Non-linear impulsive dynamical systems. Part II: Stability of feedback interconnections and optimality

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In a companion paper (Nonlinear Impulsive Dynamical Systems. Part I: Stability and Dissipativity) Lyapunov and invariant set stability theorems and dissipativity theory were developed for non-linear impulsive dynamical systems. In this paper we build on these results to develop general stability criteria for feedback interconnections of non-linear impulsive systems. In addition, a unified framework for hybrid feedback optimal and inverse optimal control involving a hybrid non-linear-non-quadratic performance functional is developed. It is shown that the hybrid cost functional can be evaluated in closed-form as long as the cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the non-linear closed-loop impulsive system. Furthermore, the Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton–Jacobi–Bellman equation.

1. Introduction

In a companion paper (Haddad et al. 2001) stability and dissipativity theory for non-linear impulsive dynamical systems were developed. Using the concepts of dissipativity and exponential dissipativity for impulsive systems, in this paper we develop feedback interconnection stability results for non-linear impulsive dynamical systems. The feedback system can be impulsive, non-linear, and either dynamic or static. General stability criteria are given for Lyapunov, asymptotic and exponential stability of feedback impulsive systems. In the case of quadratic supply rates involving net system power and input–output energy, these results generalize the positivity and small gain theorems to the case of non-linear impulsive dynamical systems. In particular, we show that if the non-linear impulsive dynamical systems \( G \) and \( G_c \) are dissipative (respectively, exponentially dissipative) with respect to quadratic supply rates corresponding to net system power, or, weighted input and output energy, then the negative feedback interconnection of \( G \) and \( G_c \) is Lyapunov (respectively, asymptotically) stable.

Next, using the stability results developed in the first part of the paper, we consider a hybrid feedback optimal control problem over an infinite horizon involving a hybrid non-linear-non-quadratic performance functional. The performance functional involves a continuous-time cost for addressing performance of the continuous-time system dynamics and a discrete-time cost for addressing performance at the resetting instants. Furthermore, the hybrid cost functional can be evaluated in closed-form as long as the non-linear-non-quadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the non-linear closed-loop impulsive system. This Lyapunov function is shown to be a solution of a steady-state, hybrid Hamilton–Jacobi–Bellman equation and thus guaranteeing both optimality and stability of the feedback controlled impulsive system. The overall framework provides the foundation for extending linear-quadratic feedback control methods to non-linear impulsive dynamical systems.

We note that the optimal control framework for impulsive dynamical systems developed herein is quite different from the quasivariational inequality methods for impulsive and hybrid control developed in the literature (e.g. Barles 1985a, b, Bardi and Dolcetta 1997, Branicky et al. 1998). Specifically, quasivariational methods do not guarantee asymptotic stability via Lyapunov functions and do not necessarily yield feedback controllers. In contrast, the proposed approach provides hybrid feedback controllers guaranteeing closed-loop stability via an underlying Lyapunov function.

The contents of the paper are as follows. Feedback interconnection stability results for non-linear impulsive dynamical systems are given in §2. In §3 we address an optimal control problem with respect to a hybrid non-linear-non-quadratic performance functional for impulsive dynamical systems. To avoid the complexity in solving the hybrid Hamilton–Jacobi–Bellman equation, in §4 we specialize the results of §3 to address an inverse optimal control problem for non-linear affine (in the control) impulsive systems. In §5 we specialize the results of §4 to linear impulsive systems controlled by non-linear controllers that minimize polynomial and multilinear cost functionals. In §6, we apply the results developed in this paper and Part I of this paper to the control of thermoacoustic instabilities in combustion processes. The overall framework demonstrates that hybrid controllers provide an extremely effective way for dissipating energy in combustion systems. Finally, we draw conclusions in §7.
2. Feedback interconnections of dissipative impulsive dynamical systems

In this section we consider stability of feedback interconnections of dissipative impulsive dynamical systems. Specifically, using the notion of dissipative and exponentially dissipative impulsive dynamical systems (Haddad et al. 2001), with appropriate storage functions and supply rates, we construct Lyapunov functions for interconnected impulsive dynamical systems by appropriately combining storage functions for each subsystem. Here, we restrict our attention to input/state-dependent impulsive dynamical systems (Haddad et al. 2001). Analogous results, with the exception of results requiring the impulsive invariance principle (Haddad et al. 2001), hold for time-dependent impulsive dynamical systems. In this paper we use the notation and assumptions established in Part I of this paper (Haddad et al. 2001). We begin by considering the non-linear dynamical system $G$ given by

\[ \dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \]

\[ x(0) = x_0, \quad (x(t), u_c(t)) \notin Z \]

\[ \Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in Z \]

\[ y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t), \quad (x(t), u_c(t)) \notin Z \]

\[ y_d(t) = h_d(x(t)) + J_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in Z \]

where $t \geq 0$, $x(t) \in D \subseteq \mathbb{R}^d$, $D$ is an open set with $0 \in D$, $x(t)$ is $x(t')$, $u_c(t) \in U_c \subseteq \mathbb{R}^{m_c}$, $u_d(t_k) \in U_d \subseteq \mathbb{R}^{m_d}$, $t_k$ denotes the $k^{\text{th}}$ instant of time at which $(x(t), u_c(t))$ intersects $Z$ for a particular trajectory $x(t)$ and input $u_c(t)$, $y_c(t) \in \mathbb{R}^e$, $y_d(t) \in \mathbb{R}^g$, $f_c : D \rightarrow \mathbb{R}^d$ is Lipschitz continuous and satisfies $f_c(0) = 0$, $G_c : D \rightarrow \mathbb{R}^{dx}$, $f_d : D \rightarrow \mathbb{R}^e$ is continuous, $G_d : D \rightarrow \mathbb{R}^{dx}$, $h_c : D \rightarrow \mathbb{R}^e$ and satisfies $h_c(0) = 0$, $J_c : D \rightarrow \mathbb{R}^{dx}$, $h_d : D \rightarrow \mathbb{R}^g$, $J_d : D \rightarrow \mathbb{R}^{dx}$ and \( Z = Z_x \times Z_u \), where $Z_x \subset D$ and $Z_u \subset U_c$, is the resetting set. Furthermore, consider the impulsive non-linear feedback system $G_c$ given by

\[ x_c(t) = f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t), \]

\[ x_c(0) = x_c, \quad (x_c(t), u_c(t)) \not\in Z \]

\[ \Delta x_c(t) = f_d(x_c(t)) + G_d(u_d(t), x_c(t))u_d(t), \quad (x_c(t), u_c(t)) \in Z \]

\[ y_c(t) = h_c(x_c(t)) + J_c(u_c(t), x_c(t))u_c(t), \quad (x_c(t), u_c(t)) \not\in Z \]

\[ y_d(t) = h_d(x_c(t)) + J_d(u_d(t), x_c(t))u_d(t), \quad (x_c(t), u_c(t)) \in Z \]

where $t \geq 0$, $\Delta x_c(t) = x_c(t^+) - x_c(t)$, $x_c(t) \in \mathbb{R}^e$, $u_c(t) \in U_c \subseteq \mathbb{R}^{m_c}$, $u_d(t_k) \in U_d \subseteq \mathbb{R}^{m_d}$, $y_c(t) \in \mathbb{R}^e$, $y_d(t_k) \in \mathbb{R}^g$, $f_c : \mathbb{R}^e \rightarrow \mathbb{R}^e$ is Lipschitz continuous and satisfies $f_c(0) = 0$, $G_c : \mathbb{R}^{dxm_c} \times \mathbb{R}^e \rightarrow \mathbb{R}^{dx}$ is continuous, $G_d : \mathbb{R}^{dxm_d} \times \mathbb{R}^g \rightarrow \mathbb{R}^{dx}$ and satisfies $h_c(0) = 0$, $J_c : \mathbb{R}^{dxm_c} \times \mathbb{R}^e \rightarrow \mathbb{R}^e$, $h_d : \mathbb{R}^{dxm_d} \times \mathbb{R}^g \rightarrow \mathbb{R}^g$, $m_c = l_c$, $m_d = l_d$, $l_c = m_c$, $l_d = m_d$ and $Z = Z_x \times Z_u$. Furthermore, even though the input–output pairs of the feedback interconnection shown on Figure 1 consist of two-vector inputs/two-vector outputs, at any given instant of time a single-vector input/single-vector output is active. Here, we assume that the negative feedback interconnection of $G$ and $G_c$ is well posed; that is

$$\det [I_{m_c} + J_c(y_c, x_c)J_c(x)] \neq 0$$

and

$$\det [I_{m_d} + J_d(y_d, x_c)J_d(x)] \neq 0$$

for all $y_c, y_d, x$ and $x_c$. The following results give sufficient conditions for Lyapunov, asymptotic and exponential stability of the negative feedback interconnection given by Figure 1. In contrast to Haddad et al. (2001), in this paper we represent the resetting time $\tau(x_0)$ for a state-dependent impulsive dynamical system by $t_k$. This minor abuse of notation considerably simplifies the presentation. Furthermore, for the results of this section we define the closed-loop resetting set

$$\hat{Z} = \hat{Z}_x \times \hat{Z}_u \cup \{(x, x_c) : (\mathcal{F}_c(x), \mathcal{F}_c(x_c)) \in Z_{cu} \times Z_{uc}\}$$

where $\mathcal{F}_c(\cdot)$ and $\mathcal{F}_c(\cdot)$ are functions of $x$ and $x_c$ arising from the algebraic loops due to $u_{ce}$ and $u_t$, respectively. Note that since the feedback interconnection of $G$ and $G_c$ is well posed, it follows that $\hat{Z}$ is well defined and depends on the closed-loop states $\hat{x} = [x^T \hat{x}_c^T]^T$. In the special case where $J_c(x) \equiv 0$ and $J_c(u_{ce}(x_c) \equiv 0$ it follows that $\hat{Z} = \hat{Z} = \hat{Z}_x \times \hat{Z}_u \cup \{(x, x_c) : (h_c(x), h_{uc}(x_c)) \in Z_{cu} \times Z_{uc}\}$. Furthermore, note that in the case where
\( Z = \emptyset \); that is, the plant is a continuous-time dynamical system without any resetting, it follows that \( \hat{Z}_c = \hat{Z}_{ce} \cup \{(x, x_c); h_c(x) \in \hat{Z}_{ce}\} \) and hence knowledge of \( x_c \) and \( y_c \) is sufficient to determine whether or not the closed-loop state vector is in the set \( \hat{Z}_c \). For the statement of the results of this section let \( T_{x_o, u_o} \) denote the set of resetting times of \( G \), let \( T_{x_o, u_o, t} \) denote the complement of \( T_{x_o, u_o, t}^c \); that is, \([0, \infty) \setminus T_{x_o, u_o, t}^c \); and let \( \hat{Z}_{c} \) denote the complement of \( T_{x_o, u_o, t}^c \); that is, \([0, \infty) \setminus T_{x_o, u_o, t}^c \).

