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GUARANTEED DOMAINS OF ATTRACTION FOR MULTIVARIABLE LURÉ SYSTEMS VIA OPEN LYAPUNOV SURFACES

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SUMMARY

In this paper we provide guaranteed stability regions for multivariable Luré-type systems. Specifically, using the Luré–Postnikov Lyapunov function a guaranteed subset of the domain of attraction for a feedback system whose forward path contains a dynamic linear time-invariant system and whose feedback path contains multiple sector-bounded time-invariant memoryless nonlinearities is constructed via open Lyapunov surfaces. It is shown that the use of open Lyapunov surfaces yields a considerable improvement over closed Lyapunov surfaces in estimating the domain of attraction of the zero solution of the nonlinear system. An immediate application of this result is the computation of transient stability regions for multimachine power systems and computation of stability regions of anti-windup controllers for systems subject to input saturation. © 1997 by John Wiley & Sons, Ltd.

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1. INTRODUCTION

Absolute stability theory guarantees stability of feedback systems whose forward path contains a dynamic linear time-invariant system and whose feedback path contains a memoryless (possibly time-varying) sector bounded nonlinearity. In the case where the sector bounded nonlinearity is known to be time-invariant, the Popov criterion[1–4] provides a sufficient condition for global absolute stability. If, however, the feedback nonlinearity satisfies the sector constraint for only a finite or semi-infinite range of its argument the feedback interconnection can only be guaranteed to be locally asymptotically stable. In this case, it is important to provide a guaranteed subset of the domain of attraction for the nonlinear system to ensure that the system trajectories do not leave the local stability region.

The problem of providing a guaranteed domain of attraction for locally stable nonlinear systems has received considerable attention (see, for example, References 5–7 and the references therein). The motivation for estimating a subset of the domain of attraction of a nonlinear system is the fact that there is no guarantee that a system trajectory starting in a subset of the state space

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$\mathscr{R}$ will remain in $\mathscr{R}$ even though the system trajectories move from one Lyapunov level surface to an inner Lyapunov level surface. For single-input/single-output (SISO) Luré and Luré-type systems the authors in References 8–10 use the Luré–Postnikov Lyapunov function to provide an estimate of domain of attraction. Specifically, References 8 and 9 use the Luré–Postnikov Lyapunov function to compute a region of asymptotic stability using closed Lyapunov surfaces while Reference 10, using a series of formal geometric arguments, provides a considerably improved region of asymptotic stability using open Lyapunov surfaces. In this paper using the multivariable Popov criterion[11,12] we extend the results of References 8–10 to multi-input/multi-output (MIMO) systems.

An immediate application of the results of this paper is the computation of transient stability regions for multimachine power systems[13,14] and computation of stability regions of anti-windup controllers for systems subjected to input saturation.[15,16] Specifically, it can be shown that the dynamic equations describing this class of systems can be cast into a multivariable Luré problem where the feedback nonlinearities satisfy sector constraints for a finite or semi-infinite range of their arguments. Hence, the results of the present paper can be used to provide a guaranteed subset of the domain of attraction for this class of systems.

2. GUARANTEED DOMAINS OF ATTRACTION FOR SISO LURÉ SYSTEMS

In this section we consider a Luré-type absolute stability problem of the nonlinear system

$$\dot{x}(t) = Ax(t) - B\phi(y), \quad x(0) = x_0, \quad t \geqslant 0 \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where (A, B) is controllable, (A, C) is observable, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is a time-invariant sector-bounded memoryless nonlinearity. In this section we restrict our attention to single-input/single-output systems, i.e., $m = 1$. This constraint is removed in the next section. For the single-input/single-output case we assume $\phi \in \Phi_s$ where Φ_s is defined by

$$\Phi_s \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R} : 0 \leqslant \phi(y)y \leqslant ky^2, \quad y \in \mathbb{R}\}$$

where k is a positive scalar. In this case it follows from the Popov criterion[3,4,17] that the zero solution of (1), (2) is globally asymptotically stable for all $\phi \in \Phi_s$ if there exists a constant $\alpha \in \mathbb{R}$, $\alpha \geqslant 0$, such that $\frac{1}{k} + (1 + \alpha s)G(s)$ is strictly positive real where $G(s) \triangleq C(sI - A)^{-1}B$. Furthermore, absolute stability can be proven by the Luré–Postnikov Lyapunov function

$$V(x) = x^{\mathrm{T}}Px + \alpha \int_0^y \phi(\sigma)\,\mathrm{d}\sigma \tag{3}$$

where $P \in \mathbb{R}^{n \times n}$, $P > 0$, satisfies the Kalman–Yakubovich–Popov Equations[4] arising from the strict positive real condition on $\frac{1}{k} + (1 + \alpha s)G(s)$. In what follows, we assume that there exists $\alpha \geqslant 0$ such that $\frac{1}{k} + (1 + \alpha s)G(s)$ is strictly positive real and hence there exists $P \in \mathbb{R}^{n \times n}$, $P > 0$, satisfying the corresponding Kalman–Yakubovich–Popov equations. Next, if the input nonlinearity ϕ is contained in Φ_s for a finite or semi-infinite range of its argument, that is,

$$\phi \in \hat{\Phi}_s \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R} : 0 \leqslant \phi(y)y \leqslant ky^2, \quad \underline{y} \leqslant y \leqslant \bar{y}\} \tag{4}$$

where $\underline{y} < 0$ and $\bar{y} > 0$ are given, then (3) can be used to provide a guaranteed subset of the domain of attraction for the nonlinear system (1), (2). Since $P > 0$ satisfies the Kalman–Yakubovich–Popov conditions arising from the strict positive real condition on $\frac{1}{k} + (1 + \alpha s)G(s)$ it follows that $\dot{V}(x) \triangleq V'(x)[Ax - B\phi(Cx)] < 0$ for all $x \in \mathscr{X}/\{0\}$ where $\mathscr{X} \triangleq \{x \in \mathbb{R}^n : \underline{y} \leqslant Cx \leqslant \bar{y}\}$.

Next, define the hyperplanes

$$\Gamma^{+} \triangleq \{x \in \mathbb{R}^n : Cx = \bar{y}\} \tag{5}$$

$$\Gamma^{-} \triangleq \{x \in \mathbb{R}^n : Cx = \underline{y}\} \tag{6}$$

with associated minimum Lyapunov values, respectively,

$$V_{\Gamma^{+}} \triangleq \min_{x \in \Gamma^{+}} V(x) \tag{7}$$

$$V_{\Gamma^{-}} \triangleq \min_{x \in \Gamma^{-}} V(x) \tag{8}$$

Proposition 2.1

Let V_{Γ^+} and V_{Γ^-} be given by (7) and (8), respectively. Then

$$V_{\Gamma^{+}} = \frac{\bar{y}^2}{CP^{-1}C^{\mathrm{T}}} + \alpha \mathscr{Y}(\bar{y}) \tag{9}$$

$$V_{\Gamma^{-}} = \frac{\underline{y}^2}{CP^{-1}C^{\mathrm{T}}} + \alpha \mathscr{Y}(\underline{y}) \tag{10}$$

where $\mathscr{Y}(z) \triangleq \int_0^z \phi(\sigma)\,\mathrm{d}\sigma$.

