# A Universal Feedback Controller for Discontinuous Dynamical Systems Using Nonsmooth Control Lyapunov Functions

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*The consideration of nonsmooth Lyapunov functions for proving stability of feedback discontinuous systems is an important extension to classical stability theory since there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein to show the existence of a feedback stabilizing controller. A constructive feedback control law based on a universal construction of smooth control Lyapunov functions was given by Sontag. Even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function, many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller. However, the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. In this paper, we develop a constructive feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives.* [DOI: 10.1115/1.4028593]

## 1 Introduction

Numerous engineering applications give rise to discontinuous dynamical systems. Specifically, in impact mechanics the motion of a dynamical system is subject to velocity jumps and force discontinuities leading to nonsmooth dynamical systems [1,2]. In mechanical systems subject to unilateral constraints on system positions [3], discontinuities occur naturally through system–environment interaction. Alternatively, control of networks and control over networks with dynamic topologies also give rise to discontinuous systems [4]. For these systems, the vector field defining the dynamical system is a discontinuous function of the state, and system stability can be analyzed using nonsmooth Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz continuous functions and proximal subdifferentials of lower semicontinuous functions [5].

The consideration of nonsmooth Lyapunov functions for proving stability of feedback discontinuous systems is an important extension to classical stability theory since, as shown in Ref. [6], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein [7] to show the existence of a feedback stabilizing controller. A constructive feedback control law based on smooth control Lyapunov functions was given in Ref. [8].

Even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function, many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller [7,9]. However, as shown in Ref. [9], the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. Furthermore, Rifford [10] addresses the problem of stabilization of globally asymptotically controllable systems wherein the system vector field is locally Lipschitz continuous in the state and uniformly in the control. For the aforementioned class of systems, Rifford [10] constructs a discontinuous control law using semiconcave control Lyapunov functions in the sense of proximal subdifferentials. However, we will not need to consider semiconcavity in what follows. Finally, the work in Ref. [11] also provides discontinuous controllers using a Filippov solution framework; however, Hirschorn [11] uses a special closed lower bounded control Lyapunov function which we also do not require here.

In this paper, we build on the results of Refs. [9–12] to develop a constructive universal feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients [13] and set-valued Lie derivatives [14]. Specifically, we address the problem of discontinuous stabilization for dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps and admitting Filippov solutions with absolutely continuous curves.

## 2 Notation and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\overline{\mathbb{Z}}_+$ denotes the set of non-negative integers, and $(\cdot)^{\mathrm{T}}$ denotes transpose. We write $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ to denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm on $\mathbb{R}^n$, $\mathcal{B}_\varepsilon(\alpha), \alpha \in \mathbb{R}^n, \varepsilon > 0$, for the *open ball centered* at $\alpha$ with *radius* $\varepsilon$, $\mathrm{dist}(p, \mathcal{M})$ for the distance from a point $p$ to the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \| p - x \|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to

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denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$. Finally, the notions of openness, convergence, continuity, and compactness that we use throughout the paper refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

In this paper, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0 \tag{1}$$

where for every $t \geq t_0, x(t) \in \mathcal{D} \subseteq \mathbb{R}^n, u(t) \in U \subseteq \mathbb{R}^m, F\colon \mathcal{D} \times U \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [15] with respect to $x$ (i.e., $F$ is bounded on a bounded neighborhood of every point $x$), continuous with respect to $u$, and admits an equilibrium point at $x_e \in \mathcal{D}$ for some $u_e \in U$; that is, $F(x_e, u_e) = 0$. The control $u(\cdot)$ in Eq. (1) is restricted to the class of *admissible* controls consisting of all measurable and locally essentially bounded functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$. For each value $u \in U$, we define the function $F_u$ by $F_u(x) = F(x, u)$.

A measurable function $\phi\colon \mathcal{D} \to U$ satisfying $\phi(x_e) = u_e$ is called a *control law*. If $u(t) = \phi(x(t))$, where $\phi$ is a control law and $x(t)$ satisfies Eq. (1), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in $U$. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, the *closed-loop system* is given by

$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0 \tag{2}$$

Analogous to the open-loop case, we define the function $F_\phi$ by $F_\phi(x) = F(x, \phi(x))$. Note that an arc $x(\cdot)$ (i.e., an absolutely continuous function from $[t_0, t]$ to $\mathcal{D}$) satisfies Eq. (1) for an admissible control $u(t) \in U$ if and only if [15, p. 152]

$$\dot{x}(t) \in \mathcal{F}(x(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0 \tag{3}$$

where $\mathcal{F}(x) \triangleq F(x, U)$, that is, $\mathcal{F}(x) \triangleq \{F(x, u) : u \in U\}$.

Here, $\mathcal{F} : \mathcal{D} \to 2^{\mathbb{R}^n}$ is a *set-valued map* that assigns sets to points and $2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$. The set $\mathcal{F}(x)$ captures all of the directions in $\mathbb{R}^n$ that can be generated at $x$ with inputs $u = u(t) \in U$. The inputs $u(\cdot)$ can be selected as either $u$: $[t_0,\infty) \to U$ or $u : \mathcal{D} \to U$. We assume that $\mathcal{F}(x)$ is an upper semicontinuous, nonempty, convex, and compact set for all $x \in \mathbb{R}^n$. That is, for every $x \in \mathcal{D}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\| z - x \| \leq \delta, \mathcal{F}(z) \subseteq \mathcal{F}(x) + \overline{\mathcal{B}}_\varepsilon(0)$. This assumption is mainly used to guarantee the existence of Filippov solutions to Eq. (2) [15].