**Theorem 1:** Consider the closed-loop system consisting of the non-linear impulsive dynamical systems \( G \) given by (1)–(4) and \( G_c \) given by (5)–(8) with input-output pairs \( (u_c, u_d; y_c, y_d) \) and \( (u_c, u_{dc}; y_c, y_{dc}) \), respectively and with \( (u_{cc}, u_{dc}) = (y_c, y_d) \) and \( (y_{cc}, y_{dc}) = (-u_c, -u_d) \). Assume \( G \) and \( G_c \) are zero-state observable and dissipative with respect to the supply rates \( (r_c(u_c, y_c), r_d(u_d, y_d)) \) and \( (r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) \) and with continuously differentiable positive definite, radially unbounded storage functions \( V_c(i) \) and \( V_{sc}(i) \), respectively, such that \( V_c(0) = 0 \), \( V_{sc}(0) = 0 \). Furthermore, assume there exists a scalar \( \sigma > 0 \) such that \( r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) \leq 0 \) and \( r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) \leq 0 \).

Then the following statements hold:

(i) The negative feedback interconnection of \( G \) and \( G_c \) is Lyapunov stable.

(ii) If \( G \) is strongly zero-state observable, \( G_c \) is exponentially dissipative with respect to the supply rate \( (r_c(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) \) and \( \text{rank } [G_{cc}(u_{cc}, 0)] = m_{cc}, u_{cc} \in \mathcal{U}_{cc} \), then the negative feedback interconnection of \( G \) and \( G_c \) is globally asymptotically stable.

(iii) If \( G \) and \( G_c \) are exponentially dissipative with respect to supply rates \( (r_c(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) \) and \( (r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) \), respectively and \( V_c(\cdot) \) and \( V_{sc}(\cdot) \) are such that there exist constants \( \alpha, \alpha_c, \beta, \beta_c > 0 \) such that

\[
\alpha \|x\|^2 \leq V_c(x) \leq \beta \|x\|^2, \quad x \in \mathbb{R}^n
\]

where \( \alpha \|x\|^2 \leq V_{sc}(x_c) \leq \beta \|x\|^2, \quad x_c \in \mathbb{R}^{n_c} \)

then the negative feedback interconnection of \( G \) and \( G_c \) is globally exponentially stable.

**Proof:** Let \( T_{x_o, u_o, t} = T_{x_o, u_o, t}^c \cup T_{x_o, u_o, t}^c \) and \( t_k \in T_{x_o, u_o, t}, k \in \mathcal{N} \). First, note that it follows from Assumptions A1 and A2 of Haddad et al. (2001) that the resetting times \( \tau_k(\tilde{x}_0) \) for the feedback system are well defined and distinct for every closed-loop trajectory. (i) Consider the Lyapunov function candidate \( \hat{V}(x_c, x_c) = V_c(x) + \sigma V_{sc}(x_c) \). Now, the corresponding Lyapunov derivative of \( \hat{V}(x_c, x_c) \) along the state trajectories \( (x(t), x_c(t)), t \in (t_k, t_{k+1}) \), is given by
contains no solution other than the trivial solution \((x(t), x_c(t)) \equiv (0, 0)\). Hence, it follows from Theorem 4 of Haddad et al. (2001) that \(x(t), x_c(t) \rightarrow M = \{(0, 0)\}\) as \(t \rightarrow \infty\). Now, global asymptotic stability of the negative feedback interconnection of \(G\) and \(G_c\) follows from the fact that \(V_\epsilon(\cdot)\) and \(V_{sc}(\cdot)\) are, by assumption, radially unbounded. (iii) Finally, if \(G\) and \(G_c\) are exponentially dissipative and (9) and (10) hold, it follows that

\[
\dot{V}(x(t), x_c(t)) = \dot{V}_\epsilon(x(t)) + \sigma \dot{V}_{sc}(x_c(t))
\]

\[
\leq -\epsilon_v V_\epsilon(x(t)) - \epsilon_{sc} V_{sc}(x_c(t)) + r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_c(t), y_{cc}(t))
\]

\[
\leq -\min\{\epsilon_v, \epsilon_{sc}\} V(x(t), x_c(t)),
\]

\[
(x(t), x_c(t)) \notin \bar{Z}_\epsilon, \quad t_k < t < t_{k+1}
\]  

(15)

and \(\Delta V(x(t_k), x_c(t_k)), (x(t_k), x_c(t_k)) \in \bar{Z}_\epsilon, \ k \in \mathcal{N}\), satisfies (14). Now, Theorem 2 of Haddad et al. (2001) implies that the negative feedback interconnection of \(G\) and \(G_c\) is globally exponentially stable. 

The next result presents Lyapunov, asymptotic and exponential stability of dissipative feedback systems with quadratic rates.

**Theorem 2:** Let \(Q_c \in \mathbb{S}^l, S_c \in \mathbb{R}^{l \times m_c}, R_c \in \mathbb{S}^m, Q_d \in \mathbb{S}^l, S_d \in \mathbb{R}^{l \times m_d}, R_d \in \mathbb{S}^m, S_{sc} \in \mathbb{R}^{l \times m_{sc}}, R_{sc} \in \mathbb{S}^l, S_{cc} \in \mathbb{R}^{l \times m_{cc}}, R_{cc} \in \mathbb{S}^l, Q_{dc} \in \mathbb{S}^l, S_{dc} \in \mathbb{R}^{l \times m_{dc}}\) and \(R_{dc} \in \mathbb{S}^l\). Consider the closed-loop system consisting of the nonlinear impulsive dynamical systems \(\dot{x} \in \mathbb{R}^n, \dot{y} \in \mathbb{R}^m\) given by (1)–(4) and \(G_c\) given by (5)–(8) and assume \(G\) and \(G_c\) are zero-state observable. Furthermore, assume \(G\) is dissipative with respect to the quadratic supply rate

\[
(r_c(u_c(t), y_c(t)), r_d(u_d(t), y_d(t))) = (y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c, y_d^T Q_d y_d + 2y_d^T S_d u_d + u_d^T R_d u_d)
\]

and has a continuously differentiable, radially unbounded storage function \(V_\epsilon(\cdot)\) and \(G_c\) is dissipative with respect to the quadratic supply rate

\[
(r_{rc}(u_{cc}(t), y_{cc}(t)), r_{dc}(u_{dc}(t), y_{dc}(t))) = (y_{cc}^T Q_{cc} y_{cc} + 2y_{cc}^T S_{cc} u_{cc} + u_{cc}^T R_{cc} u_{cc}, y_{dc}^T Q_{dc} y_{dc} + 2y_{dc}^T S_{dc} u_{dc} + u_{dc}^T R_{dc} u_{dc})
\]

and has a continuously differentiable, radially unbounded storage function \(V_{sc}(\cdot)\). Finally, assume there exists a scalar \(\sigma > 0\) such that

\[
\dot{Q}_c = \begin{bmatrix}
Q_c + \sigma R_{cc} & -S_c + \sigma S_{cc}^T \\
-S_c^T + \sigma S_{cc} & R_c + \sigma Q_{cc}
\end{bmatrix} \leq 0
\]

\[
\dot{Q}_d = \begin{bmatrix}
Q_d + \sigma R_{dc} & -S_d + \sigma S_{dc}^T \\
-S_d^T + \sigma S_{dc} & R_d + \sigma Q_{dc}
\end{bmatrix} \leq 0
\]

Then the following statements hold:

(i) The negative feedback interconnection of \(G\) and \(G_c\) is Lyapunov stable.

(ii) If \(G\) is strongly zero-state observable, \(G_c\) is exponentially dissipative with respect to the supply rate \((r_c(u_{cc}, y_{cc}), r_d(u_{dc}, y_{dc}))\) and \(\text{rank}[G_{cc}(u_{cc}, 0)] = m_{cc}, u_{cc} \in U_{cc},\) then the negative feedback interconnection of \(G\) and \(G_c\) is globally asymptotically stable.

(iii) If \(G\) and \(G_c\) are exponentially dissipative with respect to supply rates \((r_c(u_{cc}, y_{cc}), r_d(u_{dc}, y_{dc}))\) and \(\text{rank}[G_{cc}(u_{cc}, 0)] = m_{cc}, u_{cc} \in U_{cc},\) there exist constants \(\alpha, \beta, \gamma > 0\) such that (9) and (10) hold, then the negative feedback interconnection of \(G\) and \(G_c\) is globally exponentially stable.

(iv) If \(\dot{Q}_c < 0\) and \(\dot{Q}_d < 0\), then the negative feedback interconnection of \(G\) and \(G_c\) is globally asymptotically stable.

**Proof:** (i)–(iii) are a direct consequence of Theorem 2 by noting

\[
r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) = \begin{bmatrix} y_c \\ y_{cc} \end{bmatrix}^T \begin{bmatrix} Q_c & r_c \\ r_{cc} & Q_{cc} \end{bmatrix} \begin{bmatrix} y_c \\ y_{cc} \end{bmatrix}
\]

\[
r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) = \begin{bmatrix} y_d \\ y_{dc} \end{bmatrix}^T \begin{bmatrix} Q_d & r_d \\ r_{dc} & Q_{dc} \end{bmatrix} \begin{bmatrix} y_d \\ y_{dc} \end{bmatrix}
\]

and hence \(r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) \leq 0\) and \(r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) \leq 0\). To show (iv) consider the Lyapunov function candidate \(V(x, x_c) = V_\epsilon(x) + \sigma V_{sc}(x_c)\). Noting that \(u_{cc} = y_{cc}\) and \(y_{cc} = -u_{cc}\), it follows that the corresponding Lyapunov derivative satisfies

\[
\dot{V}(x(t), x_c(t)) = \dot{V}_\epsilon(x(t)) + \sigma \dot{V}_{sc}(x_c(t))
\]

\[
\leq r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t))
\]

\[
= y_c^T Q_c y_c(t) + 2y_c^T S_c u_c(t) + u_c^T R_c u_c(t)
\]

\[
+ \sigma y_{cc}^T Q_{cc} y_{cc}(t) + 2y_{cc}^T S_{cc} u_{cc}(t) + u_{cc}^T R_{cc} u_{cc}(t)
\]

\[
= \begin{bmatrix} y_c(t) \\ y_{cc}(t) \end{bmatrix}^T \begin{bmatrix} Q_c & r_c \\ r_{cc} & Q_{cc} \end{bmatrix} \begin{bmatrix} y_c(t) \\ y_{cc}(t) \end{bmatrix}
\]

\[
\leq 0, \quad (x(t), x_c(t)) \notin \bar{Z}_\epsilon, \quad t_k < t < t_{k+1}
\]  

(19)
and, similarly, the Lyapunov difference satisfies
\[
\Delta V(x(t_k), x_c(t_k)) = \begin{bmatrix} y_a(t_k) \\ y_{dc}(t_k) \end{bmatrix}^T \tilde{Q}_d \begin{bmatrix} y_a(t_k) \\ y_{dc}(t_k) \end{bmatrix} 
\leq 0, \quad (x(t_k), x_c(t_k)) \in \tilde{Z}_x, \quad k \in \mathcal{N}^*
\]
which implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable. Next, let
\[
\mathcal{R} = \{(x, x_c) \in \mathbb{R}^q \times \mathbb{R}^{q_c} : (x, x_c) \not\in \tilde{Z}_x, \quad \dot{V}(x, x_c) = 0\}
\cup \{(x, x_c) \in \mathbb{R}^q \times \mathbb{R}^{q_c} : (x, x_c) \in \tilde{Z}_x, \quad \Delta V(x, x_c) = 0\}
\]
where $\dot{V}(x, x_c)$ and $\Delta V(x, x_c)$ denote the total derivative and difference of $V(x, x_c)$ of the closed-loop system for all $(x, x_c) \not\in \tilde{Z}_x$ and $(x, x_c) \in \tilde{Z}_x$, respectively. Now, note that $\dot{V}(x, x_c) = 0$ for all $(x, x_c) \in \mathbb{R}^q \times \mathbb{R}^{q_c} \setminus \tilde{Z}_x$ if and only if $(y_c, y_{cc}) = (0, 0)$ and $\Delta V(x, x_c) = 0$ for all $(x, x_c) \in \tilde{Z}_x$ if and only if $(y_d, y_{dc}) = (0, 0)$. Since $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable it follows that
\[
\mathcal{R} = \{(x, x_c) \in \mathbb{R}^q \times \mathbb{R}^{q_c} : (x, x_c) \not\in \tilde{Z}_x, (h_d(x), h_{cc}(x)) = (0, 0)\}
\]
which contains no solution other than the trivial solution $(x(t), x_c(t)) \equiv (0, 0)$. Hence, it follows from Theorem 4 of Haddad et al. (2001) that $(x(t), x_c(t)) \rightarrow \mathcal{M} = \{(0, 0)\}$ as $t \rightarrow \infty$. Finally, global asymptotic stability follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded and hence $V(x, x_c) \rightarrow \infty$ as $\|x(x, x_c)\| \rightarrow \infty$.