Proof. Suppose $x_+ \in \Gamma^+$ solves (7). Since $\int_0^{\bar{y}} \phi(\sigma)\,\mathrm{d}\sigma$ is a constant for all $x \in \Gamma^+$, x_+ solves (7) if and only if x_+ solves the minimization problem $\min_{x\in\Gamma^+} \mathscr{Q}(x)$ where $\mathscr{Q}(x) \triangleq x^{\mathrm{T}} P x$ is the positive-definite quadratic part of $V(x)$. The existence and uniqueness of $\min_{x\in\Gamma^+} \mathscr{Q}(x)$ and $\min_{x\in\Gamma^-} \mathscr{Q}(x)$ follows from the fact that $\mathscr{Q}(x)$ is a strictly convex function on Γ^+ and Γ^-. Next, to minimize $\mathscr{Q}(x)$ subject to $x \in \Gamma^+$ form the Lagrangian $\mathscr{L}(x, \lambda) \triangleq \mathscr{Q}(x) + \lambda(Cx - \bar{y})$ where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Now if x_+ solves (7) then

$$0 = \left.\frac{\partial \mathscr{L}}{\partial x}\right|_{x = x_+} = 2x_+^{\mathrm{T}} P + \lambda C \tag{11}$$

and hence $2x_+^{\mathrm{T}} P x_+ = -\lambda C x_+ = -\lambda \bar{y}$. Next, forming (11) $P^{-1}C^{\mathrm{T}}$ yields

$$0 = 2x_+^{\mathrm{T}} C^{\mathrm{T}} + \lambda CP^{-1}C^{\mathrm{T}} = 2\bar{y} + \lambda CP^{-1}C^{\mathrm{T}}$$

and hence, since $CP^{-1}C^{\mathrm{T}} > 0$, $x_+^{\mathrm{T}} P x_+ = \dfrac{\bar{y}^2}{CP^{-1}C^{\mathrm{T}}}$. Thus,

$$V_{\Gamma^{+}} = V(x_+) = x_+^{\mathrm{T}} P x_+ + \alpha \int_0^{Cx_+} \phi(\sigma)\,\mathrm{d}\sigma = \frac{\bar{y}^2}{CP^{-1}C^{\mathrm{T}}} + \alpha \mathscr{Y}(\bar{y})$$

Equation (10) follows by carrying out the identical steps with x_+ and Γ^+ replaced by x_- and Γ^-, respectively. □

Remark 2.1

Note that the closed subset $\mathscr{D}_{\mathrm{cl}}$ of $\mathscr{X}$ defined by $\mathscr{D}_{\mathrm{cl}} \triangleq \{x \in \mathscr{X} : V(x) \leqslant V_\Gamma\}$, where $V_\Gamma \triangleq \min\{V_{\Gamma^+}, V_{\Gamma^-}\}$, is a subset of the domain of attraction for (1) and (2) since $\dot{V}(x) < 0$ for all $x \in \mathscr{D}_{\mathrm{cl}}/\{0\} \subseteq \mathscr{X}/\{0\}$.

Remark 2.2

It is important to note that unlike standard Luré absolute stability problems where the nonlinearity $\phi \in \Phi_s$ is not explicitly known our results for estimating a subset of the domain of attraction require the exact functional form of $\phi \in \Phi_s$. This restriction can be eliminated if we restrict our consideration to non-Luré–Postnikov type Lyapunov functions of the form $V(x) = x^T P x$. However, this will introduce conservatism to both the stability and domain of attraction predictions since fixed quadratic-type Lyapunov functions guarantee stability for the much larger class of arbitrary time-varying nonlinearities.[11] For further details see Reference 15.

The following lemma is needed for the main result of this section.

Lemma 2.1

Let $x_+ \in \Gamma^+$ and $x_- \in \Gamma^-$ be such that $V_{\Gamma^+} = V(x_+)$ and $V_{\Gamma^-} = V(x_-)$. Then

$$CAx_+ - CB\phi(\bar{y}) < 0 \tag{12}$$

and

$$CAx_- - CB\phi(\underline{y}) > 0 \tag{13}$$

Proof. Since by assumption x_+ minimizes $V(x)$ on Γ^+ it follows that $\frac{\partial \mathscr{L}}{\partial x}|_{x=x_+} = 0$ where $\mathscr{L}(x, \lambda) \triangleq V(x) + \lambda(Cx - \bar{y})$ is a Lagrangian and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Hence,

$$0 = \left.\frac{\partial \mathscr{L}}{\partial x}\right|_{x=x_+} = V'(x_+) + \lambda C \tag{14}$$

Next, forming $(14)x_+$ yields $V'(x_+)x_+ + \lambda\bar{y} = 0$ which implies that $\lambda = -(V'(x_+)x_+/\bar{y}) < 0$ since $V'(x_+)x_+ = 2x_+^T P x_+ + \alpha\phi(\bar{y})\bar{y} > 0$. Furthermore, since $\dot{V}(x) < 0$ for all $x \in \mathscr{X}/\{0\}$, it follows that

$$V'(x_+)[Ax_+ - B\phi(Cx_+)] = -\lambda(CAx_+ - CB\phi(\bar{y})) < 0$$

and hence $CAx_+ - CB\phi(\bar{y}) < 0$. Equation (13) follows by carrying out the identical steps with x_+ and Γ^+ replaced by x_- and Γ^-, respectively. □

Next, define the intersections of Γ^+ and Γ^- and the hyperplanes $CAx - CB\phi(\bar{y}) = 0$ and $CAx - CB\phi(\underline{y}) = 0$, respectively, by

$$\mathscr{S}^+ \triangleq \{x \in \Gamma^+ : CAx - CB\phi(\bar{y}) = 0\} \tag{15}$$

$$\mathscr{S}^- \triangleq \{x \in \Gamma^- : CAx - CB\phi(\underline{y}) = 0\} \tag{16}$$

with associated minimum Lyapunov values

$$V_{\mathscr{S}^+} \triangleq \begin{cases} \min_{x \in \mathscr{S}^+} V(x) & \text{if } \mathscr{S}^+ \neq \emptyset \\ \infty & \text{if } \mathscr{S}^+ = \emptyset \end{cases} \tag{17}$$

$$V_{\mathscr{S}^-} \triangleq \begin{cases} \min_{x \in \mathscr{S}^-} V(x) & \text{if } \mathscr{S}^- \neq \emptyset \\ \infty & \text{if } \mathscr{S}^- = \emptyset \end{cases} \tag{18}$$

Proposition 2.2

Suppose

$$0 \neq (CP^{-1}C^T)(CAP^{-1}A^TC^T) - (CAP^{-1}C^T)^2 \tag{19}$$

and let $V_{\mathscr{S}^+}$ and $V_{\mathscr{S}^-}$ be given by (17) and (18), respectively. Then

$$V_{\mathscr{S}^+} = \Psi(\bar{y}) + \alpha\mathscr{Y}(\bar{y}) \tag{20}$$

and

$$V_{\mathscr{S}^-} = \Psi(\underline{y}) + \alpha\mathscr{Y}(\underline{y}) \tag{21}$$

where $\Psi : \mathbb{R} \to \mathbb{R}$ is given by

$$\Psi(z) \triangleq \frac{C[Az - BC\phi(z)]P^{-1}[Az - CB\phi(z)]^{\mathrm{T}}C^{\mathrm{T}}}{(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - (CAP^{-1}C^{\mathrm{T}})^2}$$

Proof. Once again note that x_+ solves $\min_{x\in\mathscr{S}^+} V(x)$ if and only if x_+ solves $\min_{x\in\mathscr{S}^+} \mathscr{Q}(x)$ where $\mathscr{Q}(x) = x^{\mathrm{T}}Px$ since $\int_0^{cx} \phi(\sigma)\,\mathrm{d}\sigma$ is a constant for all $x \in \mathscr{S}^+$. Furthermore, since $\mathscr{Q}(x)$ is a positive-definite quadratic function the existence and uniqueness of $\min_{x\in\mathscr{S}^+}\mathscr{Q}(x)$ and $\min_{x\in\mathscr{S}^-} \mathscr{Q}(x)$ follows from the fact that $\mathscr{Q}(x)$ is a strictly convex function $\mathscr{S}^+$ and $\mathscr{S}^-$. Next, to minimize $\mathscr{Q}(x)$ subject to $x \in \mathscr{S}^+$ form the Lagrangian

$$\mathscr{L}(x, \lambda_1, \lambda_2) \triangleq \mathscr{Q}(x) + \lambda_1(Cx - \bar{y}) + \lambda_2(CAx - CB\phi(\bar{y}))$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the Lagrange multipliers. Now, if x_+ solves $\min_{x\in\mathscr{S}^+} V(x)$ then

$$0 = \left.\frac{\partial\mathscr{L}}{\partial x}\right|_{x=x_+} = 2x_+^{\mathrm{T}}P + \lambda_1 C + \lambda_2 CA \tag{22}$$