An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a *Filippov solution* [15] of Eq. (2) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[F_\phi](x(t)), \quad \text{a.e.} \quad t \in [t_0, \tau] \tag{4}$$

where the *Filippov set-valued map* $\mathcal{K}[F_\phi] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{K}[F_\phi](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S}) = 0} \overline{\mathrm{co}}\{F_\phi(\mathcal{B}_\delta(x) \backslash \mathcal{S})\}, \quad x \in \mathcal{D} \tag{5}$$

$\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\mathrm{co}}$" denotes convex closure, and $\cap_{\mu(S)} = 0$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.[1] Note that since $F$ is locally essentially bounded, $\mathcal{K}[F_\phi](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values. Thus, Filippov solutions are limits of solutions to $\mathcal{G}$ with $F$ averaged over progressively smaller neighborhoods around the solution point, and hence, allow solutions to be defined at points where $F$ itself is not defined. Hence, the tangent vector to a Filippov solution, when it exists, lies in the convex closure of the limiting values of the system vector field $F(\cdot, \cdot)$ in progressively smaller neighborhoods around the solution point. Dynamical systems of the form given by Eqs. (3) and (4) are called *differential inclusions* in the literature [16] and, for every state $x \in \mathbb{R}^n$, they specify a *set* of possible evolutions of $\mathcal{G}$ rather than a single one.

Since the Filippov set-valued map given by Eq. (5) is upper semicontinuous with nonempty, convex, and compact values, and $\mathcal{K}[F_\phi](\cdot)$ is also locally bounded [15, p. 85], it follows that Filippov solutions to Eq. (2) exist [15, Theorem 1, p. 77]. Recall that the Filippov solution $t \mapsto x(t)$ to Eq. (2) is a *right maximal solution* if it cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal Filippov solutions to Eq. (2) exist on $[t_0, \infty)$, and hence, we assume that (2) is *forward complete*. Recall that (2) is forward complete if and only if the Filippov solutions to Eq. (2) are uniformly globally sliding time stable [17, Lemma 1, p. 182]. An *equilibrium point* of Eq. (2) is a point $x_e \in \mathbb{R}^n$ such that $0 \in \mathcal{K}[F_\phi](x_e)$. It is easy to see that $x_e$ is an equilibrium point of Eq. (2) if and only if the constant function $x(\cdot) = x_e$ is a Filippov solution of Eq. (2). We denote the set of equilibrium points of Eq. (2) by $\mathcal{E}$. Since the set-valued map $\mathcal{K}[F_\phi](\cdot)$ is upper semicontinuous, it follows that $\mathcal{E}$ is closed.

To develop discontinuous controllers for discontinuous dynamical systems given by Eq. (1), we need to introduce the notion of generalized derivatives and gradients. Here, we focus on Clarke generalized derivatives and gradients [13].

Definition 2.1. *[13,14] Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke upper generalized derivative of $V(\cdot)$ at $x$ in the direction of $v \in \mathbb{R}^n$ is defined by*

$$V^o(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h} \tag{6}$$

*The Clarke generalized gradient $\partial V\colon \mathbb{R}^n \to 2^{\mathbb{R}^{1\times n}}$ of $V(\cdot)$ at $x$ is the set*

$$\partial V(x) \triangleq \mathrm{co}\left\{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x, x_i \notin \mathcal{N} \cup \mathcal{S} \right\} \tag{7}$$

*where co denotes the convex hull, $\nabla$ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero of points where $\nabla V$ does not exist, $\mathcal{S}$ is any subset of $\mathbb{R}^n$ of measure zero, and the unbounded sequence $\{x_i\}_{i \in \overline{\mathbb{Z}}_+} \subset \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$.*

Note that Eq. (6) always exists. Furthermore, note that it follows from Definition 2.1 that the generalized gradient of $V$ at $x$ consists of all convex combinations of all the possible limits of the gradient at neighboring points where $V$ is differentiable. In addition, note that since $V(\cdot)$ is Lipschitz continuous, it follows from Rademacher's theorem [18, Theorem 6, p. 281] that the gradient $\nabla V(\cdot)$ of $V(\cdot)$ exists almost everywhere. Moreover, for every $x \in \mathbb{R}^n$, every constant $\varepsilon > 0$, and every Lipschitz constant $L$ for $V$ on $\overline{\mathcal{B}}_\varepsilon(x)$, we have $\partial V(x) \subseteq \overline{\mathcal{B}}_L(0)$. Since $\partial V(x)$ is convex, closed, and bounded, it follows that $\partial V(x)$ is compact.

