beta$$

as $t \to \infty$. Now, setting $Q = -\Sigma_0$ it follows from Eq. (13) that $S = (1/2) I_{(N-1)}$, and hence, $\|q_1(t)\|_2 = \|x(t) - \mathbf{e}_N \overline{x}\|_2 \leq \beta$, $t \geq 0$, where

$$\beta = \frac{2\sigma_{\max}((1/2)\Sigma_0)\sqrt{N} r}{\lambda_{\min}(-\Sigma_0)} = \frac{\lambda_N(\mathcal{L})\sqrt{N} r}{\lambda_2(\mathcal{L})} \tag{15}$$

■

Remark 4.1. It is of importance to note that if all the sensor uncertainties are identical, that is, $d_i(t) = d_0(t)$ for all $i = 1, \ldots, N$, then it follows from Theorem 4.1 that all the agents reach exact agreement since in this case $\mathcal{L} d(t) = \mathcal{L} \mathbf{e}_N d_0(t) = 0$ in Eq. (8).

Note that since, by Theorem 4.1, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq ((\lambda_N(\mathcal{L})\sqrt{N}r)/(\lambda_2(\mathcal{L})))$, it follows that as the number of agents increases the uncertainty plays a prominent effect on the system. It is also important to note that the bound $((\lambda_N(\mathcal{L})\sqrt{N}r)/(\lambda_2(\mathcal{L})))$ depends on the ratio of $\lambda_N(\mathcal{L})$ and $\lambda_2(\mathcal{L})$. For example, consider a set of agents on a line graph. In this case, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 1.41r$ for $N = 2$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 5.19r$ for $N = 3$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 11.66r$ for $N = 4$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 21.17r$ for $N = 5$, and $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 34.77r$ for $N = 6$. Now, consider a set of agents on an all-to-all graph. In this case, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 1.41r$ for $N = 2$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 1.73r$ for $N = 3$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 2.00r$ for $N = 4$, $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 2.23r$ for $N = 5$, and $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2 \leq 2.44r$ for $N = 6$.

It is clear from the previous examples that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2$ increases with the size of the network. It is also interesting to note that a network designer can introduce additional connectivity between of agents to keep the bound on $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}(t)\|_2$ small as the size of the network is increased. This is clearly demonstrated in the above examples, wherein less conservative bounds are obtained for an all-to-all graph topology, which has a higher degree of connectivity between agents as compared to agents with a line graph topology.

Next, we apply Theorem 4.1 to an all-to-all connected graph network. Note that in this case, $\mathcal{L} = N I_N - E_N$, where $E_N \triangleq \mathbf{e}_N \mathbf{e}_N^{\mathrm{T}}$ denotes the *ones matrix* of order $N \times N$. Since rank $E_N = 1$, E_N has only one nonzero eigenvalue equal to N with corresponding eigenvector $\mathbf{e}_N$. Next, note that

$$\det[\lambda I_N - \mathcal{L}] = \det[\lambda I_N - (N I_N - E_N)] = \det[(\lambda - N) I_N + E_N]$$

Hence, the eigenvalues of $\mathcal{L}$ are the eigenvalues of $-E_N$ shifted by N, that is, $\mathrm{spec}(-E_N) = \{0, N, \ldots, N\}$. Now, with $\lambda_2(\mathcal{L}) = \cdots = \lambda_N(\mathcal{L}) = N$, it follows from Theorem 4.1 that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \overline{x}\|_2 \leq \sqrt{N} r$.

Alternatively, we can arrive at the same result directly by considering the update protocol for the ith agent given by

$$\begin{aligned} \dot{x}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} \left[\left(x_j(t) - d_j(t)\right) - \left(x_i(t) - d_i(t)\right) \right] \\ &= \overline{x}(t) - x_i(t) - \overline{d}(t) + d_i(t) \\ x_i(0) &= x_{i0}, \quad t \geq 0, \quad i = 1, \ldots, N \end{aligned} \tag{16}$$

where $\overline{x}(t) \triangleq (1/N)\sum_{j=1}^{N} x_j(t) \equiv \overline{x}$ and $\overline{d}(t) \triangleq (1/N)\sum_{j=1}^{N} d_j(t)$. First, note that it can be shown that $\limsup_{t\to\infty} \|x_i(t) - x_j(t)\|_2 \leq 2r$ for every $i, j = 1, \ldots, N$.

To see this, for $i, j = 1, \ldots, N$, it follows from Eq. (16) that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|x_i(t) - x_j(t)\|_2^2\right) \\ &= \left(x_i(t) - x_j(t)\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t}\left(x_i(t) - x_j(t)\right) \\ &= \left(x_i(t) - x_j(t)\right)^{\mathrm{T}} \left[\overline{x} - x_i(t) - \overline{d}(t) + d_i(t) - \left(\overline{x} - x_j(t) - \overline{d}(t) + d_j(t)\right)\right] \\ &= -\|x_i(t) - x_j(t)\|_2^2 + \left(x_i(t) - x_j(t)\right)^{\mathrm{T}} \left(d_i(t) - d_j(t)\right) \\ &\leq -\|x_i(t) - x_j(t)\|_2^2 + 2r\|x_i(t) - x_j(t)\|_2 \\ &\quad x_i(0) - x_j(0) = x_{i0} - x_{j0}, \ t \geq 0 \end{aligned}$$

where the last inequality follows from the fact that

$$\|d_i(t) - d_j(t)\|_2 \leq \|d_i(t)\|_2 + \|d_j(t)\|_2 \leq 2r, \quad t \geq 0$$

Hence, $\|x_i(t) - x_j(t)\|_2$ is a decreasing function of time as long as $\|x_i(t) - x_j(t)\|_2 > 2r$, $t \geq 0$. Now, it follows that $\|x_i(t) - x_j(t)\|_2 \leq 2r$ as $t \to \infty$ for all $i, j = 1, \ldots, N$.