Next, forming (22)$P^{-1}C^{\mathrm{T}}$ and (22)$P^{-1}A^{\mathrm{T}}C^{\mathrm{T}}$ yields, respectively,

$$0 = 2x_+^{\mathrm{T}}C^{\mathrm{T}} + \lambda_1(CP^{-1}C^{\mathrm{T}}) + \lambda_2(CAP^{-1}C^{\mathrm{T}}) \tag{23}$$

$$0 = 2x_+^{\mathrm{T}}A^{\mathrm{T}}C^{\mathrm{T}} + \lambda_1(CP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) + \lambda_2(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) \tag{24}$$

which further implies that

$$\lambda_1 = -2\frac{\bar{y}(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - \phi(\bar{y})B^{\mathrm{T}}C^{\mathrm{T}}(CAP^{-1}C^{\mathrm{T}})}{(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - (CP^{-1}A^{\mathrm{T}}C^{\mathrm{T}})^2}$$

and

$$\lambda_2 = 2\frac{\bar{y}(CP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - \phi(\bar{y})B^{\mathrm{T}}C^{\mathrm{T}}(CP^{-1}C^{\mathrm{T}})}{(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - (CP^{-1}A^{\mathrm{T}}C^{\mathrm{T}})^2}$$

Now, forming (22)x_+ and substituting for λ_1 and λ_2 yields

$$x_+^{\mathrm{T}}Px_+ = \frac{C[A\bar{y} - BC\phi(\bar{y})]P^{-1}[A\bar{y} - BC\phi(\bar{y})]^{\mathrm{T}}C^{\mathrm{T}}}{(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - (CP^{-1}A^{\mathrm{T}}C^{\mathrm{T}})^2} = \Psi(\bar{y})$$

Hence, $V_{\mathscr{S}^+} = V(x_+) = \Psi(\bar{y}) + \alpha\mathscr{Y}(\bar{y})$. Equation (21) follows by carrying out the identical steps with x_+ and $\mathscr{S}^+$ replaced by x_- and $\mathscr{S}^-$, respectively. □

Remark 2.3

Note that if $(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) - (CAP^{-1}C^{\mathrm{T}})^2 = 0$ then

$$0 = CAP^{-1/2}((CP^{-1}C^{\mathrm{T}})I_n - P^{-1/2}C^{\mathrm{T}}CP^{-1/2})P^{-1/2}A^{\mathrm{T}}C^{\mathrm{T}} \tag{25}$$

In this case, since $\sigma_{\max}(P^{-1/2}C^{\mathrm{T}}CP^{-1/2}) \leqslant CP^{-1}C^{\mathrm{T}}$, where $\sigma_{\max}(\cdot)$ denotes the maximum singular value, it follows that $P^{-1/2}((CP^{-1}C^{\mathrm{T}})I_n - P^{-1/2}C^{\mathrm{T}}CP^{-1/2})P^{-1/2}$ is non-negative definite and hence (25) yields

$$0 = CAP^{-1/2}((CP^{-1}C^{\mathrm{T}})I_n - P^{-1/2}C^{\mathrm{T}}CP^{-1/2})P^{-1/2}$$

which further implies that $CA = (CAP^{-1}C^{\mathrm{T}}/CP^{-1}C^{\mathrm{T}})\,C$. Thus, since C is a left eigenvector of A, the hyperplanes Γ^+ and $\{x \in \mathbb{R}^n : CAx - CB\phi(\bar{y}) = 0\}$ (respectively, Γ^- and $\{x \in \mathbb{R}^n : CAx - CB\phi(\underline{y}) = 0\}$) are either identical or parallel to each other. In the former case $\mathcal{S}^+ = \Gamma^+$ (resp., $\mathcal{S}^- = \Gamma^-$) and hence $V_{\mathcal{S}^+} = V_{\Gamma^+}$ (respectively, $V_{\mathcal{S}^-} = V_{\Gamma^-}$) while in the latter case $\mathcal{S}^+ = \emptyset$ (respectively, $\mathcal{S}^- = \emptyset$) and hence $V_{\mathcal{S}^+} = \infty$ (respectively, $V_{\mathcal{S}^-} = \infty$).

Next, we present the main result of this section for providing a guaranteed subset of the domain of attraction of (1), (2) when the input nonlinearity ϕ is contained in $\hat{\Phi}_{\mathrm{s}}$ for a finite range of its argument. For the statement of this result define

$$\mathcal{D}_{\mathrm{A}} \triangleq \{x \in \mathbb{R}^n : V(x) < V_{\mathcal{S}}, \underline{y} \leqslant Cx \leqslant \bar{y}\} \tag{26}$$

where $V_{\mathcal{S}} \triangleq \min\{V_{\mathcal{S}^+}, V_{\mathcal{S}^-}\}$.

Theorem 2.1

Let $\mathcal{D}_{\mathrm{A}}$ be given by (26). Then $\mathcal{D}_{\mathrm{A}}$ is a subset of the domain of attraction for (1), (2).

Proof. First we show that $\mathcal{D}_{\mathrm{A}}$ is an invariant set for (1), (2). Suppose $V_{\mathcal{S}} < \infty$. In this case in order to show that $\mathcal{D}_{\mathrm{A}}$ is an invariant set for (1), (2) it suffices to show that $CAx - CB\phi(Cx) < 0$ for all $x \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^+$ and $CAx - CB\phi(Cx) > 0$ for all $x \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^-$. Note that $\mathcal{D}_{\mathrm{A}} \cap \Gamma^+ = \{x \in \mathbb{R}^n : V(x) < V_{\mathcal{S}}, Cx = \bar{y}\}$ is a convex set and hence connected. Now, suppose that there exists $x \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^+$ such that $CAx - CB\phi(\bar{y}) \geqslant 0$. Then, since by Lemma 2.1, there exists $x_+ \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^+$ such that $CAx_+ - CB\phi(\bar{y}) < 0$, it follows from continuity that there exists $x^* \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^+$ such that $CAx^* - CB\phi(\bar{y}) = 0$ and hence $V(x^*) \geqslant V_{\mathcal{S}}$ which is a contradiction. Hence, $CAx - CB\phi(\bar{y}) < 0$ for all $x \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^+$. Using similar arguments it can be shown that $CAx - CB\phi(\underline{y}) > 0$ for all $x \in \mathcal{D}_{\mathrm{A}} \cap \Gamma^-$. Next, suppose $V_{\mathcal{S}} = \infty$. In this case $\mathcal{D}_{\mathrm{A}} = \mathcal{X}$ and hence in order to show that $\mathcal{D}_{\mathrm{A}}$ is an invariant set for (1), (2) it suffices to show that $CAx - CB\phi(Cx) < 0$ for all $x \in \Gamma^+$ and $CAx - CB\phi(Cx) > 0$ for all $x \in \Gamma^-$ which can be shown as above using the fact that there exists $x \in \Gamma^+$ such that $CAx - CB\phi(Cx) < 0$ and $x \in \Gamma^-$ such that $CAx - CB\phi(Cx) > 0$. Hence, $\mathcal{D}_{\mathrm{A}}$ is an invariant set for (1), (2). Finally, since $\dot{V}(x) < 0$ for all $x \in \mathcal{D}_{\mathrm{A}}/\{0\} \subseteq \mathcal{X}/\{0\}$ it follows that $x(t) \to 0$ as $t \to \infty$ where $x(t)$ is the solution to (1). □

Remark 2.4

Theorem 2.1 gives an estimate of the domain of attraction for (1) and (2) by constructing Lyapunov surfaces that are not necessarily closed. For the single-input/single-output case a similar result was reported in Reference 10 where the proof was given using a series of formal geometric arguments.