The following result generalizes the classical positivity and small gain theorems to the case of impulsive dynamical systems. For this result note that if a non-linear dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c(u, y_c), r_d(u, y_d)) = (2u^2, 2u_d^2, 2y_d^2, 2y_{dc}^2)$, then, with $(\kappa_c(y_c), \kappa_d(y_d)) = (-k_c y_c, -k_{dc} y_d)$, where $k_c, k_{dc} > 0$, it follows that $(r_c(u, y_c), r_d(u, y_d)) = (-k_c y_c, -k_{dc} y_d) < (0, 0)$, $(y_c, y_d) \neq (0, 0)$. Alternatively, if a non-linear dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c(u, y_c), r_d(u, y_d)) = (\gamma_c^2 u^2, \gamma_{dc}^2 u_d^2, \gamma_d^2 y_d^2)$, where $\gamma_c, \gamma_d > 0$, then, with $(\kappa_c(y_c), \kappa_d(y_d)) = (0, 0)$, it follows that $(r_c(u, y_c), r_d(u, y_d)) = (-\gamma_c y_c, -\gamma_{dc} y_d) < (0, 0)$, $(y_c, y_d) \neq (0, 0)$. Hence, if $\mathcal{G}$ is zero-state observable it follows from Theorem 7 of Haddad et al. (2001) that all storage functions of $\mathcal{G}$ are positive definite.

**Corollary 1:** Consider the closed-loop system consisting of the non-linear impulsive dynamical systems $\mathcal{G}$ given by (1)–(4) and $\mathcal{G}_c$ given by (5)–(8). Assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable. Then the following statements hold:

1. If $\mathcal{G}$ is passive and strongly zero-state observable, $\mathcal{G}_c$ is exponentially passive and $\text{rank} \{\mathcal{G}_{cc}(u_{cc}, 0) = m_{cc}, u_{cc} \in \mathcal{U}_{cc}\}$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable.

2. If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially passive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (9) and (10) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is exponentially stable.

3. **Optimal control for impulsive dynamical systems**

In this section we consider an optimal control problem for non-linear impulsive dynamical systems involving a notion of optimality with respect to a hybrid non-linear-non-quadratic performance functional. Specifically, we consider the following impulsive optimal control problem.

**Impulsive optimal control problem:** Consider the non-linear impulsive controlled system given by
\[ \dot{x}(t) = F_c(x(t), u_c(t), t), \quad x(t_0) = x_0, \] 
\[ x(t_f) = x_f, \quad u_c(t) \in U_c, \quad (t, x(t)) \notin S_x \]  
(21)

\[ \Delta x(t) = F_d(x(t), u_d(t), t), \quad u_d(t) \in U_d, \quad (t, x(t)) \in S_x \]  
(22)

where \( t \geq 0, x(t) \in \mathcal{D} \subseteq \mathbb{R}^n \) is the state vector, \( \mathcal{D} \) is an open set with \( 0 \in \mathcal{D}, \quad (u_c(t), u_d(t)) \in U_c \times U_d \subseteq \mathbb{R}^n \times \mathbb{R}^m \), \( t \in [t_0, t_f], \quad k \in \mathcal{N}_{[t_0, t_f]} \) is the hybrid control input, \( x(t_0) = x_0 \) is given, \( x(t_f) = x_f \) is fixed, \( F_c : \mathcal{D} \times U_c \times \mathbb{R} \to \mathbb{R}^n \) is Lipschitz continuous and satisfies \( F_c(0, 0, 0) = 0 \), \( F_d : \mathcal{S}_x \times U_d \to \mathbb{R}^n \) is continuous and \( \mathcal{S}_x \subseteq [0, \infty) \times \mathcal{D} \). Then determine the control inputs \( (u_c(t), u_d(t_k)) \in U_c \times U_d, \quad t \in [t_0, t_f], \quad k \in \mathcal{N}_{[t_0, t_f]} \), such that the hybrid performance functional

\[ J(x_0, u_c(\cdot), u_d(\cdot), t_0) = \int_{t_0}^{t_f} L_c(x(t), u_c(t), t) \, dt \]  
\[ + \sum_{k \in \mathcal{N}_{[t_0, t_f]}} L_d(x(t_k), u_d(t_k), t_k) \]  
(23)

is minimized, where \( L_c : \mathcal{D} \times U_c \times \mathbb{R} \to \mathbb{R} \) and \( L_d : \mathcal{S}_x \times U_d \to \mathbb{R} \) are given.

Next, we present a hybrid version of Bellman’s principle of optimality which provides necessary and sufficient conditions, with a given hybrid control \( (u_c(t), u_d(t_k)) \in U_c \times U_d, \quad t \geq t_0, \quad k \in \mathcal{N}_{[t_0, t_f]} \), for minimizing the performance functional (23).

**Lemma 1:** Let \( (u_c(t), u_d(t_k)) \in U_c \times U_d, \quad t \in [t_0, t_f], \quad k \in \mathcal{N}_{[t_0, t_f]} \), be an optimal hybrid control that generates the trajectory \( x(t), \quad t \in [t_0, t_f] \), with \( x(t_0) = x_0 \). Then the trajectory \( x(\cdot) \) from \( (t_0, x_0) \) to \( (t_f, x_f) \) is optimal if and only if for all \( t' \geq t_0, \quad t'' \leq t_f \) and \( u_c(t) \), \( t \in [t', t''] \), \( u_d(t_k), \quad k \in \mathcal{N}_{[t', t'']} \), the portion of the trajectory \( x(\cdot) \) going from \( (t', x(t')) \) to \( (t'', x(t'')) \) optimizes the same cost functional over \( [t', t''] \), where \( x(t') = x_1 \) is a point on the optimal trajectory generated by \( (u_c(t), u_d(t_k)), \quad t \in [t_0, t'], \quad k \in \mathcal{N}_{[t_0, t']} \).

**Proof:** Let \( u_c(t) \), \( t \in [t_0, t_f], \quad u_d(t_k), \quad k \in \mathcal{N}_{[t_0, t_f]} \), solve the impulsive optimal control problem and let \( x(t), \quad t \in [t_0, t_f] \), be the solution to (21) and (22). Next, ad absurdum, suppose there exist \( t' \geq t_0, \quad t'' \leq t_f \) and \( u_c(t), \quad t \in [t', t''] \), \( u_d(t_k), \quad k \in \mathcal{N}_{[t', t'']} \), such that

\[ \int_{t'}^{t''} L_c(x(t), u_c(t), t) \, dt + \sum_{k \in \mathcal{N}_{[t', t'']}} L_d(x(t_k), u_d(t_k), t_k) \]  
\[ < \int_{t'}^{t''} L_c(x(t), u_c(t), t) \, dt + \sum_{k \in \mathcal{N}_{[t', t'']}} L_d(x(t_k), u_d(t_k), t_k) \]  
(24)

which is a contradiction.
Conversely, if \((u_c(t), u_d(t))\) minimizes \(J^*(\cdot, \cdot)\) over \([t', t'']\) and \(k \in \mathcal{N}_{[t', r']}\) for all \(t' \geq t_0\) and \(t'' \leq t_f\), then it minimizes \(J^*(\cdot, \cdot)\) over \([t_0, t_f]\).

Next, let \((u_c^k(t), u_d^k(t))\), \(t \in [t_0, t_f]\), \(k \in \mathcal{N}_{[t_0, t_f]}\), solve the impulsive optimal control problem and define the optimal cost \(J^*(x_0, t_0) = J^*(x_0, u_c^k(\cdot), u_d^k(\cdot), t_0)\). Furthermore, define, for \(p: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(q: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\), the Hamiltonians

\[
H_c(x, u_c, p, t) = L_c(x, u_c, t) + p^T F_c(x, u_c, t)
\]

and

\[
H_d(x, u_d, q(x, t), t) = L_d(x, u_d, t) + q(x + F_d(x, u_d, t), t) - q(x, t)
\]

**Theorem 3:** Let \(J^*(x, t)\) denote the minimal cost for the impulsive optimal control problem with \(x_0 = x\) and \(t_0 = t\) and assume that \(J^*(\cdot, \cdot)\) is continuously differentiable in \(x\). Then

\[
0 = \frac{\partial J^*(x(t), t)}{\partial t} + \min_{u_c(\cdot) \in \mathcal{U}_c} H_c(x(t), u_c(t), p(x(t), t), t),
\]

\((t, x(t)) \not\in S_x\) \hspace{1cm} (25)

\[
0 = \min_{u_d(\cdot) \in \mathcal{U}_d} H_d(x(t), u_d(t), q(x(t), t), t), \quad (t, x(t)) \in S_x
\]

where \(p(x(t), t) = \left(\frac{\partial J^*(x(t), t)}{\partial x}\right)^T\) and \(q(x(t), t) = J^*(x(t), t)\). Furthermore, if \((u_c^k(\cdot), u_d^k(\cdot))\) solves the impulsive optimal control problem, then

\[
0 = \frac{\partial J^*(x(t), t)}{\partial t} + H_c(x(t), u_c^k(t), p(x(t), t), t),
\]

\((t, x(t)) \not\in S_x\) \hspace{1cm} (26)

\[
0 = H_d(x(t), u_d^k(t), q(x(t), t), t), \quad (t, x(t)) \in S_x
\]