In order to state the main results of this paper, we need some additional notation and definitions. Specifically, the upper right directional Dini derivative of $V(x)$ along the Filippov state trajectories $\psi(t, x, u)$ of Eq. (1) through $x \in \mathcal{D}$ with $u(\cdot) \in U$ at $t = 0$ is defined as

$$\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t, x, u))\bigg|_{t=0} \triangleq \limsup_{h \to 0^+} \frac{V(\psi(h, x, u)) - V(x)}{h} \tag{8}$$

for every $x \in \mathbb{R}^n$ such that the limit in Eq. (8) exists. Furthermore, given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, the *set-valued Lie derivative* $\mathcal{L}_f V : \mathbb{R}^n \to 2^{\mathbb{R}}$ of $V$ with respect to $f$ at $x$ [14,19] is defined as

[1]Alternatively, we can consider Krasovskii solutions of Eq. (2) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[F_\phi](x)$ is replaced with $\mathcal{K}[F_\phi](x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\{F_\phi(\mathcal{B}_\delta(x))\}$ and where $F_\phi$ is assumed to be locally bounded.

$$\mathcal{L}_f V(x) \overset{\Delta}{=} \{a \in \mathbb{R}: \text{there exists } v \in \mathcal{K}[f](x) \text{ such that } p^{\mathrm{T}} v = a \text{ for all } p^{\mathrm{T}} \in \partial V(x)\} \subseteq \bigcap_{p^{\mathrm{T}} \in \partial V(x)} p^{\mathrm{T}} \mathcal{K}[f](x). \tag{9}$$

Since $\mathcal{K}[f](x)$ is convex with compact values, it follows that for each $x \in \mathbb{R}^n$, the set $\mathcal{L}_f V(x)$ is a closed and bounded, but possibly empty, interval in $\mathbb{R}$. If $V(\cdot)$ is continuously differentiable at $x$, then $\mathcal{L}_f V(x) = \{\nabla V(x) : v \in \mathcal{K}[f](x)\}$. In the case where $\mathcal{L}_f V(x)$ is nonempty, we use the notation $\max \mathcal{L}_f V(x)$ (respectively, $\min \mathcal{L}_f V(x)$) to denote the largest (respectively, smallest) element of $\mathcal{L}_f V(x)$. Furthermore, we adopt the convention $\max \emptyset = -\infty$. Finally, recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ [13, Definition 2.3.4] if, for all $v \in \mathbb{R}^n$, the right directional derivative $V'_+(x, v) \overset{\Delta}{=} \lim_{h \to 0^+} \frac{1}{h}[V(x + hv) - V(x)]$ exists and $V'_+(x, v) = V^o(x, v)$. $V$ is called *regular* on $\mathbb{R}^n$ if it is regular at every $x \in \mathbb{R}^n$.

## 3 Nonmooth Control Lyapunov Functions

In this section, we consider a feedback control problem and introduce the notion of *control Lyapunov functions* for discontinuous dynamical systems. Furthermore, using the concept of control Lyapunov functions we provide necessary and sufficient conditions for stabilization of discontinuous nonlinear dynamical systems. To address the problem of control Lyapunov functions for discontinuous dynamical systems, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set and let $U \subseteq \mathbb{R}^m$, where $0 \in \mathcal{D}$ and $0 \in U$. Next, consider the controlled nonlinear discontinuous dynamical system (1), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$ for almost all $t \geq 0$ and the constraint set $U$ is given. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop dynamical system is given by Eq. (2).

The following stability theorem is needed for the main result of this paper. In addressing the stability properties of a Filippov solution of a discontinuous dynamical system, the usual stability definitions are valid [5,20,21]. Here, we state the stability theorem for only the local case; the global stability theorems are similar except for the additional assumption of properness on the Lyapunov function and nonrestricting the domain of analysis. For the remainder of the paper, the adjective "weak" is used in reference to a stability property when the stability property is satisfied by at least one Filippov solution starting from every initial condition in $\mathcal{D}$, whereas "strong" is used when the stability property is satisfied by all Filippov solutions starting from every initial condition in $\mathcal{D}$. Our main result will be based on applying the following theorem to the system given by Eq. (2) with $\tilde{F}(x) = F_\phi(x) = F(x, \phi(x))$.

THEOREM 3.1. *[14,21] Consider the discontinuous nonlinear dynamical system $\dot{x} = \tilde{F}(x)$, where $\tilde{F} : \mathcal{D} \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded and admits an equilibrium point $x_e$, and $\mathcal{D} \subseteq \mathbb{R}^n$ is an open and connected set with $x_e \in \mathcal{D}$. If $V : \mathcal{D} \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous, and regular function such that $\max \mathcal{L}_{\tilde{F}} V(x) \leq 0$ (respectively, $\max \mathcal{L}_{\tilde{F}} V(x) < 0, x \neq x_e$) for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_{\tilde{F}} V(x) \neq \emptyset$, then $x_e$ is strongly Lyapunov (respectively, strongly asymptotically) stable.*

The following definitions are required for stating the main result of this section.