Remark 2.5

Since $\mathcal{S}^+ \subseteq \Gamma^+$ and $\mathcal{S}^- \subseteq \Gamma^-$ it follows that $V_{\mathcal{S}^+} \geqslant V_{\Gamma^+}$ and $V_{\mathcal{S}^-} \geqslant V_{\Gamma^-}$. Hence, the domain of attraction predicted by (26) will always be larger than or equal to the domain of attraction predicted by (26) with $V_{\mathcal{S}} = \min\{V_{\Gamma^+}, V_{\Gamma^-}\}$.

Remark 2.6

As shown in Remark 2.3, if $(CP^{-1}C^{\mathrm{T}})(CAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}}) = (CAP^{-1}C^{\mathrm{T}})^2$, $\mathscr{S}^+ \neq \Gamma^+$, and $\mathscr{S}^- \neq \Gamma^-$ then $V_{\mathscr{S}^+} = V_{\mathscr{S}^-} = \infty$. In this case $V_{\mathscr{S}} = \infty$ and hence the system (1), (2) is asymptotically stable for all $x_0 \in \mathscr{X}$.

3. GUARANTEED DOMAINS OF ATTRACTION FOR MIMO LURÉ SYSTEMS

In this section we extend the result of Section 2 to multivariable systems, i.e., $m > 1$. Specifically, we consider the nonlinear system (1), (2) where $\phi \in \Phi_{\mathrm{m}}$ belongs to a class of component-decoupled time-invariant sector-bounded monotonic nonlinearities, that is,

$$\Phi_{\mathrm{m}} \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R}^m : 0 \leqslant \phi_i(y_i)y_i \leqslant k_i y_i^2, \quad \phi_i(\cdot) \text{ is differentiable}$$
$$\phi_i'(y_i) \geqslant 0, y_i \in \mathbb{R}, i = 1, \dots, m\} \tag{27}$$

where $k_i, i = 1, \dots, m$, are positive scalars and y_i denotes the ith element of $y \in \mathbb{R}^m$. Note that the nonlinear functions considered, $\phi \in \Phi_{\mathrm{m}}$, are identical to the nonlinearities considered by multivariable extensions of the Popov criterion[11,12] with the additional constraint of monotonicity. In this case global absolute stability of (1), (2) is guaranteed by the multivariable Popov criterion[11,12] which requires that $K^{-1} + (I + Ns)G(s)$ be strictly positive real where $K = \mathrm{diag}(k_1, \dots, k_m)$ and $N = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$ where $\alpha_i \in \mathbb{R}$, $\alpha_i \geqslant 0$, $k_i > 0$, $i = 1, \dots, m$. Furthermore, the Lyapunov function guaranteeing stability is given by

$$V(x) = x^{\mathrm{T}}Px + \sum_{i=1}^{m} \alpha_i \int_0^{C_i x} \phi_i(\sigma)\,\mathrm{d}\sigma \tag{28}$$

where $P \in \mathbb{R}^{n \times n}$, $P > 0$, satisfies the Kalman–Yakubovich–Popov equations[11] arising from the strict positive real condition on $K^{-1} + (I + Ns)G(s)$ and $C_i \neq 0$ denotes the ith row of C. As in the single-input/single-output case, we assume that there exists a non-negative-definite diagonal matrix $N \in \mathbb{R}^{m \times m}$ such that $K^{-1} + (I + Ns)G(s)$ is strictly positive real and hence there exists $P > 0$ satisfying the corresponding Kalman–Yakubovich–Popov equations. Next, we assume that the input nonlinearity ϕ is contained in $\hat{\Phi}_{\mathrm{m}}$ for a finite or semi-infinite range of its argument, that is,

$$\phi \in \hat{\Phi}_{\mathrm{m}} \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R}^m : 0 \leqslant \phi_i(y_i)y_i \leqslant k_i y_i^2, \quad \phi_i(\cdot) \text{ is differentiable}$$
$$\phi_i'(y_i) \geqslant 0, \underline{y}_i \leqslant y_i \leqslant \bar{y}_i, i = 1, \dots, m\} \tag{29}$$

where $\underline{y}_i < 0$ and $\bar{y}_i > 0$, $i = 1, \dots, m$, are given. Since $P > 0$ satisfies the Kalman–Yakubovich–Popov equations arising from the strict positive real condition on $K^{-1} + (I + Ns)G(\mathrm{s})$ it follows that $\dot{V}(x) < 0$ for all $x \in \mathscr{X}/\{0\}$ where $\mathscr{X} \triangleq \bigcap_{i=1}^{m} \mathscr{X}_i$ with $\mathscr{X}_i \triangleq \{x \in \mathbb{R}^n : \underline{y}_i \leqslant C_i x \leqslant \bar{y}_i\}$. The following lemma is required for our main result.

Lemma 3.1

Let $V(x)$ be given by (28) and let $\mathscr{C} \subseteq \mathbb{R}^n$ be a convex set. Then $V(x)$ is a strictly convex function on $\mathscr{C}$.

Proof. The strict convexity of the first term of $V(x)$, $x^{\mathrm{T}}Px$, is immediate. Next let $v_i:\mathbb{R}\to\mathbb{R}$ be defined by $v_i(\theta_i)\triangleq\int_0^{\theta_i}\phi_i(\sigma)\mathrm{d}\sigma$, $i=1,\ldots,m$. Now note that since $\mathrm{d}^2v_i/\mathrm{d}\theta_i^2=\phi_i'(\theta_i)\geqslant 0$, $v_i(\theta_i)$, $i=1,\ldots,m$, is a convex function on $\mathbb{R}$. Furthermore, since

$$\sum_{i=1}^{m}\alpha_i v_i(C_i(\varepsilon x_1+(1-\varepsilon)x_2))\leqslant\varepsilon\left[\sum_{i=1}^{m}\alpha_i v_i(C_i x_1)\right]+(1-\varepsilon)\left[\sum_{i=1}^{m}\alpha_i v_i(C_i x_2)\right]$$

for all $\varepsilon\in[0,1]$ and $x_1,x_2\in\mathscr{C}$, it follows that $\sum_{i=1}^{m}\alpha_i\int_0^{C_i x}\phi_i(\sigma)\mathrm{d}\sigma$ is a convex function on $\mathscr{C}$. The result now follows from the fact that the sum of a strictly convex function and a convex function is a strictly convex function. □