**Proof:** Let \((t, x(t)) \not\in S_x\). It follows from Lemma 1 that for small enough \(\varepsilon > 0\) and \(t' \in [t, t + \varepsilon]\)

\[
J^*(x(t), t) = \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{U}_c \times \mathcal{U}_d} \left[ \int_{t'}^t L_c(x(s), u_c(s), s) \, ds + \sum_{k \in \mathcal{N}_{[t', r']}} L_d(x(t_k), u_d(t_k), t_k) \right]
\]

\[
= \min_{u_c(\cdot) \in \mathcal{U}_c} \int_{t'}^t L_c(x(s), u_c(s), s) \, ds + \min_{u_d(\cdot) \in \mathcal{U}_d} \left[ \int_{t'}^t L_c(x(s), u_c(s), s) \, ds + J^*(x(t'), t') \right]
\]

or, equivalently

\[
0 = \min_{u_c(\cdot) \in \mathcal{U}_c} \left[ \frac{1}{t' - t} \int_{t'}^t J^*(x(t'), t') \, dt' \right]
\]

\[
- J^*(x(t), t) + \frac{1}{t' - t} \int_{t'}^t L_c(x(s), u_c(s), s) \, ds
\]

Letting \(t' \rightarrow t\) yields

\[
0 = \min_{u_c(\cdot) \in \mathcal{U}_c} \frac{dJ^*(x(t), t)}{dt} + L_c(x(t), u_c(t), t)
\]

Now, (25) and (27) follow by noting that

\[
\frac{dJ^*(x(t), t)}{dt} = \frac{\partial J^*(x(t), t)}{\partial t} + \frac{\partial J^*(x(t), t)}{\partial x} F_c(x(t), u_c(t), t)
\]

Next, let \((t, x(t)) \in S_x\). It follows from Lemma 1 that

\[
J^*(x(t), t) = \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{U}_c \times \mathcal{U}_d} \left[ L_d(x(t), u_d(t), t) + \int_{t'}^t L_c(x(s), u_c(t), t) \, ds + \sum_{k \in \mathcal{N}_{[t', r']}} L_d(x(t_k), u_d(t_k), t_k) \right]
\]

\[
= \min_{u_c(\cdot) \in \mathcal{U}_c} \int_{t'}^t L_c(x(s), u_c(s), s) \, ds + \min_{u_d(\cdot) \in \mathcal{U}_d} \left[ \int_{t'}^t L_c(x(s), u_c(t), t) \, ds + J^*(x(t'), t') \right]
\]

which implies (26) and (28).

Next, we provide a converse result to Theorem 3.

**Theorem 4:** Suppose there exists a continuously differentiable function \(V: \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}\) and an optimal control \((u_c^k(\cdot), u_d^k(\cdot))\) such that \(V(x(t_f), t_f) = 0\)

\[
0 = \frac{\partial V(x, t)}{\partial t} + H_c \left( x, u_c^k(t), \frac{\partial V(x, t)}{\partial x}, t \right), \quad (t, x) \not\in S_x
\]

\[
0 = H_d(x, u_d^k(t), V(x, t), t), \quad (t, x) \in S_x
\]

\[
H_c \left( x, u_c^k(t), \frac{\partial V(x, t)}{\partial x}, t \right) \leq H_c \left( x, u_c(t), \frac{\partial V(x, t)}{\partial x}, t \right), \quad (t, x) \not\in S_x, \quad u_c(t) \in \mathcal{U}_c
\]

\[
H_d(x, u_d^k(t), V(x, t), t) \leq H_d(x, u_d(t), V(x, t), t), \quad (t, x) \in S_x, \quad u_d(t) \in \mathcal{U}_d
\]

Then \((u_c^k(\cdot), u_d^k(\cdot))\) solves the impulsive optimal control problem; that is
Next, for all $t \geq t_0$, satisfy (21) and (22) and, for all $(t, x(t)) \notin S_x$, define
\[ V(x(t), t) = \frac{\partial V(x(t), t)}{\partial t} + \frac{\partial V(x(t), t)}{\partial x} F(x(t), u_c(t), t) \]
Then, with $u_c(t) = u_c^+(t)$, it follows from (29) that
\[ 0 = V(x(t), t) + L_c(x(t), u_c^+(t), t), \quad (t, x(t)) \notin S_x \tag{35} \]
Furthermore, it follows from (30) that
\[ 0 = V(x(t), t) + F_d(x(t), u_d^+(t), t) - V(x(t), t) + L_d(x(t), u_d^+(t), t), \quad (t, x(t)) \in S_x \tag{36} \]
Now, noting that $V(x(t_f), t_f) = 0$, it follows from (35) and (36) that
\[
J^*(x_0, t_0) = J(x_0, u_c^+(\cdot), u_d^+(\cdot), t_0)
= \int_{t_0}^{t_f} L_c(x(t), u_c^+(t), t) \, dt + \sum_{k \in N_{[0, \infty)}} L_d(x(t_k), u_d^+(t_k), t_k)
= V(x_0, t_0)
\]
Next, for all $(u_c^-(\cdot), u_d^-(\cdot)) \in \tilde{U}_c \times \tilde{U}_d$ it follows from (29)–(32) that
\[
J(x_0, u_c^-(\cdot), u_d^-(\cdot), t_0) = \int_{t_0}^{t_f} L_c(x(t), u_c^-(t), t) \, dt
+ \sum_{k \in N_{[0, \infty)}} L_d(x(t_k), u_d^-(t_k), t_k)
= \int_{t_0}^{t_f} \left[ -V(x(t), t) + \frac{\partial V(x(t), t)}{\partial t} \right] \, dt
+ H_c(x(t), u_c(t), \frac{\partial V(x(t), t)}{\partial x}, t) \]}
\[ + \sum_{k \in N_{[0, \infty)}} \left[ V(x(t_k), t_k) - V(x(t_k)) \right] + F_d(x(t), u_d(t_k), t_k)
+ H_d(x(t_k), u_d(t_k), V(x(t_k), t_k), t_k) \]
which completes the proof. \qed

Next, we use Theorem 4 to characterize optimal hybrid feedback controllers for non-linear impulsive dynamical systems. In order to obtain time-invariant controllers, we restrict our attention to state-dependent impulsive dynamical systems with non-Zeno solutions and optimality notions over the infinite horizon with an infinite number of resetting times. Hence, the impulsive optimal control problem becomes
\[ \dot{x}(t) = F_c(x(t), u_c(t)), \quad x(0) = x_0, \quad x(t) \notin Z_x \tag{37} \]
\[ \Delta x(t) = F_d(x(t), u_d(t)), \quad x(t) \in Z_x \tag{38} \]
where $Z_x \subset D$ and $u_c(\cdot)$ and $u_d(\cdot)$ are restricted to the class of admissible hybrid controls consisting of measurable functions such that $(u_c(t), u_d(t)) \in \mathcal{U}_c \times \mathcal{U}_d$ for all $t \geq 0$ and $k \in N_{[0, \infty)}$, where the constraint set $\mathcal{U}_c \times \mathcal{U}_d$ is given with $(0, 0) \in \mathcal{U}_c \times \mathcal{U}_d$. To address the optimal nonlinear hybrid feedback control problem let $\phi_c : D \to \mathcal{U}_c$ be such that $\phi_c(0) = 0$ and let $\phi_d : Z_x \to \mathcal{U}_d$. If $(u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))$, where $x(t), t \geq 0$, satisfies (21) and (22), then $(u_c(\cdot), u_d(\cdot))$ is a hybrid feedback control. Given the hybrid feedback control $(u_c(t), u_d(t)) = (\phi_c(x(t)), \phi_d(x(t)))$, the closed-loop state-dependent impulsive dynamical system has the form
\[ \dot{x}(t) = F_c(x(t), \phi_c(x(t))), \quad x(0) = x_0, \quad x(t) \notin Z_x \tag{39} \]
\[ \Delta x(t) = F_d(x(t), \phi_d(x(t))), \quad x(t) \in Z_x \tag{40} \]
Now, we present the main theorem for characterizing hybrid feedback controllers that guarantee closed-loop stability and minimize a hybrid non-linear-non-quadratic performance functional over the infinite horizon. For the statement of this result, recall that with $S_x = [0, \infty) \times Z_x$ it follows from Assumptions A1 and A2 of Haddad et al. (2001) that the resetting times $t_k(= \tau_k(x_0))$ are well defined and distinct for every tra-
jectory of (39) and (40). Furthermore, define the set of regulation hybrid controllers by
\[ C(x_0) = \{ (u_c(\cdot), u_d(\cdot)) \} \]
where \(H\) given by (37) and (38) satisfies \(x(t) \to 0\) as \(t \to \infty\)\)

**Theorem 5:** Consider the non-linear controlled impulsive system (37) and (38) with hybrid performance functional
\[ J(x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty L_c(x(t), u(t)) \, dt + \sum_{k \in N_{0,\infty}} L_d(x(t_k), u_d(t_k)) \]  
where \((u_c(\cdot), u_d(\cdot))\) is an admissible hybrid control. Assume there exists a continuously differentiable function \(V : D \to \mathbb{R}\) and a hybrid control law \(\phi_c : D \to U_c\) and \(\phi_d : Z \to U_d\) such that \(V(0) = 0\), \(V(x) > 0\), \(x \neq 0\), \(\phi_c(0) = 0\) and
\[ V'(x) F_c(x, \phi_c(x)) < 0, \quad x \notin Z_x, \quad x \neq 0 \]  
(42)
\[ V(x + F_d(x, \phi_d(x))) - V(x) \leq 0, \quad x \in Z_x \]  
(43)
\[ H_c(x, \phi_c(x)) = 0, \quad x \notin Z_x \]  
(44)
\[ H_d(x, \phi_d(x)) = 0, \quad x \in Z_x \]  
(45)
\[ H_c(x, u_c) \geq 0, \quad x \notin Z_x, \quad u_c \in U_c \]  
(46)
\[ H_d(x, u_d) \geq 0, \quad x \in Z_x, \quad u_d \in U_d \]  
(47)
where
\[ H_c(x, u_c) = L_c(x, u_c) + V'(x) F_c(x, u_c) \]  
(48)
\[ H_d(x, u_d) = L_d(x, u_d) + V(x + F_d(x, u_d)) - V(x) \]  
(49)
Then, with the hybrid feedback control \(u_c(\cdot), u_d(\cdot)) = (\phi_c(x(\cdot)), \phi_d(x(\cdot)))\), there exists a neighbourhood of the origin \(D_0 \subseteq D\) such that if \(x_0 \in D_0\), the zero solution \(x(t) \equiv 0\) of the closed-loop system (39) and (40) is locally asymptotically stable. Furthermore
\[ J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = V(x_0), \quad x_0 \in D_0 \]  
(50)
In addition, if \(x_0 \in D_0\) then the hybrid feedback control \(u_c(\cdot), u_d(\cdot)) = (\phi_c(x(\cdot)), \phi_d(x(\cdot)))\) minimizes \(J(x_0, u_c(\cdot), u_d(\cdot))\) in the sense that
\[ J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in C(x_0)} J(x_0, u_c(\cdot), u_d(\cdot)) \]  
(51)
Finally, if \(D = \mathbb{R}^c, U_c = \mathbb{R}^{n_c}, U_d = \mathbb{R}^{n_d}\) and \(V(x) \to \infty\) as \(\|x\| \to \infty\), then the zero solution \(x(t) \equiv 0\) of the closed-loop system (39) and (40) is globally asymptotically stable.