DEFINITION 3.1. *Let $\phi : \mathcal{D} \to U$ be a measurable mapping on $\mathcal{D} \backslash \{0\}$ with $\phi(0) = 0$. Then, Eq. (1) is feedback asymptotically stabilizable if the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous nonlinear dynamical system (2) is asymptotically stable.*

DEFINITION 3.2. *Consider the controlled discontinuous nonlinear dynamical system given by Eq. (1). A locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ satisfying*

$$\inf_{u \in U} [\max \mathcal{L}_{F_u} V(x)] < 0, \quad \text{a.e.} \quad x \in \mathcal{D} \backslash \{0\}, \tag{10}$$

*is called a control Lyapunov function.*

Note that if $V(\cdot)$ is continuously differentiable at $x$, then $\mathcal{L}_{F_u} = \{\nabla V(x) v : v \in \mathcal{K}[F_u](x)\}$. If, in addition, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is locally Lipschitz continuous in $x$ uniformly in $u$, then Eq. (10) collapses to the standard control Lyapunov definition given in Ref. [7].

If Eq. (10) holds, then there exists a feedback control law $\phi : \mathcal{D} \to U$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0, x \in \mathcal{D}, x \neq 0$, and hence, Theorem 3.1 implies that if there exists a control Lyapunov function for the discontinuous nonlinear dynamical system (1), then there exists a feedback control law $\phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the closed-loop system (2) is strongly asymptotically stable. Conversely, if there exists a feedback control law $u = \phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the discontinuous nonlinear dynamical system (1) is strongly asymptotically stable, then, since $\mathcal{L}_{F_\phi} V(x) \subseteq \{p^{\mathrm{T}} v : p^{\mathrm{T}} \in \partial V(x)$ and $v \in \mathcal{K}[F_\phi](x)\}$, it follows from Theorem 2.7 of Ref. [9] that there exists a locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0$ for almost all nonzero $x \in \mathcal{D}$ or, equivalently, there exists a control Lyapunov function for the discontinuous nonlinear dynamical system (1). Hence, a given discontinuous dynamical system of the form (1) is strongly feedback asymptotically stabilizable if and only if there exists a control Lyapunov function satisfying (10). Finally, in the case where $\mathcal{D} = \mathbb{R}^n$ and $U = \mathbb{R}^m$ the zero Filippov solution $x(t) \equiv 0$ to (1) is globally strongly asymptotically stabilizable if and only if $V(x) \to \infty$ as $||x|| \to \infty$.

Next, we consider the special case of discontinuous nonlinear systems affine in the control, and we construct state feedback controllers that globally asymptotically stabilize the zero Filippov solution of the discontinuous nonlinear dynamical system under the assumption that the system has a radially unbounded control Lyapunov function. Specifically, we consider discontinuous nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0 \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n, G : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. We assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. Note that Eq. (11) is a special case of Eq. (1) with $F(x, u) = f(x) + G(x)u$. We use the notation $f + Gu$ to denote the function $F_u(x) = f(x) + G(x)u$ for each $u \in \mathbb{R}^m$.

Note that Eq. (11) includes piecewise continuous dynamical systems as well as switched dynamical systems as special cases. For example, if $f(\cdot)$ and $G(\cdot)$ are piecewise continuous, then Eq. (11) can be represented as a differential inclusion involving Filippov set-valued maps of piecewise-continuous vector fields given by $\mathcal{K}[f](x) = \overline{\mathrm{co}}\{\lim_{i \to \infty} f(x_i) : x_i \to x, x_i \notin \mathcal{S}_f\}$, where $\mathcal{S}_f$ has measure zero and denotes the set of points where $f$ is discontinuous [22], and similarly for $G(\cdot)$. Here, we assume that $\mathcal{K}[f](\cdot)$ has at least one equilibrium point so that, without loss of generality, $0 \in \mathcal{K}[f](0)$.

Next, define

$$\mathcal{L}_G V(x) \overset{\Delta}{=} \{q \in \mathbb{R}^{1 \times m}: \text{there exists } v \in \mathfrak{G}(x) \text{ such that } p^{\mathrm{T}} v = q \text{ for all } p^{\mathrm{T}} \in \partial V(x)\},$$

where $\mathfrak{G}(x) \overset{\Delta}{=} \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S}) = 0} \overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x) \backslash \mathcal{S})\}, x \in \mathbb{R}^n$, and $\cap_{\mu(\mathcal{S}) = 0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Finally, we assume that the set $\mathcal{L}_G V(x)$ is single-valued[2] for almost all $x \in \mathbb{R}^n$ and that $\mathcal{L}_G V(x) \neq \emptyset$ at all other points $x$.

[2] The assumption that $\mathcal{L}_G V(x)$ is single-valued is necessary. Specifically, as will be seen later in the paper, the requirement that there exists $\bar{z} \in \mathcal{L}_G V(x)$ such that, for all $u \in \mathbb{R}^m, \max[\mathcal{L}_G V(x) u] = \bar{z} u$ holds if and only if $\mathcal{L}_G V(x)$ is a singleton. To see this, let $q, r \in \mathcal{L}_G V(x)$, with $q \neq r$, and assume, *ad absurdum*, that the required $\bar{z}$ exists. Then, either $q - \bar{z} \neq 0$ or $r - \bar{z} \neq 0$. Assume $q - \bar{z} \neq 0$ and let $u^{\mathrm{T}} = q - \bar{z}$. Then, $qu - \bar{z}u = (q - \bar{z})u = (q - \bar{z})(q - \bar{z})^{\mathrm{T}} = \| q - \bar{z} \|_2^2 > 0$. Hence, $qu > \bar{z}u$, which leads to a contradiction.