Next, for $i\in\{1,\ldots,m\}$, define the hyperplanes

$$\Gamma_i^{+}\triangleq\{x\in\mathbb{R}^n: C_i x=\bar{y}_i\}\tag{30}$$

$$\Gamma_i^{-}\triangleq\{x\in\mathbb{R}^n: C_i x=\underline{y}_i\}\tag{31}$$

with associated minimum Lyapunov values, respectively,

$$V_{\Gamma_i^+}\triangleq\min_{x\in\Gamma_i^+}V(x)\tag{32}$$

$$V_{\Gamma_i^-}\triangleq\min_{x\in\Gamma_i^-}V(x)\tag{33}$$

Proposition 3.1

For $i\in\{1,\ldots,m\}$ let $x_+\in\Gamma_i^+$ and $x_-\in\Gamma_i^-$ be such that $V(x_+)=V_{\Gamma_i^+}$ and $V(x_-)=V_{\Gamma_i^-}$. Then

$$0=2x_+^{\mathrm{T}}P+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_j-\frac{2\bar{y}_i+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_jP^{-1}C_i^{\mathrm{T}}}{C_iP^{-1}C_i^{\mathrm{T}}}C_i\tag{34}$$

and

$$0=2x_-^{\mathrm{T}}P+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_j-\frac{2\underline{y}_i+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_jP^{-1}C_i^{\mathrm{T}}}{C_iP^{-1}C_i^{\mathrm{T}}}C_i\tag{35}$$

Proof. First, the existence and uniqueness of $\min_{x\in\Gamma_i^+}V(x)$ and $\min_{x\in\Gamma_i^-}V(x)$ follows from the fact that $V(x)$ is a strictly convex function on Γ_i^+ and Γ_i^-. Next, to minimize $V(x)$ subject to $x\in\Gamma_i^+$ form the Lagrangian $\mathscr{L}(x,\lambda)\triangleq V(x)+\lambda(C_ix-\bar{y}_i)$ where $\lambda\in\mathbb{R}$ is a Lagrange multiplier. Now if x_+ solves (32) then

$$0=\left.\frac{\partial\mathscr{L}}{\partial x}\right|_{x=x_+}=2x_+^{\mathrm{T}}P+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_j+\lambda C_i\tag{36}$$

Next, forming $(36)P^{-1}C_i^{\mathrm{T}}$ yields

$$\lambda=-\frac{2\bar{y}_i+\sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_jP^{-1}C_i^{\mathrm{T}}}{C_iP^{-1}C_i^{\mathrm{T}}}$$

and hence

$$0 = 2x_+^{\mathrm{T}} P + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j - \frac{2\bar{y}_i + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j P^{-1} C_i^{\mathrm{T}}}{C_i P^{-1} C_i^{\mathrm{T}}} C_i \tag{37}$$

Equation (35) follows by carrying out the identical steps with x_+ and Γ_i^+ replaced by x_- and Γ_i^-, respectively. □

Remark 3.1

As in the single-input/single-output case, note that the closed subset $\mathscr{D}_{\mathrm{cl}}$ of $\mathscr{X}$ defined by $\mathscr{D}_{\mathrm{cl}} \triangleq \{x \in \mathscr{X} : V(x) \leqslant V_\Gamma\}$, where $V_\Gamma \triangleq \min\{\min_{i=1,\ldots,m} V_{\Gamma_i^+}, \min_{i=1,\ldots,m} V_{\Gamma_i^-}\}$, is a subset of the domain of attraction for (1) and (2).

Remark 3.2

As in the single-input/single-output case Proposition 3.1 requires the exact functional form of $\phi \in \hat{\Phi}_{\mathrm{m}}$. However, in the case where we restrict our consideration to non-Luré–Postnikov type functions of the form $V(x) = x^{\mathrm{T}} Px$ this restriction can once again be eliminated. Furthermore, in this case the nonlinearity set $\hat{\Phi}_{\mathrm{m}}$ can be generalized to non-monotonic fully coupled nonlinearities. As mentioned in Remark 2.2 however, this will introduce conservatism to both stability and domain of attraction predictions since stability is assured for a much larger class of arbitrary time-varying nonlinearities.

Since in the multivariable Luré problem $\sum_{i=1}^{m} \alpha_i \int_0^{C_i x} \phi_i(\sigma)\mathrm{d}\sigma$ is no longer a constant on Γ_i^+ and Γ_i^- for $i \in \{1, \ldots, m\}$, the resulting minimization problem considered in Proposition 3.1 results in a set of nonlinear algebraic equations. However, since $\alpha_i \int_0^{C_i x} \phi^i(\sigma)\mathrm{d}\sigma \geqslant 0$, $i = 1, \ldots, m$, a lower bound involving a closed-form expression for the minimum Lyapunov values (32) and (33) can be computed.

Corollary 3.1

For $i \in \{1, \ldots, m\}$ let $V_{\Gamma_i^+}$ and $V_{\Gamma_i^-}$ be given by (32) and (33), respectively. Then

$$V_{\Gamma_i^+} \geqslant \frac{\bar{y}_i^2}{C_i P^{-1} C_i^{\mathrm{T}}} + \alpha_i \mathscr{Y}_i(\bar{y}_i) \tag{38}$$

$$V_{\Gamma_i^-} \geqslant \frac{\underline{y}_i^2}{C_i P^{-1} C_i^{\mathrm{T}}} + \alpha_i \mathscr{Y}_i(\underline{y}_i) \tag{39}$$

where $\mathscr{Y}_i(z) \triangleq \int_0^z \phi_i(\sigma)\mathrm{d}\sigma$.

Proof. Note that $V(x) \geqslant \hat{V}(x) \triangleq x^{\mathrm{T}} Px + \alpha_i \int_0^{C_i x} \phi_i(\sigma)\mathrm{d}\sigma$ and hence $\min_{x \in \Gamma_i^+} V(x) \geqslant \min_{x \in \Gamma_i^+} \hat{V}(x)$ and $\min_{x \in \Gamma_i^-} V(x) \geqslant \min_{x \in \Gamma_i^-} \hat{V}(x)$. Now since $\alpha_i \int_0^{C_i x} \phi_i(\sigma)\mathrm{d}\sigma$ is a constant on Γ_i^+ and Γ_i^- the result follows as in the proof of Proposition 2.1. □

The following lemma is needed for the main result of this section.

Lemma 3.2

For $i \in \{1, \ldots, m\}$ let $x_+ \in \Gamma_i^+$ and $x_- \in \Gamma_i^-$ be such that $V_{\Gamma_i^+} = V(x_+)$ and $V_{\Gamma_i^-} = V(x_-)$. Then

$$C_i A x_+ - C_i B \phi(C x_+) < 0 \tag{40}$$

and

$$C_i A x_- - C_i B\phi(Cx_-) > 0 \tag{41}$$

Proof. The proof is similar to that of Lemma 2.1 and hence is omitted. □

Next, for $i \in \{1, \ldots, m\}$, define the intersections of Γ_i^+ and Γ_i^- and the hyperplanes $C_i A x - C_i B\phi(Cx) = 0$ by

$$\mathscr{S}_i^+ \triangleq \{x \in \Gamma_i^+ : C_i A x - C_i B\phi(Cx) = 0\} \tag{42}$$

$$\mathscr{S}_i^- \triangleq \{x \in \Gamma_i^- : C_i A x - C_i B\phi(Cx) = 0\} \tag{43}$$

with associated minimum Lyapunov values, respectively,

$$V_{\mathscr{S}_i^+} \triangleq \begin{cases} \min_{x \in \mathscr{S}_i^+} V(x) & \text{if } \mathscr{S}_i^+ \neq \emptyset \\ \infty & \text{if } \mathscr{S}_i^+ = \emptyset \end{cases} \tag{44}$$

$$V_{\mathscr{S}_i^-} \triangleq \begin{cases} \min_{x \in \mathscr{S}_i^-} V(x) & \text{if } \mathscr{S}_i^- \neq \emptyset \\ \infty & \text{if } \mathscr{S}_i^- = \emptyset \end{cases} \tag{45}$$

For the statement of the next result let $\Theta_i : \mathbb{R}^n \to \mathbb{R}^{m \times m}$ and $\beta_i : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\Theta_i(x) \triangleq \operatorname{diag}[\phi_1'(C_1 x), \ldots, \phi_m'(C_m x)]$$

$$\beta_i(x) \triangleq (C_i P^{-1} C_i^{\mathrm{T}})[C_i A P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}} - C_i B \Theta_i(x) C P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}}] - (C_i P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}})[C_i A P^{-1} C_i^{\mathrm{T}} - C_i B \Theta_i(x) C P^{-1} C_i^{\mathrm{T}}]$$