**Proof:** Local and global asymptotic stability is a direct consequence of (42) and (43) by applying Theorem 2 of Haddad et al. (2001) to the closed-loop system (39), (40). Conditions (50) and (51) are a direct consequence of Theorem 4 with \(V(x, t) = V(x), \ t_0 = 0, \ t_r \to \infty\) and using the fact that \(\lim_{t \to \infty} V(x(t)) = 0\) and \(\lim_{k \to \infty} V(x(t_k)) = 0\).

**Remark 2:** Note that (44) and (45) are the steady state hybrid Hamilton–Jacobi–Bellman equations for the non-linear hybrid system (37) and (38) with the hybrid performance criterion \(J(x_0, u_c(\cdot), u_d(\cdot))\) given by (41). Furthermore, Theorem 5 guarantees optimality with respect to the set of admissible stabilizing hybrid controllers \(C(x_0)\). However, it is important to note that an explicit characterization of \(C(x_0)\) is not required. In addition, the optimal stabilizing hybrid feedback control law \(u_c, u_d = (\phi_c(x), \phi_d(x))\) is independent of the initial condition \(x_0\). Finally, in order to assure asymptotic stability of the hybrid closed-loop system (39) and (40), Theorem 5 requires that \(V(\cdot)\) satisfy (42) and (43) which implies that \(V(\cdot)\) is a Lyapunov function for the hybrid closed-loop system (39) and (40).

Next, we specialize Theorem 5 to linear impulsive systems. For the following result let \(A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times m_c}, A_d \in \mathbb{R}^{n_d \times n_d}, B_d \in \mathbb{R}^{n_d \times m_d}, R_{1c} \in \mathbb{R}^{n_c \times n_c}, R_{1d} \in \mathbb{R}^{n_d \times n_d}, R_{2c} \in \mathbb{R}^{n_c \times n_c}\) be given, where \(R_{1c}, R_{2c}, R_{1d}\) and \(R_{2d}\) are positive definite.

**Corollary 2:** Consider the linear controlled impulsive system
\[ \dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(0) = x_0, \quad x(t) \notin Z_x \]  
(52)
\[ \Delta x(t) = (A_d - I_n)x(t) + B_d u_d(t), \quad x(t) \in Z_x \]  
(53)
with quadratic hybrid performance functional
\[ J(x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty [x^T(t) R_{1c} x(t) + u_c^T(t) R_{2c} u_c(t)] \, dt + \sum_{k \in N_{0,\infty}} [x^T(t_k) R_{1d} x(t_k)] + u_d^T(t_k) R_{2d} u_d(t_k) \]  
(54)
where \((u_c(\cdot), u_d(\cdot))\) is an admissible hybrid control. Furthermore, assume there exists a positive-definite matrix \(P \in \mathbb{R}^{n \times n}\) such that
Theorem 5: The result is a direct consequence of Theorem 2.\( x(t) \in \mathbb{R}^n \) and (55) hold for all \( x \neq 0 \) and \( x \in \mathbb{Z}_x \). Hence, \( \mathcal{C}(x) = \{ x \in \mathbb{R}^n : Hx = 0 \} \), where \( H \in \mathbb{R}^{m \times n} \), it follows that (56) holds when \( P > 0 \) with a specific internal matrix structure. This of course reduces the number of free elements in \( P \) satisfying (55) and (56). Alternatively, to avoid the complexity in solving (55) and (56), an inverse optimal control problem can be solved wherein \( R_1, R_2, R_1d, R_2d \) are arbitrary. In this case, (55) and (56) are implied by

\[
0 = A_c^TP + PA_c + R_{1c} - PB_c R_{2c}^2 B_c^TP x, \quad x \notin \mathbb{Z}_x
\]

\[
0 = x^T(A_c^TPA_d - P + R_{1d} - A_d^T P B_d R_{2d}^2 B_d^T P A_d)x, \quad x \in \mathbb{Z}_x
\]

and

\[
H_c(x,u_c) = x^TR_{1c}x + u_c^TR_{2c}u_c + x^T(A_c^TP + PA_c)x
\]

\[
+ 2x^TPB_cu_c, \quad x \notin \mathbb{Z}_x
\]

\[
H_d(x,u_d) = x^TR_{1d}x + u_d^TR_{2d}u_d + (A_d x + B_d u_d)^T P (A_d x + B_d u_d)
\]

\[
- x^TPx, \quad x \in \mathbb{Z}_x
\]

Remark 4: For given \( R_{1c}, R_{2c}, R_{1d} \) and \( R_{2d}, (55) \) and (56) can be solved using constrained non-linear programming methods using the structure of \( \mathbb{Z}_x \). For example, in the case where \( \mathbb{Z}_x \) is characterized by the hyperplane \( \mathbb{Z}_x = \{ x \in \mathbb{R}^n : Hx = 0 \} \), where \( H \in \mathbb{R}^{m \times n} \), it follows that (56) holds when \( \partial H_d/\partial u_d = R_{2d} + B_d^T P B_d \neq 0 \). Now, \( \partial H_d/\partial u_d = 2R_{2d}u_d + 2B_d^TP x = 0 \), \( x \notin \mathbb{Z}_x \) and \( \partial H_d/\partial u_d = (2R_{2d} + B_d^TP B_d)u_d + 2B_d^TP A_d x = 0 \), \( x \in \mathbb{Z}_x \), give the unique global minimum of \( H_c(x,u_c), x \notin \mathbb{Z}_x \) and \( H_d(x,u_d), x \in \mathbb{Z}_x \), respectively.

Remark 3: The optimal hybrid feedback control law \((\phi_c(x), \phi_d(x))\) in Corollary 2 is derived using the properties of \( H_c(x,u_c) \) and \( H_d(x,u_d) \) as defined in Theorem 5. Specifically, since

\[
\mathcal{C}(x) = \{ x \in \mathbb{R}^n : Hx = 0 \}
\]

where \( \mathcal{C}(x) \) is the set of regulation hybrid controllers for (52) and (53) and \( x \notin \mathbb{R}^n \).

Proof: The result is a direct consequence of Theorem 5 with \( F_c(x, u_c) = A_c x + B_c u_c, \quad L_c(x, u_c) = x^T R_{1c} x + u_c^T R_{2c} u_c \), for \( x \notin \mathbb{Z}_x \), \( F_d(x, u_d) = (A_d - I_n)x + B_d u_d, \quad L_d(x, u_d) = x^T R_{1d} x + u_d^T R_{2d} u_d \), for \( x \in \mathbb{Z}_x \), \( V(x) = x^TPx, \quad D = \mathbb{R}^n \) and \( \mathcal{C}(x) = \{ x \in \mathbb{R}^n : Hx = 0 \} \).

Similarly, it follows from (55) that \( H_c(x, \phi_c(x)) = 0 \), \( x \notin \mathbb{Z}_x \) and hence \( V(x) F_c(x, \phi_c(x)) < 0 \) for all \( x \neq 0 \) and \( x \notin \mathbb{Z}_x \). Similarly, it follows from (56) that \( H_d(x, \phi_d(x)) = 0 \), \( x \in \mathbb{Z}_x \) and hence \( V(x + F_d(x, \phi_d(x)) - V(x) < 0 \) for all \( x \neq 0 \) and \( x \notin \mathbb{Z}_x \). Thus,

\[
H_c(x,u_c) = H_c(x,u_c) - H_c(x,\phi_c(x))
\]

\[
= [u_c - \phi_c(x)]^T R_{2c} [u_c - \phi_c(x)] \geq 0, \quad x \notin \mathbb{Z}_x
\]

and

\[
H_d(x,u_d) = H_d(x,u_d) - H_d(x,\phi_d(x))
\]

\[
= [u_d - \phi_d(x)]^T (R_{2d} + B_d^T P B_d) [u_d - \phi_d(x)]
\]

\[
\geq 0, \quad x \in \mathbb{Z}_x
\]

so that all conditions of Theorem 5 are satisfied. Finally, since \( V(x) \) is radially unbounded, the zero solution \( x(t) \equiv 0 \) to (52) and (53) with \( u_c(t) = \phi_c(x(t)) = -R_{2c} B_c^T P x(t), \quad x(t) \notin \mathbb{Z}_c \) and \( u_d(t) = \phi_d(x(t)) = -(R_{2d} + B_d^T P B_d)^{-1} B_d^T P A_d x(t), \quad x(t) \in \mathbb{Z}_x \), is globally asymptotically stable. \( \square \)
4. Inverse optimal control for nonlinear affine impulsive systems

In this section we use the results of §3 to obtain controllers that are predicated on an inverse optimal hybrid control problem. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton–Jacobi–Bellman equation we do not attempt to minimize a given cost functional, but rather, we parametrize a family of stabilizing hybrid controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the non-linear impulsive system dynamics, the Lyapunov function of the closed-loop system and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints. In addition, as shown by Haddad et al. (2000), the inverse optimal hybrid controllers guarantee hybrid disc, sector and gain margins to multiplicative input uncertainty. Hence, the inverse optimal hybrid controllers obtained in this section additionally provide robustness guarantees to multiplicative input uncertainty and thus guarantee robustness to unmodelled actuator dynamics.

Consider the state-dependent affine (in the control) impulsive dynamical system

\[
\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t),
\]

\[
x(0) = x_0, \quad x(t) \not\in Z_x \quad (62)
\]

\[
\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad x(t) \in Z_x \quad (63)
\]

Furthermore, we consider performance integrands

\[
L_c(x, u_c) \text{ and } L_d(x, u_d)
\]

of the form

\[
L_c(x, u_c) = L_{1c}(x) + u_c^T R_{2c}(x)u_c \quad (64)
\]

\[
L_d(x, u_d) = L_{1d}(x) + u_d^T R_{2d}(x)u_d \quad (64)
\]

where \(L_{1c} : \mathbb{R}^n \to \mathbb{R}\) and satisfies \(L_{1c}(x) \geq 0, x \in \mathbb{R}^n, R_{2c} : \mathbb{R}^n \to \mathbb{P}^{m_c}, L_{1d} : \mathcal{Z} \to \mathbb{R}\) and satisfies \(L_{1d}(x) \geq 0, x \in \mathbb{R}^n\) and \(R_{2d} : \mathcal{Z} \to \mathbb{P}^{m_d}\) so that (23) becomes

\[
J(x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty \left[L_{1c}(x(t)) + u_c^T(t) R_{2c}(x(t))u_c(t)\right] dt + \sum_{k \in \mathcal{N}_{x(x)}^+} [L_{1d}(x(t_k))] + u_d^T(t_k) R_{2d}(x(t_k))u_d(t_k)] \quad (65)
\]