THEOREM 3.2. *Consider the controlled discontinuous nonlinear dynamical system given by Eq. (11). Then a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ is a control Lyapunov function for Eq. (11) if and only if*

$$\max \mathcal{L}_f V(x) < 0, \quad \text{a.e.} \quad x \in \mathcal{R}, \tag{12}$$

where $\mathcal{R} \triangleq \{x \in \mathbb{R}^n \backslash \{0\} : \mathcal{L}_G V(x) = 0\}$.

*Proof.* Sufficiency is a consequence of the definition of a control Lyapunov function and the sum rule for computing the generalized gradient of locally Lipschitz continuous functions [22]. Specifically, for systems of the form (11), note that $\mathcal{L}_{f+Gu}V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u$ for almost all $x$ and all $u$, and hence, $\inf_{u\in U}[\max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u] = -\infty$ when $x \notin \mathcal{R}$ and $x \neq 0$, whereas $\inf_{u\in U}[\max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u] < 0$ for almost all $x \in \mathcal{R}$. Hence, Eq. (12) implies Eq. (10) with $F_u(x) = f(x) + G(x)u$.

To prove necessity suppose, *ad absurdum*, that $V(\cdot)$ is a control Lyapunov function and Eq. (12) does not hold. In this case, there exists a set $\mathcal{M} \subseteq \mathcal{R}$ of positive measure such that $\max \mathcal{L}_f V(x) \geq 0$ for all $x \in \mathcal{M}$. Let $x \in \mathcal{M}$ and let $\alpha \in \mathcal{L}_f V(x) \cap [0, \infty)$. From the definition of a control Lyapunov function, $x$ is such that there exists $u$ such that $\max \mathcal{L}_{f+Gu}V(x) < 0$ and, by the sum rule for generalized gradients, the inclusion $\mathcal{L}_f V(x) \subseteq \mathcal{L}_{f+Gu}V(x) + \mathcal{L}_{-Gu}V(x)$ is satisfied (since the sum rule holds for almost all $x$). Now, since $x \in \mathcal{M}$, we have $\mathcal{L}_{-Gu}V(x) = -\mathcal{L}_{Gu}V(x) \subseteq -\mathcal{L}_G V(x)u \subseteq \{0\}$. Hence, there exists a non-negative $\alpha \in \mathcal{L}_{f+Gu}V(x)$, which is a contradiction. This proves the theorem.

It follows from Theorem 3.2 that the zero Filippov solution $x(t) \equiv 0$ of a discontinuous nonlinear affine system of the form (11) is globally strongly feedback asymptotically stabilizable if and only if there exists a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying Eq. (12). Hence, Theorem 3.2 provides necessary and sufficient conditions for discontinuous nonlinear system stabilization.

Next, using Theorem 3.2 we *construct* an explicit feedback control law that is a function of the control Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \dfrac{\alpha(x) + \sqrt{\alpha^2(x) + (\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0 \\ 0, & \beta(x) = 0 \end{cases} \tag{13}$$

where $\alpha(x) \triangleq \max \mathcal{L}_f V(x), \beta(x) \triangleq (\mathcal{L}_G V(x))^{\mathrm{T}}$, and $c_0 \geq 0$ is a constant. In this case, the control Lyapunov function $V(\cdot)$ of (11) is a Lyapunov function for the closed-loop system (11) with $u = \phi(x)$, where $\phi(x)$ is given by Eq. (13). To see this, recall that using the sum rule for computing the generalized gradient of locally Lipschitz continuous functions, [22] it follows that $\mathcal{L}_{f+Gu}V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_{Gu}V(x)$ for almost all $x \in \mathbb{R}^n$.

In particular, Theorem 3.2 gives

$$\max \mathcal{L}_{F_\phi} V(x) = \max \mathcal{L}_{f+G\phi} \tag{14}$$

$$\begin{aligned} &\leq \max\left[\mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)\right] \\ &= \max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x) \\ &= \alpha(x) + \beta^{\mathrm{T}}(x)\phi(x) \\ &= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{\alpha^2(x) + (\beta^{\mathrm{T}}(x)\beta(x))^2}, & \beta(x) \neq 0, \\ \alpha(x), & \beta(x) = 0, \end{cases} \\ &< 0, \ x \in \mathbb{R}^n, \text{ a.e. } x \neq 0, \end{aligned} \tag{15}$$

which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (11), and hence, by Theorem 3.1, guaranteeing global strong asymptotic stability with $u = \phi(x)$ given by Eq. (13).