Proposition 3.2

Let $i \in \{1, \ldots, m\}$. Suppose $\mathscr{S}_i^+ \neq \emptyset$, $\mathscr{S}_i^- \neq \emptyset$ and let $x_+ \in \mathscr{S}_i^+$ and $x_- \in \mathscr{S}_i^-$ be such that $V(x_+) = V_{\mathscr{S}_i^+}$ and $V(x_-) = V_{\mathscr{S}_i^-}$. Then

$$\begin{aligned}
0 = {} & 2\beta_i(x_+) x_+^{\mathrm{T}} P + \sum_{j=1}^{m} \beta_i(x_+) \alpha_j \phi_j(C_j x_+) C_j \\
& + \Bigg[\left\{ 2x_+^{\mathrm{T}} A^{\mathrm{T}} C_i^{\mathrm{T}} + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}} \right\} \{C_i A P^{-1} C_i^{\mathrm{T}} - C_i B \bar{\Theta}_i C P^{-1} C_i^{\mathrm{T}}\} \\
& - \left\{ 2x_+^{\mathrm{T}} C_i^{\mathrm{T}} + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j P^{-1} C_i^{\mathrm{T}} \right\} \{C_i A P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}} - C_i B \bar{\Theta}_i C P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}}\} \Bigg] C_i \\
& + \Bigg[\left\{ 2x_+^{\mathrm{T}} C_i^{\mathrm{T}} + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j P^{-1} C_i^{\mathrm{T}} \right\} \{C_i P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}}\} \\
& - \left\{ 2x_+^{\mathrm{T}} A^{\mathrm{T}} C_i^{\mathrm{T}} + \sum_{j=1}^{m} \alpha_j \phi_j(C_j x_+) C_j P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}} \right\} \{C_i P^{-1} C_i^{\mathrm{T}}\} \Bigg] [C_i A - C_i B \bar{\Theta}_i C]
\end{aligned} \tag{46}$$

and

$$
\begin{aligned}
0 = {} & 2\beta_i(x_-)x_-^{\mathrm{T}}P + \sum_{j=1}^{m} \beta_i(x_-)\alpha_j\phi_j(C_jx_-)C_j \\
& + \Bigg[\Bigg\{2x_-^{\mathrm{T}}A^{\mathrm{T}}C_i^{\mathrm{T}} + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_jP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}\Bigg\}\{C_iAP^{-1}C_i^{\mathrm{T}} - C_iB\underline{\Theta}_iCP^{-1}C_i^{\mathrm{T}}\} \\
& - \Bigg\{2x_-^{\mathrm{T}}C_i^{\mathrm{T}} + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_jP^{-1}C_i^{\mathrm{T}}\Bigg\}\{C_iAP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}} - C_iB\underline{\Theta}_iCP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}\}\Bigg]C_i \\
& + \Bigg[\Bigg\{2x_-^{\mathrm{T}}C_i^{\mathrm{T}} + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_jP^{-1}C_i^{\mathrm{T}}\Bigg\}\{C_iP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}\} \\
& - \Bigg\{2x_-^{\mathrm{T}}A^{\mathrm{T}}C_i^{\mathrm{T}} + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_-)C_jP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}\Bigg\}\{C_iP^{-1}C_i^{\mathrm{T}}\}\Bigg][C_iA - C_iB\underline{\Theta}_iC] \qquad (47)
\end{aligned}
$$

where $\bar{\Theta}_i \triangleq \Theta_i(x_+)$ and $\underline{\Theta}_i \triangleq \Theta_i(x_-)$.

Proof. First, we prove the existence of $\min_{x\in\mathscr{S}_i^+}V(x)$ and $\min_{x\in\mathscr{S}_i^-}V(x)$. Since $\phi(\cdot)$ is a continuous function on $\mathbb{R}^m$ it follows that $\mathscr{S}_i^+$ is a closed subset of $\mathbb{R}^n$. Now, if $\mathscr{S}_i^+$ is bounded the existence of $\min_{x\in\mathscr{S}_i^+}V(x)$ is immediate since $V(x)$ is continuous on $\mathscr{S}_i^+$ and $\mathscr{S}_i^+$ is compact. Next, suppose $\mathscr{S}_i^+$ is unbounded. Since $V(x)$ is radially unbounded, that is, $V(x)\to\infty$ as $\|x\|\to\infty$, it follows that for every $M>0$ there exists $r>0$ such that $V(x)>M$ for all $x\in\mathscr{S}_i^+$ and $\|x\|>r$ where $\|\cdot\|$ is the Euclidean vector norm. Now let $\mathscr{S}_M \triangleq \{x\in\mathscr{S}_i^+ : V(x)\leqslant M\}$ and $\bar{\mathscr{S}}_M \triangleq \{x\in\mathscr{S}_i^+ : V(x)>M\}$ where $M>0$ is chosen such that $\mathscr{S}_M\neq\emptyset$. Next, note that $\mathscr{S}_M\subseteq\{x\in\mathbb{R}^n : \|x\|\leqslant r\}$ and hence $\mathscr{S}_M$ is compact, which proves the existence of $\min_{x\in\mathscr{S}_M}V(x)$. Furthermore, $\min_{x\in\mathscr{S}_M}V(x)\leqslant M\leqslant\inf_{x\in\bar{\mathscr{S}}_M}V(x)$ and hence $\min_{x\in\mathscr{S}_i^+}V(x)=\min_{x\in\mathscr{S}_M}V(x)$. Using similar arguments we can show the existence of $\min_{x\in\mathscr{S}_i^-}V(x)$.

Next, to minimize $V(x)$ subject to $x\in\mathscr{S}_i^+$ form the Lagrangian $\mathscr{L}(x,\lambda_1,\lambda_2)\triangleq V(x)+\lambda_1(C_ix-\bar{y}_i)+\lambda_2(C_iAx-C_iB\phi(Cx))$ where $\lambda_1,\ \lambda_2\in\mathbb{R}$ are Lagrange multipliers. Now, if x_+ solves (44) then

$$0 = \left.\frac{\partial\mathscr{L}}{\partial x}\right|_{x=x_+} = 2x_+^{\mathrm{T}}P + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_j + \lambda_1C_i + \lambda_2(C_iA - C_iB\bar{\Theta}_iC) \qquad (48)$$

Next, forming $(48)P^{-1}C_i^{\mathrm{T}}$ and $(48)P^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}$ yields, respectively,

$$0 = 2\bar{y}_i + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_jP^{-1}C_i^{\mathrm{T}} + \lambda_1C_iP^{-1}C_i^{\mathrm{T}} + \lambda_2(C_iAP^{-1}C_i^{\mathrm{T}} - C_iB\bar{\Theta}_iCP^{-1}C_i^{\mathrm{T}}) \qquad (49)$$

and

$$
\begin{aligned}
0 = {} & 2C_iB\phi(Cx) + \sum_{j=1}^{m}\alpha_j\phi_j(C_jx_+)C_jP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}} + \lambda_1C_iP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}} \\
& + \lambda_2(C_iAP^{-1}A^{\mathrm{T}}C^{\mathrm{T}} - C_iB\bar{\Theta}_iCP^{-1}A^{\mathrm{T}}C_i^{\mathrm{T}}) \qquad (50)
\end{aligned}
$$

which upon solving for λ_1 and λ_2 and substituting into (48) yields (46). Equation (47) follows by carrying out the identical steps with x_+ and $\mathscr{S}_i^+$ replaced by x_- and $\mathscr{S}_i^-$, respectively. □

Next we present the main result of this section for providing a guaranteed subset of the domain of attraction of (1), (2) when the input nonlinearity ϕ is contained in Φ_{m} for a finite range of its argument. For the statement of this result define

$$\mathscr{D}_{\mathrm{A}} \triangleq \{x \in \mathbb{R}^n : V(x) < V_{\mathscr{S}}, \underline{y}_i \leqslant C_i x \leqslant \bar{y}_i, i = 1, \ldots, m\} \tag{51}$$

where $V_{\mathscr{S}} \triangleq \min_{i=1,\ldots,m}\{\min(V_{\mathscr{S}_i^+}, V_{\mathscr{S}_i^-})\}$.