**Corollary 3:** Consider the non-linear impulsive controlled system (62) and (63) with performance functional (65). Assume there exists a continuously differentiable function \(V : \mathbb{R}^p \to \mathbb{R}\) and functions \(P_{12} : \mathcal{Z} \to \mathbb{R}^{n \times n}\) and \(P_{2} : \mathcal{Z} \to \mathbb{R}^{n \times n}\) such that \(V(0) = 0, V(x) > 0, x \in \mathbb{R}^p, x \neq 0\)

\[
V'(x)[f_c(x) - \frac{1}{2}G_c(x) R_{2c}(x) G_c^T(x) V''(x)] < 0, \quad x \not\in Z_x, x \neq 0 \quad (66)
\]

\[
V(x + f_d(x) - \frac{1}{2} G_d(x)(R_{2d}(x) + P_2(x))^{-1} P_{12}^T(x) - V(x) \leq 0 \quad x \in Z_x \quad (67)
\]

\[
V(x + f_d(x) + G_d(x)u_d) = V(x + f_d(x)) + P_{12}(x)u_d + u_d^T P_2(x) u_d \quad (68)
\]

where \(u_d\) is admissible and

\[
V(x) \to \infty \quad \text{as } \|x\| \to \infty \quad (69)
\]

Then the zero solution \(x(t) \equiv 0\) to the closed-loop system

\[
\dot{x}(t) = f_c(x(t)) + G_c(x(t))\phi_c(x(t)), \quad x(0) = x_0, \quad x(t) \not\in Z_x \quad (70)
\]

\[
\Delta x(t) = f_d(x(t)) + G_d(x(t))\phi_d(x(t)), \quad x(t) \in Z_x \quad (71)
\]

is globally asymptotically stable with the hybrid feedback control law

\[
\phi_c(x) = -\frac{1}{2} R_{2c}^{-1}(x) G_c^T(x) V''(x), \quad x \not\in Z_x \quad (72)
\]

\[
\phi_d(x) = -\frac{1}{2} R_{2d}(x) + P_2(x))^{-1} P_{12}^T(x), \quad x \in Z_x \quad (73)
\]

and performance functional (65), with

\[
L_{1c}(x) = \phi_c^T(x) R_{2c}(x) \phi_c(x) - V'(x)f_c(x) \quad (74)
\]

\[
L_{1d}(x) = \phi_d^T(x) R_{2d}(x) + P_2(x))\phi_d(x) - V(x + f_d(x)) + V(x) \quad (75)
\]

is minimized in the sense that

\[
J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{E}(x_0)} J(x_0, u_c(\cdot), u_d(\cdot)) \quad (66)
\]

\[
x_0 \in \mathbb{R}^p \quad (76)
\]

Finally

\[
J(x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^p \quad (77)
\]

**Proof:** The result is a direct consequence of Theorem 5 with \(D = \mathbb{R}^p, U_C = \mathbb{R}^{m_c}, U_d = \mathbb{R}^{m_d}, F_c(x, u_c) = f_c(x) + G_c(x)u_c, F_d(x, u_d) = f_d(x) + G_d(x)u_d, L_c(x, u_c) = L_{1c}(x) + u_c^T R_{2c}(x)u_c\) and \(L_d(x, u_d) = L_{1d}(x) + u_d^T R_{2d}(x)u_d\). Specifically, with (64) the Hamiltonians have the form
\[ H_c(x, u_c) = L_{1c}(x) + u_c^T R_{2c}(x) u_c + V'(x)(f_c(x) + G_c(x) u_c), \]
\[ x \notin \mathcal{Z}_x, \quad u_c \in \mathbb{R}^{n_u} \quad (78) \]
\[ H_d(x, u_d) = L_{1d}(x) + V(x + f_d(x)) + P_{12}(x) u_d - V(x), \]
\[ x \in \mathcal{Z}_x, \quad u_d \in \mathbb{R}^{n_u} \quad (79) \]

Now, the hybrid feedback control law (72) and (73) is obtained by setting \( \partial H_c / \partial u_c = 0 \) and \( \partial H_d / \partial u_d = 0 \). With (72) and (73), it follows that (66) and (67) imply (42) and (43), respectively. Next, since \( V(\cdot) \) is continuously differentiable and \( x = 0 \) is a local minimum of \( V(\cdot) \), it follows that \( V'(0) = 0 \) and hence \( \phi_c(0) = 0 \). Next, with \( L_{1c}(x) \) and \( L_{1d}(x) \) given by (74) and (75), respectively, and \( \phi_c(x) \) and \( \phi_d(x) \) given by (72) and (73), (44) and (45) hold. Finally, since
\[ H_c(x, u_c) = H_c(x, u_c) - H_c(x, \phi_c(x)) \]
\[ = [u_c - \phi_c(x)]^T R_{2c}(x) [u_c - \phi_c(x)], \]
\[ x \notin \mathcal{Z}_x \quad (80) \]
\[ H_d(x, u_d) = H_d(x, u_d) - H_d(x, \phi_d(x)) \]
\[ = [u_d - \phi_d(x)]^T (R_{2d}(x) + P_2(x)) [u_d - \phi_d(x)], \]
\[ x \in \mathcal{Z}_x \quad (81) \]

where \( R_{2c}(x) > 0, \ x \notin \mathcal{Z}_x \) and \( R_{2d}(x) + P_2(x) > 0, \ x \in \mathcal{Z}_x \), conditions (46) and (47) hold. The result now follows as a direct consequence of Theorem 5.

Remark 5: Note that (66) and (67) are equivalent to
\[ V(x) = V'(x)[f_c(x) + G_c(x) \phi_c(x)] < 0, \]
\[ x \notin \mathcal{Z}_x, \quad x \neq 0 \quad (82) \]
\[ \Delta V(x) = V(x + f_d(x) + G_d(x) \phi_d(x)) - V(x) \leq 0, \]
\[ x \in \mathcal{Z}_x \quad (83) \]

with \( \phi_c(x) \) and \( \phi_d(x) \) given by (72) and (73), respectively. Furthermore, conditions (82) and (83) with \( V(0) = 0 \) and \( V(x) > 0, \ x \in \mathbb{R}^n, \ x \neq 0 \), assure that \( V(x) \) is a Lyapunov function for the impulsive closed-loop system (70) and (71).

Remark 6: In the case where \( R_{2c}(x), \ x \notin \mathcal{Z}_x \), is a diagonal weighting function, the hybrid controller (72) and (73) guarantees hybrid disc, sector and gain margins to multiplicative input uncertainty of \((\frac{1}{2}, \infty), (1/1 + \theta_d, 1/1 - \theta_d))\), where \( \theta_d = \sqrt{2d/\sigma_d}, \ \sigma_d = \inf_{x \in \mathcal{Z}_x} \sigma_{\min}(R_{2d}(x)) \) and \( \sigma_d = \sup_{x \in \mathcal{Z}_x} \sigma_{\max}(R_{2d}(x) + P_2(x)) \). For details see Haddad et al. (2000).

5. Non-linear hybrid control with polynomial and multilinear performance functionals

In this section we specialize the results of §4 to impulsive systems controlled by inverse optimal non-linear hybrid controllers that minimize a derived polynomial and multilinear cost functional. These results generalize the non-linear feedback control results derived by Bass and Weber and Speyer (1976) to impulsive dynamical systems. For the results in this section we assume \( u_d(t_0) \equiv 0 \). Furthermore, let \( R_{1c} \in \mathbb{P}^m, \ R_{1d} \in \mathbb{P}^m, \ R_{2c} \in \mathbb{P}^{n_u}, \ R_{q}, \hat{R}_q \in \mathbb{N}^m, \ q = 2, \ldots, r, \) be given, where \( r \) is a positive integer and define \( S_c = B_c R_{2c}^{-1} B_c^T \).

Corollary 4: Consider the linear controlled impulsive system
\[ \dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \quad (84) \]
\[ \Delta x(t) = (A_d - I_u) x(t), \quad x(t) \in \mathcal{Z}_x \quad (85) \]
where \( u_c \) is admissible. Assume there exist \( P \in \mathbb{P}^p \) and \( M_q \in \mathbb{N}^m \), \( q = 2, \ldots, r \), such that
\[ 0 = x^T (A_c^T P + PA_c + R_{1c} - PB_c R_{2c}^{-1} B_c^T P) x, \]
\[ x \notin \mathcal{Z}_x \quad (86) \]
\[ 0 = x^T [(A_c - S_c P)^T M_q + M_q (A_c - S_c P) + \hat{R}_q] x, \]
\[ x \notin \mathcal{Z}_x, \quad q = 2, \ldots, r \quad (87) \]
\[ 0 = x^T (A_d^T P A_d - P + R_{1d}) x, \quad x \in \mathcal{Z}_x \quad (88) \]
\[ 0 = x^T (A_d^T M_q A_d - M_q + \hat{R}_q) x, \]
\[ x \in \mathcal{Z}_x, \quad q = 2, \ldots, r \quad (89) \]

Then the zero solution \( x(t) \equiv 0 \) of the closed-loop system
\[ \dot{x}(t) = A_c x(t) + B_c \phi_c(x(t)), \]
\[ x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x \quad (90) \]
\[ \Delta x(t) = (A_d - I_u) x(t), \quad x(t) \in \mathcal{Z}_x \quad (91) \]
is globally asymptotically stable with the feedback control law
\[ \phi_c(x) = -R_{2c}^{-1} B_c^T (P + \sum_{q=2}^{r} (x^T M_q x)^{q-1} M_q) x, \]
\[ x \notin \mathcal{Z}_x \quad (92) \]
and the performance functional (65), with \( R_{2c}(x) = R_{2c} \)
Non-linear impulsive dynamical systems. Part II

1671

\[ L_{1c}(x) = x^T \left( R_{1c} + \sum_{q=2}^{r} (x^T M_q x)^{q-1} R_q \right) 
+ \left[ \sum_{q=2}^{r} (x^T M_q x)^{q-1} M_q \right]^T 
\times S_c \left[ \sum_{q=2}^{r} (x^T M_q x)^{q-1} M_q \right] x \]  
\[ (93) \]

\[ L_{1d}(x) = x^T R_{1d} x + \sum_{q=2}^{r} \frac{1}{q} \left[ (x^T \hat{R}_q x)^{q-1} \sum_{j=1}^{q} (x^T M_q x)^{q-j} \right] \times (x^T A_d^T M_q A_d x)^q ] \]  
\[ (94) \]

is minimized in the sense that

\[ J(x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot) \in \mathbb{C}(x_0)} J(x_0, u_c(\cdot)), \quad x_0 \in \mathbb{R}^n \]  
\[ (95) \]

Finally,

\[ J(x_0, \phi_c(x(\cdot))) = x_0^T P x_0 + \sum_{q=2}^{r} \frac{1}{q} (x_0^T M_q x_0)^q, \quad x_0 \in \mathbb{R}^n \]  
\[ (96) \]

**Proof:** The result is a direct consequence of Corollary 3 with \( f_c(x) = A_c x, \quad f_d(x) = (A_d - I_n)x, \quad G_c(x) = B_c, \quad G_d(x) = 0, \quad u_d = 0, \quad R_{2c}(x) = R_{2c}, \quad R_{2d}(x) = I_{n_k} \) and \( V(x) = x^T P x + \sum_{q=2}^{r} (1/q) (x^T M_q x)^q \). Specifically, for \( x \notin \mathbb{Z} \) it follows from (86), (87), and (92) that

\[ V'(x) = 2 G_c(x) R_{2c}^{-1} G^T_c(x) V^T(x) - \frac{1}{2} (x^T M_q x)^q - x^T R_{1c} x - \frac{1}{q} \sum_{q=2}^{r} (x^T M_q x)^{q-1} x^T \hat{R}_q x - \phi_c(x) R_{2c} \phi_c(x) 
\]

\[ = -x^T R_{1d} x - \frac{1}{q} \sum_{q=2}^{r} (x^T M_q x)^{q-1} x^T \hat{R}_q x - \phi_c(x) R_{2c} \phi_c(x) 
\]

which implies (66). For \( x \in \mathbb{Z} \) it follows from (88) and (89) that

\[ \Delta V(x) = V(x + f_d(x)) - V(x) 
\]

\[ = -x^T R_{1d} x - \frac{1}{q} \sum_{q=2}^{r} (x^T \hat{R}_q x) \]

\[ \times \sum_{j=1}^{q} (x^T M_q x)^{j-1} (x^T A_d^T M_q A_d x)^{q-j} \]

which implies (67) with \( G_d(x) = 0 \). Finally, with \( u_d = 0, \) (68) is automatically satisfied so that all the conditions of Corollary 3 are satisfied.