*Example 3.1*. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction given by [14]

$$\dot{x}_1(t) = -\mathrm{sign}(x_2(t)) - \frac{1}{2}\mathrm{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0 \tag{16}$$

$$\dot{x}_2(t) = \mathrm{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20} \tag{17}$$

where $\mathrm{sign}(\sigma) \triangleq \sigma/|\sigma|, \sigma \neq 0$, and $\mathrm{sign}(0) \triangleq 0$. Next, consider the locally Lipschitz continuous function $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x) = \begin{cases} \{\mathrm{sign}(x_1)\} \times \{\mathrm{sign}(x_2)\}, & x_1 \neq 0, \ x_2 \neq 0 \\ \{\mathrm{sign}(x_1)\} \times [-1, 1], & x_1 \neq 0, \ x_2 = 0 \\ [-1, 1] \times \{\mathrm{sign}(x_2)\}, & x_2 \neq 0, \ x_1 = 0 \\ \overline{\mathrm{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\}, & (x_1, x_2) = (0,0) \end{cases}$$

Hence, with $f(x) = [-\mathrm{sign}(x_2) - 1/2\mathrm{sign}(x_1), \mathrm{sign}(x_1)]^{\mathrm{T}}$ and $G(x) = [0, 1]^{\mathrm{T}}$,

$$\mathcal{L}_f V(x) = \begin{cases} \{-\frac{1}{2}\}, & x_1 \neq 0, \ x_2 \neq 0 \\ \emptyset, & x_1 \neq 0, \ x_2 = 0 \\ \emptyset, & x_2 \neq 0, \ x_1 = 0 \\ \{0\}, & (x_1, x_2) = (0,0) \end{cases}$$

and

$$\mathcal{L}_G V(x) = \begin{cases} \{\mathrm{sign}(x_2)\}, & x_1 \neq 0, \ x_2 \neq 0 \\ \emptyset, & x_1 \neq 0, \ x_2 = 0 \\ \{\mathrm{sign}(x_2)\}, & x_2 \neq 0, \ x_1 = 0 \\ \emptyset, & (x_1, x_2) = (0,0) \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \backslash \{0\} : \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 3.2 that $V(x) = |x_1| + |x_2|$ is a control Lyapunov function for Eqs. (16) and (17).

Next, note that it follows from Eq. (13) that for almost all $x \in \mathbb{R}^2 \backslash \{0\}$

$$\phi(x) = -\left(c_0 + \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \mathrm{sign}^4(x_2)}}{\mathrm{sign}^2(x_2)}\right)\mathrm{sign}(x_2), \quad \mathrm{sign}(x_2) \neq 0 \tag{18}$$

where $c_0 \geq 0$, and hence, since $\mathcal{L}_{f+G\phi}V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)$ for almost all $x$,

$$\max \mathcal{L}_{f+G\phi}V(x) \ \leq \ -\left(c_0 + \frac{\sqrt{5}}{2}\right) \ < \ 0.$$

Now, it follows from Theorem 3.1 that Eq. (18) is a globally strongly stabilizing feedback controller. Figures 1 and 2 show the phase portraits of the open-loop ($u(t) \equiv 0$) and closed-loop nonsmooth harmonic oscillator with $c_0 = 0$, respectively. Finally, Figs. 3 and 4 show the state trajectories and the control trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $c_0 = 0$. △

*Example 3.2*. Consider the controlled dynamical system $\mathcal{G}$ given by Eq. (11), where $x = [x_1, x_2]^{\mathrm{T}}$, $u = [u_1, u_2]^{\mathrm{T}}$,

$$f(x) = \begin{bmatrix} |x_1|(-x_1 + |x_2|) \\ x_2(-x_1 - |x_2|) \end{bmatrix}, \quad G(x) = \begin{bmatrix} |x_1| & 0 \\ 0 & x_2 \end{bmatrix}$$

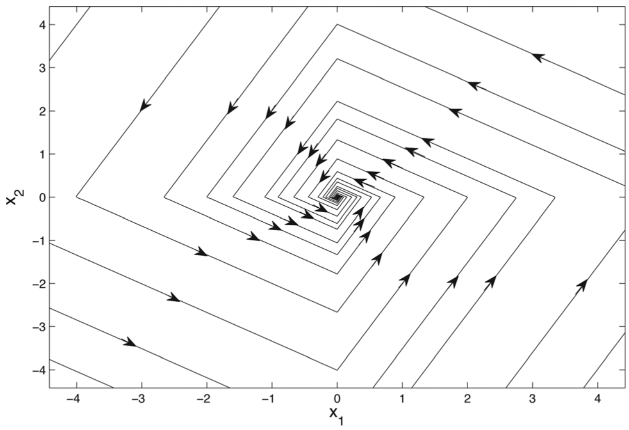


**Fig. 1 Phase portrait of the open-loop nonsmooth harmonic oscillator**

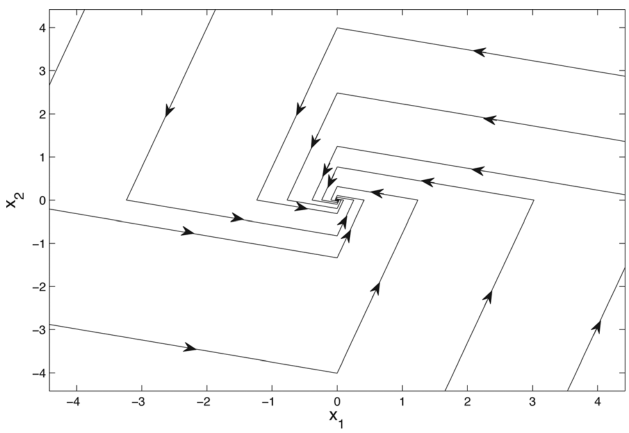


**Fig. 2 Phase portrait of the closed-loop nonsmooth harmonic oscillator**

Next, consider the locally Lipschitz continuous function $V(x) = 2|x_1| + 2|x_2|$ and note that