Theorem 3.1

Let $\mathscr{D}_{\mathrm{A}}$ be given by (51). Then $\mathscr{D}_{\mathrm{A}}$ is a subset of the domain of attraction for (1), (2).

Proof. First we show that $\mathscr{D}_{\mathrm{A}}$ is an invariant set for (1), (2). Suppose $V_{\mathscr{S}} < \infty$. In order to show that $\mathscr{D}_{\mathrm{A}}$ is an invariant set for (1), (2) it suffices to show that $C_iAx - C_iB\phi(Cx) < 0$ for all $x \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$ and $C_iAx - C_iB\phi(Cx) > 0$ for all $x \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^-$, $i = 1, \ldots, m$. Since $V(x)$ is a convex function on $\mathbb{R}^n$ it follows that $\mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$ is a convex set and hence connected for all $i = 1, \ldots, m$. Now, suppose that there exists $x \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$ for $i \in \{1, \ldots, m\}$ such that $C_iAx_+ - C_iB\phi(Cx) \geqslant 0$. Then, since by Lemma 3.2, there exists $x_+ \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$ such that $C_iAx_+ - C_iB\phi(Cx_+) < 0$, it follows from continuity that there exists $x^* \in \mathscr{D}_{\mathrm{A}} \cap \Gamma^+$ such that $CA_i{}^* - C_iB\phi(Cx^*) = 0$ and hence $V(x^*) \geqslant V_{\mathscr{S}}$ which is a contradiction. Hence, $C_iAx - C_iB\phi(Cx) < 0$ for all $x \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$, $i = 1, \ldots, m$. Using similar arguments it can be shown that $C_i\mathrm{A}x - C_iB\phi(Cx) > 0$ for all $x \in \mathscr{D}_{\mathrm{A}} \cap \Gamma_i^-$, $i = 1, \ldots, m$. Next, suppose $V_{\mathscr{S}} = \infty$. In this case $\mathscr{D}_{\mathrm{A}} = \mathscr{X}$ and hence in order to show that $\mathscr{D}_{\mathrm{A}}$ is an invariant set for (1), (2) it suffices to show that $C_iAx - C_iB\phi(Cx) < 0$ for all $x \in \Gamma_i^+$ and $C_iAx - C_iB\phi(Cx) < 0$ for all $x \in \Gamma_i^-$, $i = 1, \ldots, m$, which can be shown as above using the fact that there exists $x \in \Gamma_i^+$ such that $C_iAx - C_iB\phi(Cx) < 0$ and $x \in \Gamma_i^-$ such that $C_iAx - C_iB\phi(Cx) > 0$, $i = 1, \ldots, m$. Hence, $\mathscr{D}_{\mathrm{A}}$ is an invariant set for (1), (2). Finally, since $\dot{V}(x) < 0$ for all $x \in \mathscr{D}_{\mathrm{A}}/\{0\} \subseteq \mathscr{X}/\{0\}$ it follows that $x(t) \to 0$ as $t \to \infty$ where $x(t)$ is a solution to (1). □

Remark 3.3

Note that the monotonicity constraint on the nonlinearity $\phi \in \Phi_{\mathrm{m}}$ (i.e., $\phi_i'(y) \geqslant 0, i = 1, \ldots, m$) is required to assure that the Lyapunov function (28) is a strictly convex function which in turn assures the existence of $V_{\Gamma_i^+}$ (respectively, $V_{\Gamma_i^-}$), $V_{\mathscr{S}_i^+}$ (respectively, $V_{\mathscr{S}_i^-}$), and the convexity of $\mathscr{D}_{\mathrm{A}} \cap \Gamma_i^+$ (respectively, $\mathscr{D}_{\mathrm{A}} \cap \Gamma_i^-$). This is in contrast to the single-input/single-output case since the non-convex (integral) part of the Lyapunov function (3) is constant on Γ^+ (respectively, Γ^-) and hence a monotonicity assumption on $\phi \in \Phi_{\mathrm{s}}$ is not required.

Remark 3.4

Theorem 3.1 is a multivariable generalization of the single-input/single-output result given in Theorem 2.1.

As mentioned in the Introduction an important class of nonlinear feedback systems of the form (1), (2) where the input nonlinearities ϕ are contained in $\hat{\Phi}_{\mathrm{m}}$ for a finite or semi-infinite range of their arguments arise in the analysis and synthesis of anti-windup controllers for systems subjected to input saturation[15,16] and multimachine power systems.[13,14] For the aforementioned systems an explicit expression for estimating the domain of attraction relaxes any explicit unverifiable *a priori* assumptions on the magnitude of the control signal for the input saturation problem and allows the computation of transient stability regions for power system problems. For both classes of systems it can be shown that $C_iB = 0$, $i = 1, \ldots, m$.[14,15] In this case considerable simplification can be achieved in Proposition 3.2. Finally, as in Corollary 3.1 taking one (constant) term in (28) for a fixed $i \in \{1, \ldots, m\}$, a lower bound involving a closed-form expression for the minimum Lyapunov values (44) and (45) can be computed.

Corollary 3.2

For $i \in \{1, \ldots, m\}$ let $V_{\mathscr{S}_i^+}$ and $V_{\mathscr{S}_i^-}$ be given by (44) and (45), respectively. If $C_i B = 0$ then

$$V_{\mathscr{S}_i^+} \geqslant \Psi_i(\bar{y}_i) + \alpha_i \mathscr{Y}_i(\bar{y}_i) \tag{52}$$

$$V_{\mathscr{S}_i^-} \geqslant \Psi_i(\underline{y}_i) + \alpha_i \mathscr{Y}_i(\underline{y}_i) \tag{53}$$

where $\Psi_i : \mathbb{R} \to \mathbb{R}$ is given by

$$\Psi_i(z) \triangleq \frac{C_i A P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}} z^2}{(C_i P^{-1} C_i^{\mathrm{T}})(C_i A P^{-1} A^{\mathrm{T}} C_i^{\mathrm{T}}) - (C_i A P^{-1} C_i^{\mathrm{T}})^2}$$

Proof. The proof is similar to that of Proposition 2.1. □

Remark 3.5

For the case $C_i B = 0$, $i = 1, \ldots, m$, the estimate of the domain of attraction predicted by (51) will always be larger than or equal to the domain of attraction predicted by (51) with $V_{\mathscr{S}} = \min_{i=1,\ldots,m}\{\min(\Psi_i(\bar{y}_i) + \alpha_i \mathscr{Y}_i(\bar{y}_i), \Psi_i(\underline{y}_i) + \alpha_i \mathscr{Y}_i(\underline{y}_i))\}$. However, in the latter case considerable numerical simplification is achieved since the lower bounds for (44) and (45) are given by closed-form expressions.