Next, since $\overline{x}(t) \equiv \overline{x}$, it follows that $\|x_i(t) - \overline{x}\|_2 \leq r$ as $t \to \infty$ for all $i = 1, \ldots, N$. Furthermore, since

$$\|x(t) - \mathbf{e}_N \overline{x}\|_2^2 = \sum_{i=1}^{N} \|x_i(t) - \overline{x}\|_2^2 \leq N r^2$$

as $t \to \infty$, it follows that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N\bar{x}\|_2 \le \sqrt{N}r$, which is identical to the result obtained by applying Theorem 4.1 directly.

5 Discrete-Time Consensus With a Connected Graph Topology

In this section, we consider the discrete-time consensus problem over an undirected network with a connected graph topology. Once again, we assume that only approximate information of the location of neighboring agents is available at any given instant of time with the ith agent uncertainty satisfying $\|d_i(k)\|_2 \le r$, $k \in \overline{\mathbb{Z}}_+$, for $i = 1,\dots, N$. In particular, we consider the update protocol for agent i given by

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j\in\mathcal{N}(i)} (z_j(k) - z_i(k)),\ \ x_i(0) = x_{i0},\ \ k \in \overline{\mathbb{Z}}_+,\quad i = 1,\dots,N \tag{17}$$

where

$$z_j(k) - z_i(k) \triangleq (x_j(k) - d_j(k)) - (x_i(k) - d_i(k))$$

and $\varepsilon > 0$. In this case, it follows from Eq. (17) that

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j\in\mathcal{N}(i)} (x_j(k) - x_i(k)) + \varepsilon \sum_{j\in\mathcal{N}(i)} (d_i(k) - d_j(k)),\quad x_i(0) = x_{i0},\quad k \in \overline{\mathbb{Z}}_+,\quad i = 1,\dots,N$$

or, equivalently, in compact form

$$x(k+1) = \tilde{\mathcal{P}}x(k) + \varepsilon\tilde{\mathcal{L}}d(k),\quad x(0) = x_0,\quad k \in \overline{\mathbb{Z}}_+ \tag{18}$$

where $\tilde{\mathcal{L}} \triangleq I_n \otimes \mathcal{L} \in \mathbb{R}^{nN\times nN}$, $\tilde{\mathcal{P}} \triangleq I_n \otimes \mathcal{P} \in \mathbb{R}^{nN\times nN}$, $\mathcal{L} \in \mathbb{R}^{N\times N}$ denotes the graph Laplacian, $\mathcal{P} \triangleq I_N - \varepsilon\mathcal{L} \in \mathbb{R}^{N\times N}$ denotes the Perron matrix, $x \triangleq [x_1^1,\dots,x_N^1,\dots,x_1^n,\dots,x_N^n]^{\mathrm{T}}$, $d \triangleq [d_1^1,\dots,d_N^1,\dots, d_1^n,\dots,d_N^n]^{\mathrm{T}}$, and x_i^j and d_i^j denote the jth component of x_i and d_i, respectively.

Although our results can be directly extended to the case of Eq. (18), once again, for simplicity of exposition, we will focus on individual agent states evolving in $\mathbb{R}$ (i.e., $n = 1$). In this case, Eq. (18) becomes

$$x(k+1) = \mathcal{P}x(k) + \varepsilon\mathcal{L}d(k),\quad x(0) = x_0,\quad k \in \overline{\mathbb{Z}}_+ \tag{19}$$

For the statement of the next result, define $\Delta_{\max} \triangleq \max_{i\in\{1,\dots,N\}} \deg(i)$.

THEOREM 5.1. *Consider an undirected network of N agents with a connected graph topology given by Eq. (19) and let $\varepsilon \in (0, (1/(\Delta_{\max})))$. Then,*

$$\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N\bar{x}(k)\|_2 \le \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{1 - \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)}$$

Proof. First, define $\delta(k) \triangleq x(k) - \mathbf{e}_N\bar{x}(k)$ and note that $\bar{x}(k+1) = (1/N)\mathbf{e}_N^{\mathrm{T}}x(k+1) = (1/N)\ \ \mathbf{e}_N^{\mathrm{T}}(x(k) + \varepsilon(-\mathcal{L}x(k) + \mathcal{L}d(k))) = \bar{x}(k)$, where we used the fact that $\mathcal{L}\mathbf{e}_N = 0_N$ and $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$. Hence, $\bar{x}(k) = (1/N)\mathbf{e}_N^{\mathrm{T}}x(k) = (1/N)\mathbf{e}_N^{\mathrm{T}}x(0) = \bar{x}$, $k \in \overline{\mathbb{Z}}_+$, which shows that the centroid of the network does not change over time in the presence of time-varying interagent measurement uncertainties. Next, evaluating $\delta(k+1)$, $k \in \overline{\mathbb{Z}}_+$, yields

$$\begin{aligned}\delta(k+1) &= x(k+1) - \mathbf{e}_N\bar{x}(k+1)\\ &= \mathcal{P}x(k) + \varepsilon\mathcal{L}d(k) - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}[\mathcal{P}x(k) + \varepsilon\mathcal{L}d(k)]\\ &= \mathcal{P}\left[x(k) - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}x(k)\right] + \left[I - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right]\varepsilon\mathcal{L}d(k)\\ &= \left[\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right]\delta(k) + \varepsilon\mathcal{L}d(k),\quad \delta(0) = \delta_0,\quad k \in \overline{\mathbb{Z}}_+\end{aligned} \tag{20}$$

Now, considering a Lyapunovlike function $V : \mathbb{R}^{(N-1)} \to \mathbb{R}$ given by $V(\delta) = \|\delta\|_2$ and recalling that $\rho(M) = \|M\|_2$ for an arbitrary symmetric matrix M, it follows from Eq. (20) that

$$\begin{aligned}V(\delta(k+1)) &= \|\delta(k+1)\|_2\\ &\le \left\|\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)\delta(k)\right\|_2 + \|\varepsilon\mathcal{L}d(k)\|_2\\ &\le \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)\|\delta(k)\|_2 + \varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r\\ &= \left(\rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right) + \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{\|\delta(k)\|_2}\right)V(\delta(k)),\\ &\quad k \in \overline{\mathbb{Z}}_+\end{aligned} \tag{21}$$