**Remark 7:** As noted in Remark 4, viewing \( R_{1c}, R_{2c}, \) \( R_q \) and \( \hat{R}_q, q = 2, \ldots, r, \) as arbitrary matrices, it follows that (86)–(89) are implied by a set of Bilinear matrix inequalities (BMIs). This considerably minimizes the numerical complexity for solving (86)–(89).

Finally, we specialize the results of §4 to linear impulsive systems controlled by inverse optimal hybrid controllers that minimize a derived multilinear functional. First, however, we give several definitions involving multilinear forms. A scalar function \( \psi : \mathbb{R}^p \rightarrow \mathbb{R} \) is \( q \)-multilinear if \( q \) is a positive integer and \( \psi(x) \) is a linear combination of terms of the form \( x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \), where \( i_j \) is a non-negative integer for \( j = 1, \ldots, n \) and \( i_1 + i_2 + \cdots + i_n = q \). Furthermore, a \( q \)-multilinear function \( \psi(\cdot) \) is non-negative definite (resp., positive definite) if \( \psi(x) \geq 0 \) for all \( x \in \mathbb{R}^p \) (resp., \( \psi(x) > 0 \) for all non-zero \( x \in \mathbb{R}^p \)). Note that if \( q \) is odd then \( \psi(x) \) cannot be positive definite. If \( \psi(\cdot) \) is a \( q \)-multilinear function then \( \psi(\cdot) \) can be represented by means of Kronecker products; that is, \( \psi(x) \) is given by \( \psi(x) = \psi x^{[q]} \), where \( \Psi \in \mathbb{R}^{p \times n^q} \) and \( x^{[q]} = x \otimes x \otimes \cdots \otimes x \) (q times), \( \otimes \) denotes Kronecker product. For the next result recall the definition of \( S_c, \) let \( R_{1c} \in \mathbb{P}^p, \) \( R_{1d} \in \mathbb{P}^n, \) \( R_{2c} \in \mathbb{P}^n, \) \( R_{2d} \in \mathbb{P}^n \), \( \hat{R}_q \in \mathbb{P}^n \), \( q = 2, \ldots, r, \) be given, where \( \mathbb{P}^{n^q} = \{ \Psi \in \mathbb{R}^{p \times n^q} : \Psi x^{[q]} \geq 0, \ x \in \mathbb{R}^p \} \) and define the repeated (q times) Kronecker sum as \( \Psi = A \bigoplus A \bigoplus \cdots \bigoplus A \).

**Corollary 5:** Consider the linear controlled impulsive system (84) and (85). Assume there exist \( P \in \mathbb{P}^n \) and \( \hat{P}_q \in \mathbb{P}^n, q = 2, \ldots, r, \) such that

\[ 0 = x^T (A_c^T P + PA_c + R_{1c} - PB_c R_{2c}^{-1} B_c^T P) x, \quad x \notin \mathbb{Z} \]  
\[ (97) \]

\[ 0 = x^T (\hat{P}_q [2^q d(A_c - S_c P)] + \hat{R}_2 q) x, \quad x \notin \mathbb{Z}, \quad q = 2, \ldots, r \]  
\[ (98) \]

\[ 0 = x^T (A_d^T P A_d - P + R_{1d}) x, \quad x \in \mathbb{Z} \]  
\[ (99) \]

\[ 0 = x^T (\hat{P}_q [I_n^{[2q]} - I_n^{[2q]}] + \hat{R}_2 q) x, \quad x \in \mathbb{Z}, \quad q = 2, \ldots, r \]  
\[ (100) \]

Then the zero solution \( x(t) \equiv 0 \) of the closed-loop system (90) and (91) is globally asymptotically stable with the feedback control law

\[ \phi_c(x) = -R_{2c}^{-1} B_c^T (P x + \frac{1}{q} g^{T}(x)), \quad x \notin \mathbb{Z} \]  
\[ (101) \]

where \( g(x) = \sum_{q=2}^{r} \frac{1}{q} \hat{P}_q [2^q] \) and the performance functional (65), with \( \hat{R}_{2c}(x) = R_{2c} \) and
Finally, with $u_d = 0$, (68) is automatically satisfied so that all the conditions of Corollary 3 are satisfied.

\[ L_{1c}(x) = x^T R_{1c} x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} + \frac{1}{2} g' g'(x) S_c g'^T(x) \]  
(102)

\[ L_{1d}(x) = x^T R_{1d} x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} \]  
(103)

**Proof:** The result is a direct consequence of Corollary 3 with $f_c(x) = A_c x$, $f_d(x) = (A_d - I_d) x$, $G_c(x) = B_c$, $G_d(x) = 0$, $u_d = 0$, $R_{2c}(x) = R_{2c}$, $R_{2d}(x) = I_{u_d}$ and $V(x) = x^T P x + \sum_{q=2}^{r} \tilde{P}_q x^{[2q]}$. Specifically, for $x \not\in Z_x$ it follows from (97), (98) and (101) that

\[ V'(x)[f_c(x) - \frac{1}{2} G_c(x) R_{2c}^{-1}(x) G_c^T(x) V'^T(x)] \]

\[ = -x^T R_{1c} x - \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} 
- \frac{1}{2} g' g'(x) S_c g'^T(x) \]

which implies (67). For $x \in Z_x$ it follows from (68) and (100) that

\[ \Delta V(x) = V(x + f_d(x)) - V(x) \]

\[ = -x^T R_{1d} x - \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} \]

which implies (67) with $G_d(x) = 0$. Finally, with $u_d = 0$, (68) is automatically satisfied so that all the conditions of Corollary 3 are satisfied. \( \square \)

6. **Hybrid controllers for combustion systems**

High performance aeroengine afterburners and ram-jets often experience combustion instabilities at some operating condition. Combustion in these high energy density engines is highly susceptible to flow disturbances, resulting in fluctuations to the instantaneous rate of heat release in the combustor. This unsteady combustion provides an acoustic source resulting in self-excited oscillations. In particular, unsteady combustion generates acoustic pressure and velocity oscillations which in turn perturb the combustion even further (Culick 1976, Candel 1992). These pressure oscillations, known as thermoacoustic instabilities, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates and even component melting. Hence, the need for active control to mitigate combustion induced pressure instabilities is severe.

In this section we apply the results developed in this paper and Part I of this paper (Haddad et al. 2001) to the control of thermoacoustic instabilities in combustion processes. We stress that the combustion model we use is *not* an impulsive dynamical system and hence can be stabilized by conventional non-linear control methods. The aim here however, is to show that hybrid control provides an extremely efficient mechanism for dissipating energy in the combustion process with far superior performance than any conventional control methodology. In particular, we show that the proposed hybrid controller provides finite-time stabilization. Since, as can be seen below, the combustion system dynamics are Lipschitz continuous, finite-time stabilization cannot be achieved by any other conventional controller.

To design hybrid controllers for combustion systems we concentrate on a two-mode, non-linear time-averaged combustion model with non-linearities present due to the second-order gas dynamics. This model is developed in Culick (1988) and is given by

\[ \dot{x}_1(t) = \alpha_1 x_1(t) + \theta_1 x_2(t) - \beta x_1(t)x_3(t) \]

\[ + x_2(t)x_4(t) + u_{s1}(t), \quad x_1(0) = x_{10}, \quad t \geq 0 \]  
(106)

\[ \dot{x}_2(t) = -\theta_1 x_1(t) + \alpha_2 x_2(t) + \beta x_2(t)x_1(t) \]

\[ - x_1(t)x_4(t) + u_{s2}(t), \quad x_2(0) = x_{20} \]  
(107)

\[ \dot{x}_3(t) = \alpha_3 x_3(t) + \theta_2 x_4(t) + \beta (x_1^2(t) - x_2^2(t)) \]

\[ + u_{s3}(t), \quad x_3(0) = x_{30} \]  
(108)

\[ \dot{x}_4(t) = -\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) \]

\[ + u_{s4}(t), \quad x_4(0) = x_{40} \]  
(109)

where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = ((\gamma + 1)/8\gamma)\omega_1$, $\gamma$ denotes the ratio of specific heats, $\omega_1$ is frequency of the fundamental mode and $u_{si}, i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$ and $x_0 = [1 \ 1 \ 1 \ 1]^T$, the open-loop ($u_{si}(t) \equiv 0, i = 1, \ldots, 4$) dynamics (106)-(109) result in a limit cycle instability. Figure 2 shows the phase portrait, state response and plant energy $V_s(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$ versus time.

6.1. **Time-dependent hybrid controllers**

To design a stabilizing time-dependent hybrid controller for (106)-(109) we first design a control law $u_i = -K_i x + u_{si}, \quad K_i = \text{diag}[k_{s1}, k_{s2}, k_{s3}, k_{s4}], \quad u_{si} = [u_{s1}, u_{s2}, u_{s3}, u_{s4}]^T$ and
Non-linear impulsive dynamical systems. Part II

$u_c = [u_{c1}, u_{c2}, u_{c3}, u_{c4}]^T$. In this case, (106)-(109) are given by (1), (2) with $Z = \emptyset$ and

$$f_c(x) = \begin{bmatrix}
\alpha_1 x_1 + \theta_1 x_2 - \beta(x_1 x_3 + x_2 x_4) - k_{s1} x_1 \\
-\theta_1 x_1 + \alpha_1 x_2 + \beta(x_2 x_3 - x_1 x_4) - k_{s2} x_2 \\
\alpha_2 x_3 + \theta_2 x_4 + \beta(x_1^2 - x_2^2) - k_{s3} x_3 \\
-\theta_2 x_3 + \alpha_2 x_4 + 2\beta x_1 x_2 - k_{s4} x_4
\end{bmatrix}$$

$$G_c(x) = I_4$$

(110)

$$f_d(x) = 0, \quad G_d(x) = 0$$

(111)

Now, with $y_c = x$, $k_{s1} = k_{s2} = \alpha_1$ and $k_{s3} = k_{s4} = 0$, it follows that (1), (3), with $f_c(x)$ and $G_c(x)$ given by (110) and $h_c(x) = x$ and $J_c(x) = 0$, is passive with input $u_c$, output $y_c$ and plant energy function, or storage function, $V_c(x)$. Hence, $V_c(x)f_c(x) \leq 0, \quad x \in \mathbb{R}^4$. Furthermore, (1), (3), with $f_c(x)$ and $G_c(x)$ given by (110) and $h_c(x) = x$ and $J_c(x) = 0$, is zero-state observable. Figure 3 shows the phase portrait, state response and plant energy of the controlled system (1), (3) with $u_s = -K x + u_c$ and $u_c \equiv 0$.