4. ILLUSTRATIVE NUMERICAL EXAMPLES

To illustrate the improvement in estimating the domain of attraction using open Lyapunov surfaces over closed Lyapunov surfaces we consider two examples. For these examples the Lyapunov matrix P appearing in Proposition 3.1, Corollary 3.1 and Theorem 3.1 is computed by forming the linear matrix inequality version of the Kalman–Yakubovich–Popov conditions corresponding to the multivariable Popov criterion[11] and using the LMI toolbox.[18]

Example 4.1

First, we consider the nonlinear system (1), (2) with

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

Using the multivariable Popov criterion with $\alpha_1 = \alpha_2 = 1$ it can be shown that the zero solution of (1), (2) is globally asymptotically stable for $\phi \in \Phi_{\mathrm{m}}$ with $k_1 = k_2 = 8$.

Next, consider the nonlinearities $\phi_i(y_i)$, $i = 1, 2$, given by

$$\phi_i(y_i) = y_i^3, \qquad i = 1, 2$$

In this case the nonlinearities $\phi_i(y_i)$, $i = 1, 2$, satisfy the sector constraint only for the interval $|y_i| \leqslant 2{\cdot}8284$, $i = 1, 2$, and hence global asymptotic stability is not assured. It follows from Corollary 3.1 and Remark 3.1 that a guaranteed subset of the domain of attraction involving closed Lyapunov surfaces is $\mathscr{D}_{\rm cl} = \{x \in \mathbb{R}^n : V(x) \leqslant 59{\cdot}42\}$. Alternatively, since C is a left eigenvector of A and $C_i B = 0$, $i = 1, 2$, it follows that $\mathscr{S}_i^+ = \mathscr{S}_i^- = \emptyset$ and hence Theorem 3.1 yields $\mathscr{D}_{\rm A} = \mathscr{X} = \{x \in \mathbb{R}^n : \|y_i\| \leqslant 2{\cdot}8284,\ i = 1, 2\}$, which corresponds to a considerable improvement over the guaranteed subset of the domain of attraction predicted by Corollary 3.1.

Example 4.2

Here we consider the nonlinear system (1), (2) with

$$A = \begin{bmatrix} -1 & 0{\cdot}01 \\ 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Using the multivariable Popov criterion with $\alpha_1 = \alpha_2 = 1$ it can be shown that the zero solution of (1), (2) is globally asymptotically stable for $\phi \in \Phi_{\rm m}$ with $k_1 = k_2 = 14$.

Next, consider the nonlinearities $\phi_i(y_i)$, $i = 1, 2$, given by

$$\phi_i(y_i) = y_i^5, \qquad i = 1, 2$$

In this case, the nonlinearities $\phi_i(y_i)$, $i = 1, 2$, satisfy the sector constraint only for the interval $|y_i| \leqslant 1{\cdot}9343$, $i = 1, 2$, and hence global asymptotic stability is not assured. Now, it follows from Proposition 3.1 and Remark 3.1 that a guaranteed subset of the domain of attraction involving closed Lyapunov surfaces is $\mathscr{D}_{\rm cl} = \{x \in \mathbb{R}^n : V(x) \leqslant 19{\cdot}869\}$. Alternatively, it follows from Theorem 3.1 that $\mathscr{D}_{\rm A} = \{x \in \mathbb{R}^n : V(x) < 5276{\cdot}4,\ |y_i| \leqslant 1{\cdot}9343,\ i = 1, 2\}$, which, as shown in Figure 1, corresponds to a significant improvement over the guaranteed subset of the domain of attraction predicted by Proposition 3.1.

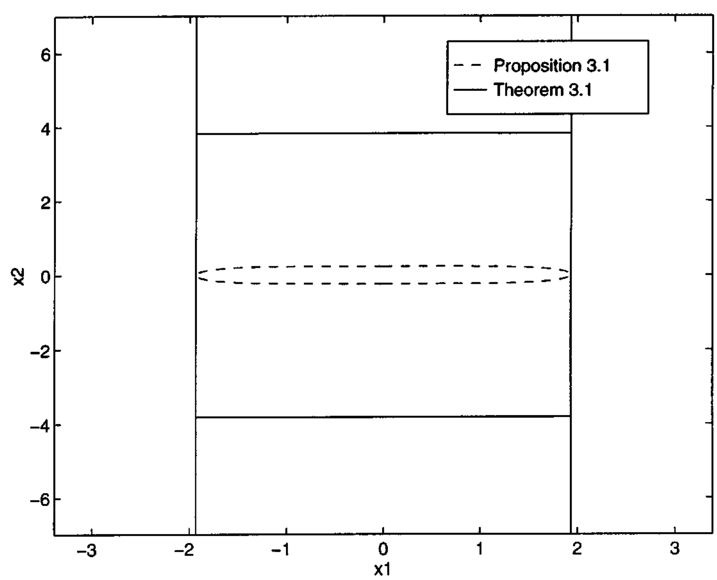

Figure 1. Comparison of estimates of the domain of attraction predicted by Proposition 3.1 and Theorem 3.1.

5. CONCLUSION

Guaranteed domains of attraction for multivariable Luré systems via open Lyapunov surfaces were developed. It was shown that the construction of open Lyapunov surfaces yields a considerable improvement over closed Lyapunov surfaces in estimating the domain of attraction of nonlinear Luré systems. Numerical examples were given to demonstrate the effectiveness of the approach.

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REFERENCES

1. Aizerman, M. A. and F. R. Gantmacher, *Absolute Stability of Regulator Systems*, Holden-Day Inc., San Francisco, 1983.
2. Lefschetz, S., *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.
3. Popov, V. M., 'Absolute stability of nonlinear systems of automatic control', *Automat. Rem. Contr.*, **22**, 628–629 (1967).
4. Popov, V. M., *Hyperstability of Control Systems*, Springer, New York, 1973.
5. Chiang, H.-D. and J. S. Thorp, 'Stability regions of nonlinear dynamical systems: A constructive methodology', *IEEE Trans. Automat. Control*, **34**, 1229–1241 (1989).
6. Khalil, H. K., *Nonlinear Systems*, Macmillan Publishing, New York, 1992.
7. Vannelli, A. and M. Vidyasagar, 'Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems', *Automatica*, **21**, 69–80 (1985).
8. Walker, J. A. and N. H. McClamroch, 'Finite regions of attraction for the problem Luré', *Int. J. Control*, **6**, 331–336 (1967).
9. Weissenberger, S., 'Application of results from the absolute stability problem to the computation of finite stability domains', *IEEE Trans. Automat. Control*, **AC-13**, 124–125 (1968).
10. Willems, J. L., 'The computation of finite stability regions by means of open Lyapunov surfaces', *Int. J. Control*, **10**, 537–544 (1969).
11. Haddad, W. M. and D. S. Bernstein, 'Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle and Popov theorems and their application to robust stability, part I: Continuous-time theory', *Int. J. Robust and Nonlinear Control*, **3**, 313–339 (1993).
12. Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1993.
13. Willems, J. L., 'Optimum Lyapunov functions and transient stability regions for multimachine power systems', *Proc. Inst. Elect. Eng.*, **117**, 573–577 (1970).
14. Willems, J. L. and J. C. Willems, 'The application of Lyapunov methods to the computation of transient stability regions for multimachine power systems', *IEEE Trans. Power Apparatus and Systems*, **PAS-89**, 795–801 (1970).
15. Haddad, W. M. and V. Kapila, 'Fixed-architecture controller synthesis for systems with input–output time-varying nonlinearities', *Int. J. Robust and Nonlinear Control*, to appear.
16. Tyan, F. and D. S. Bernstein, 'Anti-windup compensator synthesis for systems with saturating actuators', *Int. J. Robust and Nonlinear Control*, **5**, 521–537 (1995).
17. Narendra, K. S. and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*, Academic Press, New York, 1973.
18. Gahinet, P. and A. Nemirovskii, *LMI Lab: A Package for Manipulation and Solving LMI's* INRIA, France, 1993.