Hence, it follows from Eq. (21) that $V(\delta(k+1)) < V(\delta(k))$ for $\rho(\mathcal{P} - (1/N)\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) + ((\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r)/(\|\delta(k)\|_2)) < 1$ and $k \in \overline{\mathbb{Z}}_+$. Now, recalling that all the eigenvalues of the Perron matrix of an undirected connected graph with $\varepsilon \in (0, (1/(\Delta_{\max})))$ are located in the unit circle and only one eigenvalue has an absolute value of 1 [27], it follows that $\rho(\mathcal{P} - (1/N)\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) < 1$. Hence, it follows from Eq. (21) that

$$\|\delta(k)\|_2 \le \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{1 - \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)}$$

as $k \to \infty$. ■

Remark 5.1. Note that

$$\begin{aligned}\det\left[\lambda I_N - \left(\mathcal{P} - \frac{1}{N}E_N\right)\right] &= \det\left[\lambda I_N - \left(I_N - \varepsilon\mathcal{L} - \frac{1}{N}E_N\right)\right]\\ &= \det\left[(\lambda - 1)I_N - \left(-\varepsilon\mathcal{L} - \frac{1}{N}E_N\right)\right]\end{aligned} \tag{22}$$

Now, since E_N has only one nonzero eigenvalue equal to N with the corresponding eigenvector $\mathbf{e}_N$ and $\mathcal{L}$ has only one zero eigenvalue with the corresponding eigenvector $\mathbf{e}_N$, it follows that $\mathrm{spec}(-\varepsilon\mathcal{L} - (1/N)E_N) = \{-1, -\varepsilon\lambda_2(\mathcal{L}),\dots, -\varepsilon\lambda_N(\mathcal{L})\}$. Thus, it follows from Eq. (22) that $\mathrm{spec}(\mathcal{P} - (1/N)E_N) = \{0, (1 - \varepsilon\lambda_2(\mathcal{L})),\dots, (1 - \varepsilon\lambda_N(\mathcal{L}))\}$. Hence, $\rho(\mathcal{P} - (1/N)\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}) = \max\{|(1 - \varepsilon\lambda_2(\mathcal{L}))|, |(1 - \varepsilon\lambda_N(\mathcal{L}))|\}$.

Next, we apply Theorem 5.1 to an all-to-all connected graph network. Note that in this case, $\mathcal{L} = NI_N - E_N$. Now, recall that $\lambda_2(\mathcal{L}) = \cdots = \lambda_N(\mathcal{L}) = N$, and hence, for $\varepsilon \in (0, (1/N))$, it follows from Theorem 5.1 and Remark 5.1 that

$$\begin{aligned}\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N\bar{x}(k)\|_2 &\le \frac{\varepsilon\lambda_{\max}(\mathcal{L})\sqrt{N}r}{1 - \rho\left(\mathcal{P} - \frac{1}{N}\mathbf{e}_N\mathbf{e}_N^{\mathrm{T}}\right)} = \frac{\varepsilon N\sqrt{N}r}{1 - (1 - \varepsilon N)}\\ &= \sqrt{N}r\end{aligned}$$

Alternatively, we can arrive at the same result directly by considering the update protocol for the ith agent given by

$$x_i(k+1) \in \alpha\frac{1}{N}\sum_{j=1}^{N}\mathcal{X}_j(k) + (1-\alpha)x_i(k) = \mathcal{B}_{\alpha r}(\alpha\bar{x}(k)) + (1-\alpha)x_i(k)$$
$$x_i(0) = x_{i0},\ \ k \in \overline{\mathbb{Z}}_+,\ \ i = 1,\dots,N \tag{23}$$

where $\alpha \in (0, 1]$ and $\bar{x}(k) \triangleq (1/N)\sum_{i=1}^{N} x_i(k) \equiv \bar{x}$. First, note that it can be shown that $\limsup_{k\to\infty} \|x_i(k) - x_j(k)\|_2 \le 2r$ for every i, $j = 1,\dots, N$.

To see this, for $i, j = 1,\dots, N$, it follows from Eq. (23) that

$$x_i(k+1) - x_j(k+1) \in \mathcal{B}_{\alpha r}(\alpha x_{\mathrm{ave}}(k)) - \mathcal{B}_{\alpha r}(\alpha x_{\mathrm{ave}}(k)) + (1-\alpha)(x_i(k) - x_j(k)),\quad k \in \overline{\mathbb{Z}}_+ \tag{24}$$

which implies

$$\|x_i(k+1) - x_j(k+1)\|_2 \leq (1-\alpha)\|x_i(k) - x_j(k)\|_2 + 2r\alpha \quad (25)$$

Hence, since $\|x_i(k+1)-x_j(k+1)\|_2 \leq \|x_i(k)-x_j(k)\|_2$ for $\|x_i(k) - x_j(k)\|_2 \geq 2r$, it follows that $\|x_i(k)-x_j(k)\|_2 \leq 2r$ as $k \to \infty$ for all $i, j = 1,\ldots, N$. Now, using identical arguments as in Sec. 4, it follows that $\limsup_{k\to\infty} \|x(k) - \mathbf{e}_N\overline{x}\|_2 \leq \sqrt{N}r$, which is identical to the result obtained by using Theorem 5.1 directly.

6 A Set-Valued Analysis Approach to Discrete-Time Consensus

In this section, we present a set-valued approach for the discrete-time consensus protocol considered in Sec. 5. Due to its mathematical generality, set-valued analysis can prove beneficial for generalizing our results to nonlinear network architectures with a dynamic network topology. Before presenting the main results of this section, we require some additional notation and definitions. Specifically, consider the difference inclusion

$$x(k+1) \in \mathcal{F}(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \quad (26)$$

where, for every $k \in \overline{\mathbb{Z}}_+$, $x(k) \in \mathbb{R}^n$, $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a *set-valued map* that assigns sets to points, and $2^{\mathbb{R}^n}$ denotes the collection of all the subsets of $\mathbb{R}^n$. The set-valued map $\mathcal{F}$ has a *nonempty value at x* if $\mathcal{F}(x) \neq \varnothing$. It is assumed that $\mathcal{F}$ has nonempty values for ever $x \in \mathbb{R}^n$. Hence, maximal solutions to Eq. (26) are complete, and consequently, by a *solution* of Eq. (26) with initial condition $x(0) = x_0$ we mean a function $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$ that satisfies Eq. (26).