To improve the performance of the above controller, we use the flexibility in $u_c$ to design a hybrid controller. Specifically, consider the hybrid controller emulating the plant structure given by (5)-(8) with $S_c = T \times \mathbb{R}^{u_c} \times \mathbb{R}^{w_c}$

$$f_{cc}(x_c) = \begin{bmatrix}
\alpha_1 x_{c1} + \theta_1 x_{c2} - \beta(x_{c1} x_{c3} + x_{c2} x_{c4}) - k_{s1} x_{c1} \\
-\theta_1 x_{c1} + \alpha_1 x_{c2} + \beta(x_{c2} x_{c3} - x_{c1} x_{c4}) - k_{s2} x_{c2} \\
\alpha_2 x_{c3} + \theta_2 x_{c4} + \beta(x_{c1}^2 - x_{c2}^2) - k_{s3} x_{c3} \\
-\theta_2 x_{c3} + \alpha_2 x_{c4} + 2\beta x_{c1} x_{c2} - k_{s4} x_{c4}
\end{bmatrix}$$

$$G_{cc}(x_c) = I_4$$

(112)

$$f_{dc}(x_c) = \begin{bmatrix}
-x_{c1} \\
x_{c2} \\
-x_{c3} \\
-x_{c4}
\end{bmatrix}, \quad G_{dc}(x_c) = 0$$

(113)

$$h_{cc}(x_c) = -[x_{c1}, x_{c2}, x_{c3}, x_{c4}]^T$$

$$J_{cc}(x_c) = 0$$

(114)

$$h_{dc}(x_c) = 0$$

$$J_{dc}(x_c) = 0$$

where $k_{s1} > \alpha_1$, $k_{s2} > \alpha_1$, $k_{s3} > \alpha_2$ and $k_{s4} > \alpha_2$. It can be easily shown using Corollary 4 and Remark 15 of Haddad et al. (2001) that the hybrid controller (5)-(8) with dynamics given by (112)-(114), resetting set $S_c = T \times \mathbb{R}^{u_c} \times \mathbb{R}^{w_c}$, input $y_c$ and output $-u_c$ is exponentially passive with controller energy, or storage function,
\[ V_{sc}(x_c) = x_1^2 + x_2^2 + x_3^2 + x_4^2. \] Hence, \( V_{sc}(x_c)f_{dc}(x_c) \leq -\varepsilon V_{sc}(x_c), \ x_c \in \mathbb{R}^4, \) where \( \varepsilon = \min\{\alpha_1 - k_{d1}, \alpha_1 - k_{d2}, \alpha_2 - k_{d3}, \alpha_2 - k_{d4}\}. \) Furthermore, note that rank \( [G_{dc}(0)] = 4. \) Hence, stability of the closed-loop system (1), (3), (5)–(8), is guaranteed by Theorem 1. Finally, we note that the total energy of the closed-loop system (1), (3), (5)–(8) is given by
\[ V(\bar{x}) = V_s(x) + V_{sc}(x_c) \]
\[ = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \] (115)
where \( \bar{x} = [x^T \ x_c^T]^T. \)

The effect of the resetting law (6) with \( f_{dc}(x_c) \) and \( G_{dc}(x_c) \) given by (113), is to cause all controller states to be instantaneously reset to zero; that is, the resetting law (6) implies \( V_{sc}(x_c + \Delta x_c) = 0. \) The closed-loop resetting law is thus given by
\[ \Delta \bar{x} = [0 \ 0 \ 0 \ 0 - x_{c1} - x_{c2} - x_{c3} - x_{c4}]^T \] (116)

Note that since
\[ \bar{x} + \Delta \bar{x} = [x_1 \ x_2 \ x_3 \ x_4 \ 0 \ 0 \ 0 \ 0]^T \] (117)
it follows that
\[ V(\bar{x} + \Delta \bar{x}) = V_s(x) \] (118)
and
\[ V(\bar{x} + \Delta \bar{x}) - V(\bar{x}) = -V_{sc}(x_c) \leq 0 \] (119)

Now, from (119) it follows that the resetting law (6) causes the total energy to instantaneously decrease by an amount equal to the accumulated controller energy.

To illustrate the dynamic behaviour of the closed-loop system, let \( \alpha_1 = 5, \ \alpha_2 = -55, \ k_{d1} = \alpha_1, \ k_{d2} = \alpha_1, \ k_{d3} = 0, \ k_{d4} = 0, \ k_{c1} = \alpha_1 + 0.1, \ k_{c2} = \alpha_1 + 0.1, \ k_{c3} = 0, \ k_{c4} = 0 \) and \( T = \{2, 4, 6, \ldots\}, \) so that the controller resets periodically with a period of 2s. The response of the controlled system (1) and (3) with the resetting controller (5)–(8) and initial condition \( x_0 = [1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \) is shown on figure 4. Note that the control force versus time is discontinuous at the resetting times. A comparison of the plant energy, control energy and total energy is given in figure 5.

In this example the resetting times were chosen arbitrarily. However, with the same choice of controller parameters we can choose a resetting time to achieve finite-time stabilization. Specifically, the resetting time will correspond to the time at which all of the energy of the plant is drawn to the controller. This resetting time can be obtained from the energy history of the closed-loop system without resetting. In particular, the time instant when the plant and controller interchange energies such that plant energy is at zero, will correspond to the resetting time that achieves finite-time stabilization.
For this example, finite-time stability is achieved by choosing the resetting instant at $t = 1.6223$.

### 6.2. Input/state-dependent hybrid controllers

In this subsection we describe the mathematical setting and design of an input/state-dependent resetting controller. We consider the plant and resetting controller as described in §6.1 with $S_c = [0, \infty) \times Z_{cx} \times Z_{cu}$, where

$$Z_{cx} \times Z_{cu} = \{(x_c, u_c): f_{dc}(x_c) \neq 0 \text{ and } V'_{uc}(x_c)[f_{dc}(x_c) + G_{uc}(x_c)u_{uc}] \leq 0\}$$

The resetting set (120) is thus defined to be the set of all controller state and input points that represent non-increasing control energy, except those that satisfy $f_{dc}(x_c) = 0$. As mentioned in Remark 4 of Haddad et al. (2001), the states $x_c$ that satisfy $f_{dc}(x_c) = 0$ are states that do not change under the action of the resetting law and thus we need to exclude these states from the resetting set to ensure that the Assumption A1 of Haddad et al. (2001) is not violated. For the two-state, time-averaged combustion system given by (1) and (3) with $Z = \emptyset$ and dynamics (110) and $h_c(x) = x$, the input/state-dependent resetting set (120) becomes

$$Z_{cx} \times Z_{uc} = \{(x_c, u_c): f_{dc}(x_c) \neq 0 \text{ and } 2u_{cc}(\alpha_1 x_{c1} + \theta_1 x_{c2} - \beta(x_{c1} x_{c3} + x_{c2} x_{c4}) - k_{c1} x_{c1} + u_{cc}) + 2u_{cd}(-\theta_1 x_{c1} + \alpha_1 x_{c2} + \beta(x_{c2} x_{c3} - x_{c1} x_{c4}) - k_{c2} x_{c2} + u_{cd}) + 2u_{cc}(\alpha_2 x_{c3} + \theta_2 x_{c4} + \beta(x_{c1} x_{c2} - x_{c2}^2) - k_{c3} x_{c3} + u_{cc}) + 2u_{cd}(-\theta_2 x_{c3} + \alpha_2 x_{c4} + 2\beta x_{c1} x_{c2} - k_{c4} x_{c4} + u_{cd}) \leq 0\}$$

where $u_{ci}, i = 1, \ldots, 4$, represents the $i$th component of $u_c$. Now, it can be shown that Assumptions A1 and A2 of Haddad et al. (2001) are satisfied using straightforward calculations. Furthermore, since the resetting controller given in §6.1 is exponentially passive for $S_c = [0, \infty) \times \mathbb{R}^{uc} \times \mathbb{R}^{uc}$, it follows that the resetting controller is exponentially passive for $S_c = [0, \infty) \times Z_{cx} \times Z_{cu}$. Hence, asymptotic stability of the closed-loop system (1), (3) and (5)–(8) is guaranteed by Theorem 1. Finally, note that knowledge of $x_c$ and $y_c$ is sufficient to determine whether or not the closed-loop state vector $\hat{x}$ is in the resetting set $\tilde{Z}_r$, where

$$\tilde{Z}_r = Z_{cx} \cup \{(x, x_c): h_c(x_c) \in \tilde{U}_{cc} \} = \{\tilde{x}: f_{dc}(x_c) \neq 0 \text{ and } V'_{uc}(x_c)[f_{dc}(x_c) + G_{uc}(x_c)h_c(x)] \leq 0\}$$

By resetting the controller states, the plant energy can never increase. Hence, this approach allows the plant energy to ‘flow’ to the controller, where it increases the controller energy, but does not allow the controller energy to ‘flow’ back to the plant. This type of controller is referred to in Bupp et al. (2000) as a one-way resetting controller.

To illustrate the dynamics behaviour of the closed-loop system we again choose $\alpha_1 = 5$, $\alpha_2 = -55$, $k_{c1} = \alpha_1$, $k_{c2} = \alpha_1$, $k_{cd} = 0$, $k_{c1} = \alpha_1 + 0.1$, $k_{c2} = \alpha_1 + 0.1$, $k_{c3} = 0$ and $k_{c4} = 0$, with initial condition $x_0 = [1 1 1 1 0 0 0 0]^T$. The response of controlled system (1), (3), with dynamics (110) and $h_c(x) = x$ and the state-dependent resetting controller given by (5)–(8) with dynamics (112)–(114) and resetting set (121), is given in figure 6. The total energy, plant
energy and controller energy versus time are shown in Figure 7. Note that the proposed input/state-dependent resetting controller achieves finite-time stabilization.

7. Conclusion

In this paper general stability criteria were given for Lyapunov, asymptotic and exponential stability of feedback interconnections of non-linear impulsive dynamical systems. A unified framework for hybrid feedback optimal control over an infinite horizon involving a hybrid non-linear-non-quadratic performance functional was also developed. The overall framework provides the foundation for generalizing optimal linear-quadratic control methods to non-linear impulsive dynamical systems. The proposed framework was applied to the control of thermoacoustic instabilities in combustion processes demonstrating the ability of hybrid controllers to significantly enhance the removal of energy in combustion systems.

The underlying intention of this two-part paper has been to develop a unified framework for analysis and control synthesis of non-linear impulsive dynamical systems. This exposition demonstrates that stability, dissipativity, stability of feedback interconnections and optimality of continuous-time and discrete-time systems are part of the more general system theory of impulsive dynamical systems.

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