$$\partial V(x) = \begin{cases} \{2\text{sign}(x_1)\} \times \{2\text{sign}(x_2)\}, & x_1 \neq 0,\ x_2 \neq 0 \\ \{2\text{sign}(x_1)\} \times [-2, 2], & x_1 \neq 0,\ x_2 = 0 \\ [-2, 2] \times \{2\text{sign}(x_2)\}, & x_2 \neq 0,\ x_1 = 0 \\ \overline{\text{co}}\{(2,2), (-2,2), (-2,-2), (2,-2)\}, & (x_1, x_2) = (0,0) \end{cases}$$

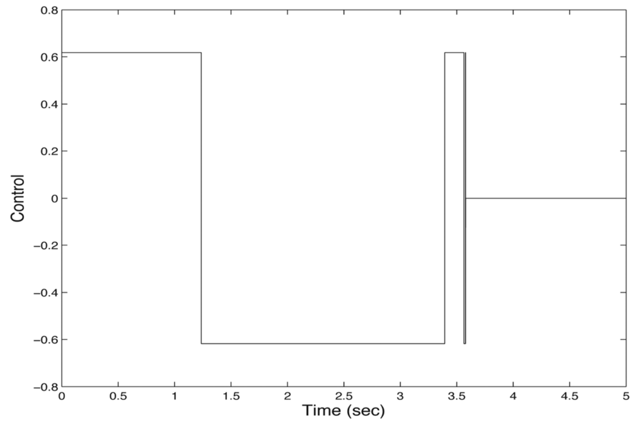


**Fig. 4 Control trajectories of the closed-loop system versus time**

Hence

$$\mathcal{L}_f V(x) = \begin{cases} \{-2x_1^2 - 2x_2^2\}, & x_1 \neq 0,\ x_2 \neq 0 \\ \{-2x_1^2\}, & x_1 \neq 0,\ x_2 = 0 \\ \{-2x_2^2\}, & x_2 \neq 0,\ x_1 = 0 \\ \{0\}, & (x_1, x_2) = (0,0) \end{cases}$$

and

$$\mathcal{L}_G V(x) = \begin{cases} \{(2x_1, 2|x_2|)\}, & x_1 \neq 0,\ x_2 \neq 0 \\ \{(2x_1, 0)\}, & x_1 \neq 0,\ x_2 = 0 \\ \{(0, 2|x_2|)\}, & x_2 \neq 0,\ x_1 = 0 \\ \{(0,0)\}, & (x_1, x_2) = (0,0) \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \setminus \{0\} : \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 3.2 that $V(x) = 2|x_1| + 2|x_2|$ is a control Lyapunov function.

Setting $\alpha(x) = \max \mathcal{L}_f V(x)$ and $\beta(x) = (\mathcal{L}_G V(x))^{\mathrm{T}}$, it follows that $\beta^{\mathrm{T}}(x)\beta(x) = 4(x_1^2 + x_2^2)$ and $\alpha^2(x) + (\beta^{\mathrm{T}}(x)\beta(x))^2 = 4(x_1^2 + x_2^2)^2 + 16(x_1^4 + x_2^4 + 2x_1^2x_2^2) = 20(x_1^4 + x_2^4) + 40x_1^2x_2^2 = 20(x_1^2 + x_2^2)^2$, and hence, Eq. (13) gives

$$\phi(x) = \begin{cases} -\left(c_0 + (\sqrt{5} - 1)\right) \begin{bmatrix} x_1 \\ |x_2| \end{bmatrix}, & (x_1, x_2) \neq (0,0) \\ 0, & (x_1, x_2) = (0,0) \end{cases} \tag{19}$$

where $c_0 \geq 0$. Thus, $\max \mathcal{L}_{f+G\phi} V(x) \leq -|x|^2$ for all $x \neq 0$. Now, it follows from Theorem 3.1 that Eq. (19) is a globally strongly stabilizing feedback controller. Figures 5 and 6 show the phase