The set-valued map $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is *outer semicontinuous at x* if, for every sequence $\{x_i\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} x_i = x$, every convergent sequence $\{y_i\}_{i=0}^{\infty}$ with $y_i \in \mathcal{F}(x_i)$ satisfies $\lim_{i\to\infty} y_i \in \mathcal{F}(x)$. $\mathcal{F}$ is *continuous at x* if $\mathcal{F}$ is outer semicontinuous at x and, for every $y \in \mathcal{F}(x)$ and every convergent sequence $\{x_i\}_{i=0}^{\infty}$, there exists $y_i \in \mathcal{F}(x_i)$ such that $\lim_{i\to\infty} y_i = y$. $\mathcal{F}(x)$ is *locally bounded at x* if there exists a neighborhood $\mathcal{N}$ of x such that $\mathcal{F}(\mathcal{N}) = \cup_{z\in\mathcal{N}}\mathcal{F}(z)$ is bounded. If $\mathcal{F}$ has compact values and is locally bounded at x, then $\mathcal{F}$ is *upper semicontinuous at x*, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\|z - x\| < \delta$, $\mathcal{F}(z) \subseteq \mathcal{F}(x) + \overline{\mathcal{B}}_{\varepsilon}(0)$, where $\overline{\mathcal{B}}_{\varepsilon}(0)$ denotes the closure of $\mathcal{B}_{\varepsilon}(0)$.

Given the function $\gamma : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$, the *positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^n$ for which there exists an increasing divergent sequence $\{k_n\}_{n=0}^{\infty}$ satisfying $\lim_{n\to\infty} \gamma(k_n) = y$. We denote the positive limit set of a solution $\psi(\cdot)$ of Eq. (26) by $\Omega(\psi)$. The positive limit set of a bounded solution of Eq. (26) is nonempty, compact, and weakly forward invariant with respect to Eq. (26) [28].

The following theorem gives a general set-valued invariance principle using the set-valued analysis tools developed in Ref. [4] and is necessary for the main result of this section.

Theorem 6.1. *Consider the difference inclusion given by Eq. (26). Assume that $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is outer semicontinuous and locally bounded with nonempty values for all $x \in \mathbb{R}^n$. Let $V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a continuous set-valued map and let $\mathcal{M} \subset \mathbb{R}^n$ be a closed set such that the following statements hold:*

(i) $V(\mathcal{F}(x)) \subseteq V(x)$ for every $x \in \mathbb{R}^n$ and
(ii) if $V(y) = V(x)$ for some $y \in \mathcal{F}(x)$, then $x \in \mathcal{M}$.

Then every bounded solution $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}^n$ of Eq. (26) converges to $\mathcal{M}$, that is, $\lim_{k\to\infty} \mathrm{dist}(x(k), \mathcal{M}) = 0$.

Proof. It follows from (i) that $V(\psi(k+1)) \subseteq V(\psi(k))$ for every solution $\psi(k)$, $k \in \overline{\mathbb{Z}}_+$, of Eq. (26). Thus, the sequence of closed sets $\{V(\psi(k)\}_{k=0}^{\infty}$ is nonincreasing, and hence, $\lim_{k\to\infty} V(\psi(k)) = \cap_{k=0}^{\infty} V(\psi(k)) \triangleq \mathcal{V}$ [28]. Next, note that since $\psi(k)$, $k \in \overline{\mathbb{Z}}_+$, is bounded, $\Omega(\psi)$ is nonempty. Now, for all $x \in \Omega(\psi)$, it follows from the definition of $\Omega(\psi)$ and the continuity of V that $V(x) = \mathcal{V}$. Moreover, the outer semicontinuity of $\mathcal{F}$ ensures that $\Omega(\psi)$ is weakly positively (and negatively) invariant. Specifically, for every $x \in \Omega(\psi)$, there exists $y \in \mathcal{F}(x)$ such that $y \in \Omega(\psi)$. Thus, for every $x \in \Omega(\psi)$, there exists $y \in \mathcal{F}(x)$ such that $V(x) = V(y) = \mathcal{V}$, and hence, $\Omega(\psi) \subseteq \mathcal{M}$. Finally, since $\mathrm{dist}(\psi(k), \Omega(\psi)) \to 0$ as $k \to 0$, it follows that $\psi(k) \to \mathcal{M}$ as $k \to \infty$. ■

Next, we illustrate Theorem 6.1 by applying it to the network system given by Eq. (23). The conclusions of the proposition below are weaker than the results obtained in Sec. 5. However, as noted above, the set-valued approach can prove beneficial for nonlinear network architectures where direct computation relying on a linear structure is not possible as well as for partial graph connectivity structures with directed information flow.

Proposition 6.1. *Consider a network of N agents with an all-to-all graph connectivity given by Eq. (23) and let $x(\cdot)$ be a bounded solution of Eq. (23). Then,* $\limsup_{k\to\infty} \|x_i(k) - x_j(k)\|_2 \leq 4r$ *for every $i, j = 1, \ldots, N$.*

Proof. Let the set-valued map $V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be given by

$$V(x) = \mathcal{B}_{\delta_1(x)}(x_{\mathrm{ave}}) \times \cdots \times \mathcal{B}_{\delta_N(x)}(x_{\mathrm{ave}})$$

where, for every $i \in \{1,\ldots, N\}$

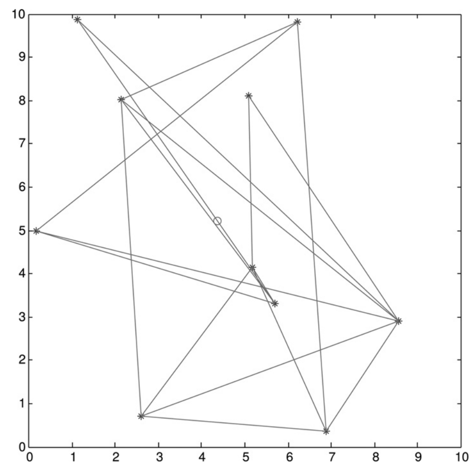

Fig. 2 Initial network configuration of ten agents with sensor accuracy of radius $r = 1$

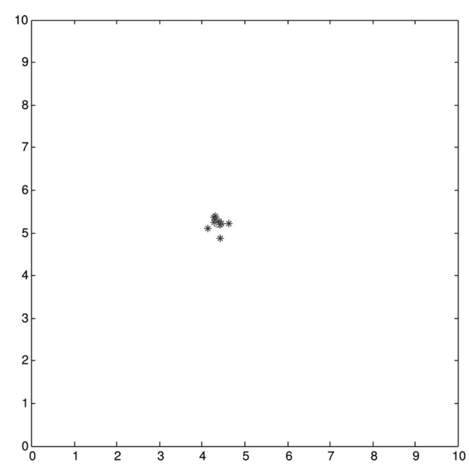

Fig. 3 Network configuration of ten agents with sensor accuracy of radius $r = 1$ at $t = 3.5$ s

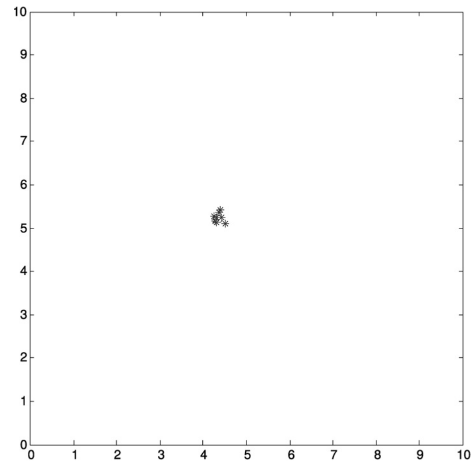

Fig. 4 Network configuration of ten agents with sensor accuracy of radius $r = 1$ at $t = 7.5$ s

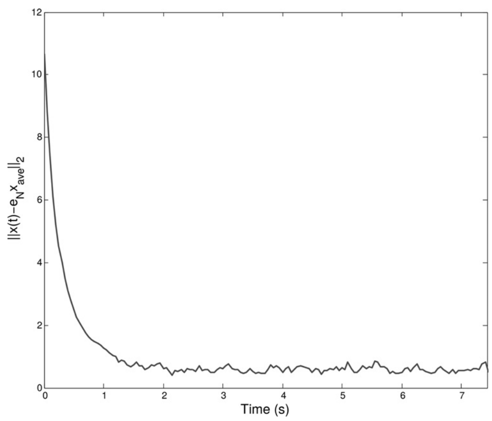

Fig. 5 Plot of $\|x(t) - e_N \bar{x}\|_2$ versus time

$$\delta_i(x) = \begin{cases} \|x_i - x_{\text{ave}}\|_2, & \|x_i - x_{\text{ave}}\|_2 \geq 2r \\ 2r, & \|x_i - x_{\text{ave}}\|_2 \leq 2r \end{cases}$$

and "$\times$" denotes Cartesian product. Note that V is continuous and has closed and bounded values. Next, it can be shown using a similar argument as in Sec. 5 that

$$x_i(k+1) - x_{\text{ave}}(k+1) \in \mathcal{B}_{\alpha r}(\alpha x_{\text{ave}}(k)) - \mathcal{B}_{\alpha r}(x_{\text{ave}}(k)) + (1-\alpha)x_i(k), \quad k \in \overline{\mathbb{Z}}_+$$

which implies

$$\|x_i(k+1) - x_{\text{ave}}(k+1)\|_2 \leq (1-\alpha)\|x_i(k) - x_{\text{ave}}(k)\|_2 + 2r\alpha$$

Hence, the function $\delta_i(\cdot)$ decreases for $\|x_i - x_{\text{ave}}\|_2 > 2r$ and remains constant for $\|x_i - x_{\text{ave}}\|_2 \leq 2r$, $i \in \{1, \ldots, N\}$, and hence, conditions (i) and (ii) of Theorem 6.1 are satisfied. Now, it follows from Theorem 6.1 that every bounded solution $x_i(\cdot)$, $i \in \{1, \ldots, N\}$, converges to $\mathcal{B}_{2r}(x_{\text{ave}})$. Hence, $\|x_i(k) - x_j(k)\|_2 \leq 4r$ as $k \to \infty$ for all $i, j = 1, \ldots, N$. ■

7 Illustrative Numerical Examples

In this section, we present two illustrative numerical examples to demonstrate the efficacy of the proposed framework.

Example 7.1. In this example, we consider a random network of ten agents with connected, undirected, and time-invariant

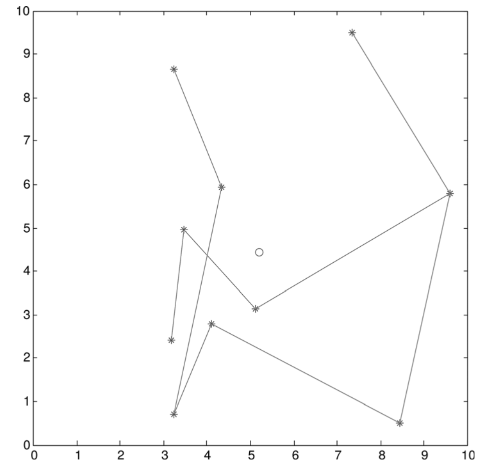

Fig. 6 Initial network configuration of ten agents with sensor accuracy of radius $r = 1$