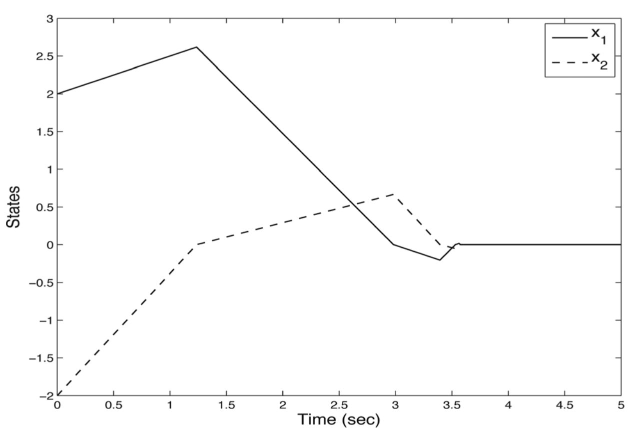


**Fig. 3 State trajectories of the closed-loop system versus time**

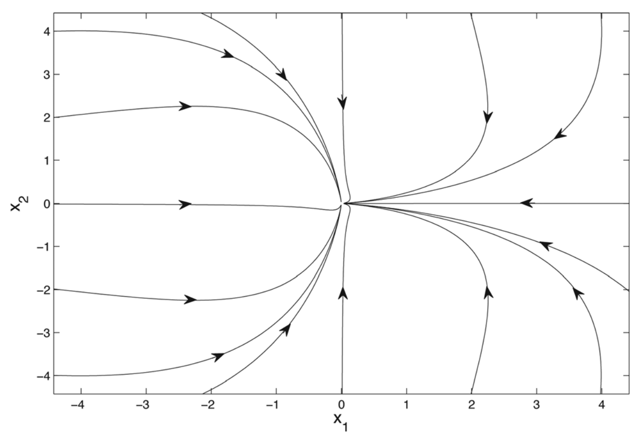


**Fig. 5 Phase portrait of the open-loop system**

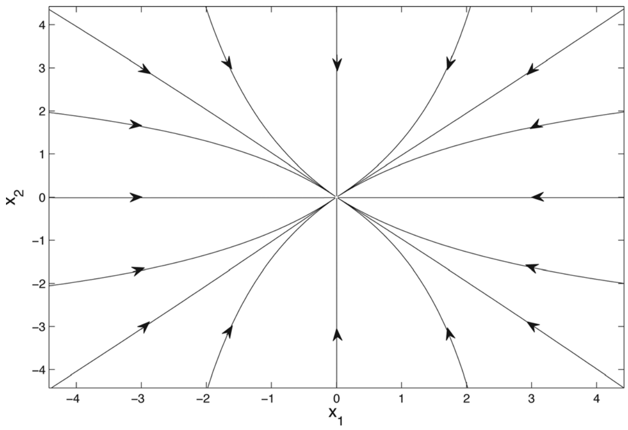


**Fig. 6 Phase portrait of the closed-loop system**

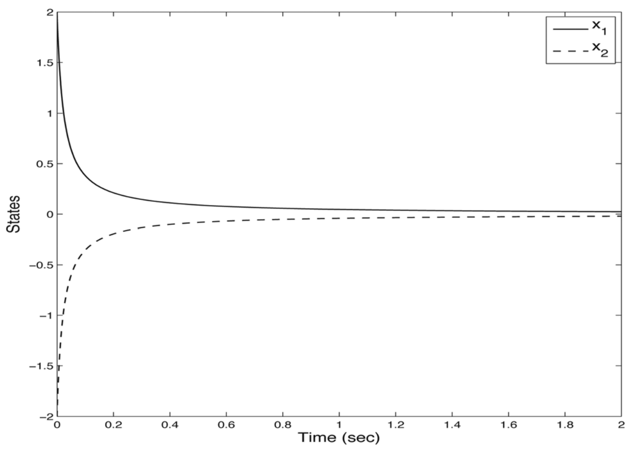


**Fig. 7 State trajectories of the closed-loop system versus time**

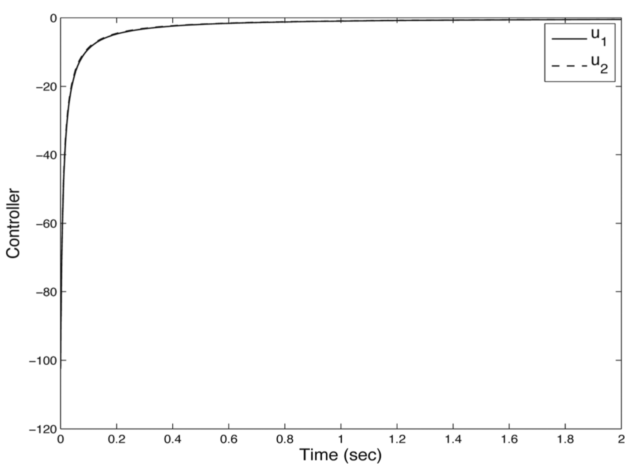


**Fig. 8 Control trajectories of the closed-loop system versus time**

portraits of the open-loop ($u(t) \equiv 0$) and closed-loop system with $c_0 = 50$, respectively. Finally, Figs. 7 and 8 show the state trajectories and the control trajectories of the closed-loop system versus time for $x(0) = [2, -2]^T$ and $c_0 = 50$. △

## 4 Conclusion

In this paper, we developed a constructive universal feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives. Specifically, we address the problem of discontinuous stabilization for dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps and admitting Filippov solutions. In the case where the system vector field is locally Lipschitz continuous and our control Lyapunov function is assumed to be continuously differentiable, our results specialize to the control Lyapunov function of Artstein [7] and our constructive universal controller specializes to Sontag's universal feedback control law [8]. The efficacy of the proposed approach is shown in two representative examples involving discontinuous dynamics and Lipschitz continuous control Lyapunov functions. Extensions of this work for addressing connections between nonsmooth control Lyapunov functions and inverse optimality is currently under development.

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