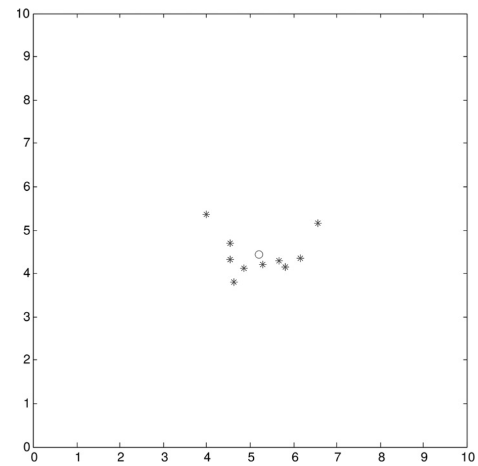

Fig. 7 Network configuration of ten agents with sensor accuracy of radius $r = 1$ at $t = 3.5$ s

communication graph network topologies and with agent dynamics given by Eq. (8). Furthermore, we assume that the ith agent uncertainty is modeled as a standard white noise process. Figures 2–5 show the initial, intermediate, and final network configurations, as well as $\|x(t) - \mathbf{e}_N \bar{x}\|_2$ versus time, of the network of agents when the agents have sensor accuracy of radius 1, $\lambda_2(\mathcal{L}) = 1.5568$, and $\lambda_N(\mathcal{L}) = 7.5704$. The circle indicates the location of the initial centroid of the agents. Note that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \bar{x}\|_2 \leq ((\lambda_N(\mathcal{L})\sqrt{N} r)/(\lambda_2(\mathcal{L}))) = 15.3775$.

Alternatively, Figs. 6–9 show the initial, intermediate, and final network configurations, as well as $\|x(t) - \mathbf{e}_N \bar{x}\|_2$ versus time, of the network of agents when the agents have sensor accuracy of radius 1, $\lambda_2(\mathcal{L}) = 0.1172$, and $\lambda_N(\mathcal{L}) = 4.3721$. Once again, the circle indicates the location of the initial centroid of the agents. Note that $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N \bar{x}\|_2 \leq ((\lambda_N(\mathcal{L})\sqrt{N} r)/(\lambda_2(\mathcal{L}))) = 117.9675$.

Finally, Figs. 10–12 show the initial, intermediate, and final configurations, respectively, of the network of ten agents when agents have sensor accuracy of radius 0.5 and the network is all-to-all connected. The simulation shows that the agents reach a consensus set with diameter less than $2r = 1$. The circle indicates a set with diameter 1 centered at the initial centroid of the agents.

Example 7.2. In this example, we use the proposed framework for pitch rate consensus of a set of commercial airplanes in the presence of inaccurate sensor measurements, which are modeled as a standard white noise process coupled with sinusoidal time-

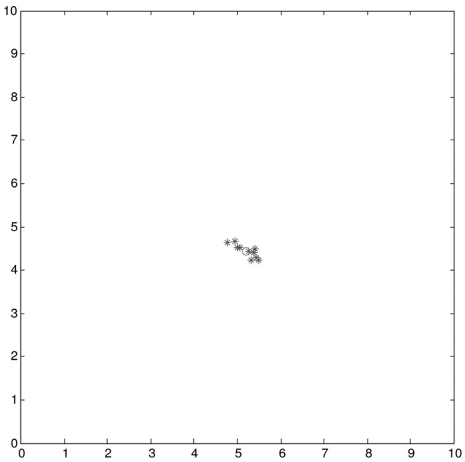

Fig. 8 Network configuration of ten agents with sensor accuracy of radius $r = 1$ at $t = 7.5$ s

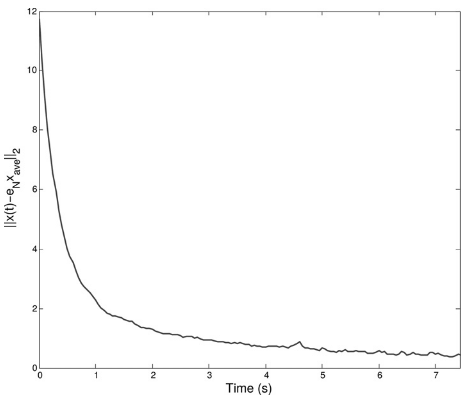

Fig. 9 Plot of $\|x(t) - e_N \bar{x}\|_2$ versus time

varying exogenous disturbances. Specifically, consider the multi-agent system representing the controlled longitudinal motion of three Boeing 747 airplanes linearized at an altitude of 40 kft and a velocity of 774 ft/s [29] given by

$$\dot{\xi}_i(t) = A\xi_i(t) + B\nu_i(t), \quad \xi_i(0) = \xi_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0 \tag{27}$$

where

$$\xi_i(t) = [v_{x_i}(t), v_{z_i}(t), q_i(t), \theta_{e_i}(t)]^{\mathrm{T}} \in \mathbb{R}^4, \quad t \geq 0 \tag{28}$$

is a state vector of agent i, $i = 1, 2, 3$, with $v_{x_i}(t)$, $t \geq 0$, representing the x-body-axis component of the velocity of the airplane center of mass with respect to the reference axes (in ft/s); $v_{z_i}(t)$, $t \geq 0$, representing the z-body-axis component of the velocity of the airplane center of mass with respect to the reference axes (in ft/s); $q_i(t)$, $t \geq 0$, representing the y-body-axis component of the angular velocity of the airplane (pitch rate) with respect to the reference axes (in crad/s); $\theta_{e_i}(t)$, $t \geq 0$, representing the pitch Euler angle of the airplane body axes with respect to the reference axes (in crad); $\nu_i(t)$, $t \geq 0$, representing the elevator control input (in crad); and

$$A = \begin{bmatrix} -0.003 & 0.039 & 0 & -0.332 \\ -0.065 & -0.319 & 7.74 & 0 \\ 0.020 & -0.101 & -0.429 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.010 \\ -0.180 \\ -1.16 \\ 0 \end{bmatrix} \tag{29}$$

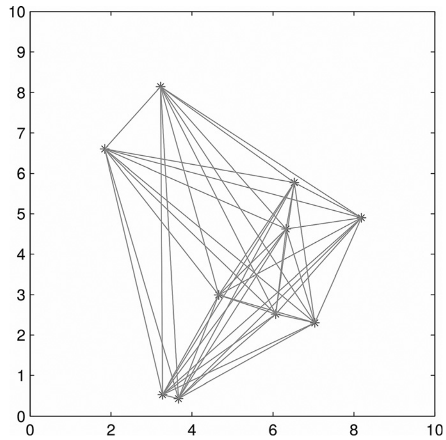

Fig. 10 Initial network configuration of ten agents with sensor accuracy of radius $r = 0.5$

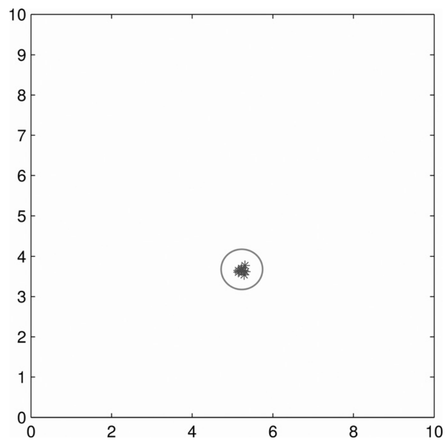

Fig. 11 Network configuration of ten agents with sensor accuracy of radius $r = 0.5$ at $t = 3.5$ s

Here, we utilize the two-level hierarchical controller proposed in Ref. [30], which is composed of a lower-level controller for command following and a higher-level controller for pitch rate consensus of the three airplanes given by Eq. (27). To address the lower-level controller design, let $x_i(t)$, $i = 1, 2, 3$, $t \geq 0$, be a command generated by Eq. (6) (i.e., the guidance command) and let $s_i(t)$, $i = 1, 2, 3$, $t \geq 0$, denote the integrator state satisfying

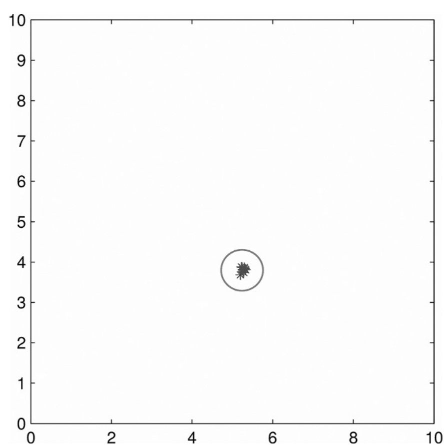

Fig. 12 Network configuration of ten agents with sensor accuracy of radius $r = 0.5$ at $t = 7.5$ s

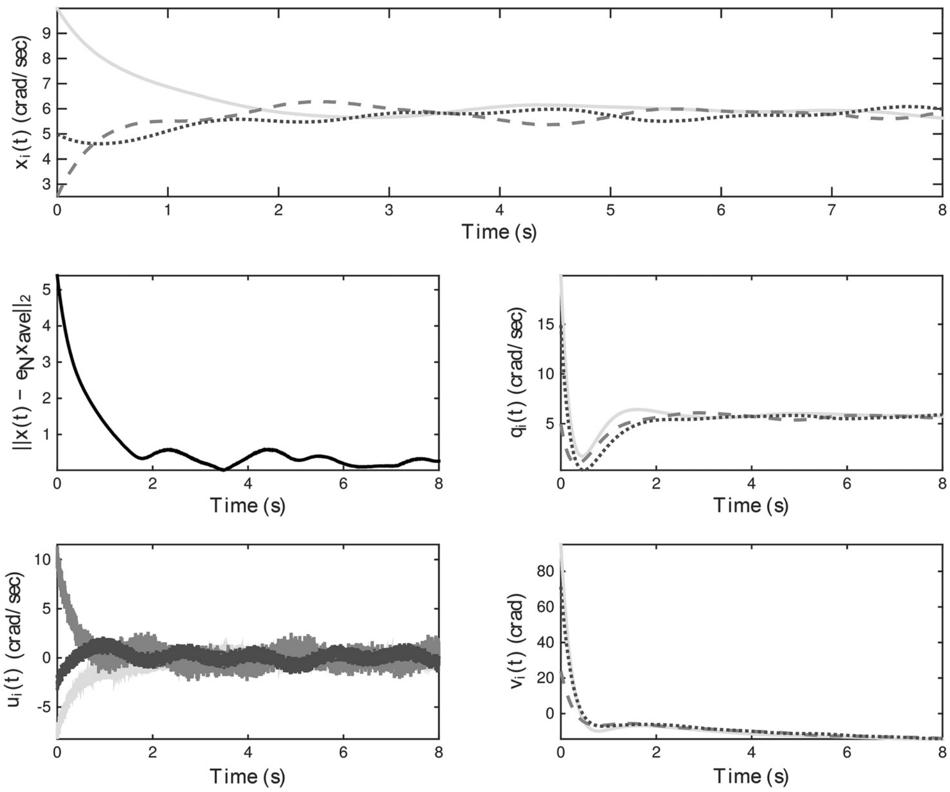

Fig. 13 Agent guidance state ($x_i(t)$, $t \geq 0$), $\|x(t) - e_N \bar{x}\|_2$, pitch rate ($q_i(t)$, $t \geq 0$), guidance input ($u_i(t)$, $t \geq 0$), and elevator control ($v_i(t)$, $t \geq 0$) responses for the three airplanes on a line graph in the presence of inaccurate sensor measurements (solid, dashed, and dotted lines denote the responses for the first, second, and third airplanes, respectively)

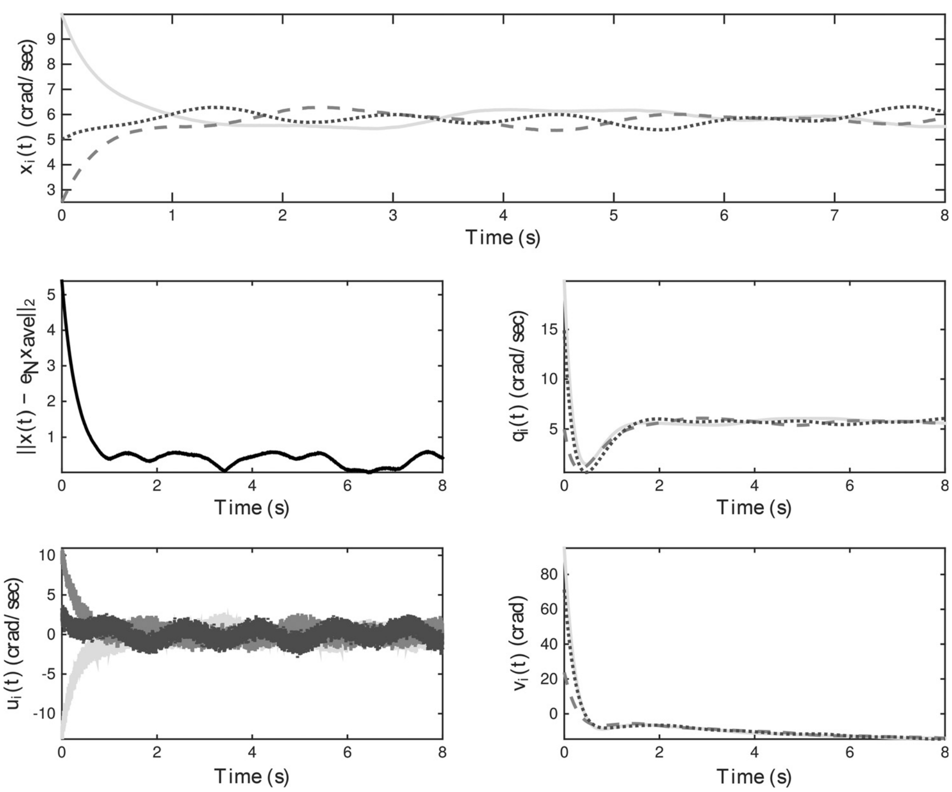

Fig. 14 Agent guidance state ($x_i(t)$, $t \geq 0$), $\|x(t) - e_N \bar{x}\|_2$, pitch rate ($q_i(t)$, $t \geq 0$), guidance input ($u_i(t)$, $t \geq 0$), and elevator control ($v_i(t)$, $t \geq 0$) responses for the three airplanes on an all-to-all graph in the presence of inaccurate sensor measurements (solid, dashed, and dotted lines denote the responses for the first, second, and third airplanes, respectively)

$$\dot{s}_i(t) = E\xi_i(t) - x_i(t), \quad s_i(0) = s_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0 \tag{30}$$

where $E = [0,\ 0,\ 1,\ 0]$. Now, defining the augmented state $\overline{\xi}(t) \triangleq [\xi^{\mathrm{T}}(t), s_i(t)]^{\mathrm{T}}$, Eqs. (27) and (30) give

$$\dot{\overline{\xi}}_i(t) = \overline{A}\,\overline{\xi}_i(t) + \overline{B}_1\nu_i(t) + \overline{B}_2 x_i(t), \quad \overline{\xi}_i(0) = \overline{\xi}_{i_0}, \quad i = 1, 2, 3, \quad t \geq 0 \tag{31}$$

where

$$\overline{A} \triangleq \begin{bmatrix} A & 0 \\ \mathcal{E} & 0 \end{bmatrix}, \quad \overline{B}_1 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \overline{B}_2 \triangleq \begin{bmatrix} 0 \\ -I \end{bmatrix} \tag{32}$$

Furthermore, let the elevator control input be given by

$$\nu(t) = -K\overline{\xi}(t), \quad K = [-0.0157, 0.0831, -4.7557, -0.1400, -9.8603], \quad t \geq 0 \tag{33}$$

which is designed using an optimal linear-quadratic regulator.

For the higher-level controller design, we use Eq. (6) to generate $x_i(t)$, $t \geq 0$, that has a direct effect on the lower-level controller design to achieve pitch rate consensus. Specifically, the lower-level controller for each agent allows for the tracking of $x_i(t)$, $t \geq 0$, whereas the higher-level controllers allow for the implementation of Eq. (6). Figures 13 and 14 present the results for all the initial conditions set to zero and $x_1(0) = 10$, $x_2(0) = 2.5$, and $x_3(0) = 5$. In particular, Fig. 13 shows that the three airplanes on a line graph achieve approximate pitch rate consensus in the presence of inaccurate sensor measurements with $r = 1$, where the collective behavior of these airplanes satisfies $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N\overline{x}\|_2 \leq ((\lambda_N(\mathcal{L})\sqrt{N}r)/(\lambda_2(\mathcal{L}))) = 5.1962$. Figure 14 shows a similar collective behavior performance for the airplanes for an all-to-all connected graph with $r = 1$, where $\limsup_{t\to\infty} \|x(t) - \mathbf{e}_N\overline{x}\|_2 \leq ((\lambda_N(\mathcal{L})\sqrt{N}r)/(\lambda_2(\mathcal{L}))) = 1.7321$ holds.

8 Conclusion

In this paper, we considered the problem of approximate consensus for multiagent systems with a connected, undirected, and time-invariant communication graph topology with uncertain interagent measurements, wherein the agents can detect the location of the neighboring agents only up to an accuracy of a ball of radius r. In addition, we presented a formulation of the problem using set-valued maps and a set-valued invariance principle. Since the agent dynamics are an element of a set-valued convex map, the set to which the agents converge is time-varying unless a collision avoidance strategy or a stopping criteria is enforced. Future extensions will focus on using the proposed set-valued framework to develop control design protocols for static and dynamic nonlinear networks with directed graph topologies and uncertain interagent measurements. Furthermore, we will identify multiagent network nodes that are more prone to uncertainties using computational analysis tools.

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