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Semistability of switched dynamical systems, Part I: Linear system theory☆

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This paper develops semistability and uniform semistability analysis results for switched linear systems. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system's initial conditions. Since solutions to switched systems are a function of the system's initial conditions as well as the switching signals, uniformity here refers to the convergence rate of the multiple solutions as the switching signal evolves over a given switching set. The main results of the paper involve sufficient conditions for semistability and uniform semistability using multiple Lyapunov functions and sufficient regularity assumptions on the class of switching signals considered.

1. Introduction

An essential feature of multiagent network systems is that these systems possess a continuum of equilibria [1,2]. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a non-isolated equilibrium cannot be asymptotically stable. Hence, asymptotic stability is not the appropriate notion of stability for systems having a continuum of equilibria. For such systems possessing a continuum of equilibria, *semistability* [3,4] is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in the neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. It is important to note that semistability is not equivalent to set stability of the equilibrium set. Indeed, it is possible for trajectories to approach the equilibrium set without any trajectory approaching any single equilibrium [4].

Since communication links among multiagent systems are often unreliable due to multipath effects and exogenous disturbances, the information exchange topologies in network systems are often dynamic. In particular, link failures or creations in network multiagent systems result in switching of the communication topology. This is the case, for example, if information between agents is exchanged by means of line-of-sight sensors that experience periodic communication dropouts due to agent motion. Variation in network topology introduces control input discontinuities, which in turn give rise to switched dynamical systems. In this case, the vector field defining the dynamical system is a discontinuous function of the state and/or time, and hence, system stability should involve analysis of semistability of switched systems having a continuum of equilibria.

Building on the results of [1,5], in this paper we develop semistability and uniform semistability analysis results for switched linear systems. Since solutions to switched systems are a function of both the system initial conditions and

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the admissible switching signals, uniformity here refers to the convergence rate to a Lyapunov stable equilibrium as the switching signal ranges over a given switching set. The main results of the paper involve sufficient conditions for semistability and uniform semistability using multiple Lyapunov functions and sufficient regularity assumptions on the class of switching signals considered. Specifically, using multiple Lyapunov functions whose derivatives are negative semidefinite, semistability of the switched linear system is established. If, in addition, the admissible switching signals have infinitely many disjoint intervals of length bounded from below and above, uniform semistability can be concluded. Finally, we note that the results of the present paper can be viewed as an extension of asymptotic stability results for switched linear systems developed in [6,7,5].

Although the results of this paper are confined to linear systems, nonlinear semistability theory for switched dynamical systems is considered in [8].

2. Switched dynamical systems

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathrm{Re}\,\lambda$ denotes the real part of $\lambda \in \mathbb{C}$, $(\cdot)^{\mathrm{T}}$ denotes the transpose, and $(\cdot)^{\mathrm{D}}$ denotes the Drazin generalized inverse. For $A \in \mathbb{R}^{n\times m}$ we write $\operatorname{rank} A$ to denote the rank of A, $\mathcal{N}(A)$ to denote the null space of A, $\mathcal{R}(A)$ to denote the range space of A, and for $A \in \mathbb{R}^{n\times n}$ we write $\operatorname{spec}(A)$ to denote the spectrum of A. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball *centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$.

In this paper, we consider *switched linear systems* $\mathcal{G}_\sigma$ given by

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in \mathcal{S},\ x(0) = x_0,\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$, $\sigma : [0, \infty) \to \mathcal{P}$ denotes a piecewise constant switching signal, and $\mathcal{S}$ denotes the set of switching signals. The switching signal σ effectively switches the right-hand side of (1) by selecting different vector fields from the parameterized family $\{A_p x : p \in \mathcal{P}\}$. The *switching times* of (1) refer to the time instants at which the switching signal σ is discontinuous. Our convention here is that $\sigma(\cdot)$ is left-continuous, that is, $\sigma(t^-) = \sigma(t)$, where $\sigma(t^-) \triangleq \lim_{h\to 0^+} \sigma(t + h)$. The pair $(x, \sigma) : [0, \infty) \times \mathcal{S} \to \mathbb{R}^n$ is a *solution* to the switched system (1) if $x(\cdot)$ is piecewise continuously differentiable and satisfies (1) for all $t \geq 0$. The set $\mathcal{S}_p[\tau, T]$, $\tau > 0$, $T \in [0, \infty]$, denotes the set of signals σ for which there is an infinite number of disjoint intervals of length no smaller than τ on which σ is constant, and consecutive intervals with this property are separated by no more than T [5] (including the initial time). Finally, a point $x_{\mathrm{e}} \in \mathbb{R}^n$ is an *equilibrium point* of (1) if and only if $A_{\sigma(t)}x_{\mathrm{e}} = 0$ for all $\sigma(t) \in \mathcal{S}$ and for all $t \geq 0$.

We assume that the following assumption holds for (1).

Assumption 1. $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) - \{0\} \neq \emptyset$.

Let $\mathcal{E} \triangleq \{x_{\mathrm{e}} \in \mathbb{R}^n : A_{\sigma(t)}x_{\mathrm{e}} = 0, \sigma(t) \in \mathcal{S}, t \geq 0\}$. Then $\mathcal{E} = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p)$ and $\mathcal{E}$ contains an element other than 0. It is important to note that our results also hold for the case where $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \{0\}$. However, due to space limitations, we do not consider this case in the paper.

Definition 2.1. (i) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *Lyapunov stable* if for every switching signal $\sigma \in \mathcal{S}$ and every $\varepsilon > 0$, there exists $\delta = \delta(\sigma, \varepsilon) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq 0$. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly Lyapunov stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\|x(t) - x_{\mathrm{e}}\| < \varepsilon$ for all $t \geq 0$.

(ii) An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *semistable* if for every switching signal $\sigma \in \mathcal{S}$, x_{e} is Lyapunov stable and there exists $\delta = \delta(\sigma) > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ and $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. An equilibrium point $x_{\mathrm{e}} \in \mathcal{E}$ of (1) is *uniformly semistable* if x_{e} is uniformly Lyapunov stable and there exists $\delta > 0$ such that for all $\|x_0 - x_{\mathrm{e}}\| \leq \delta$, $\lim_{t\to\infty} x(t) = z$ uniformly in σ and $z \in \mathcal{E}$ is a uniformly Lyapunov stable equilibrium point.

(iii) The switched system (1) is *semistable* if all the equilibrium points of (1) are semistable. The switched system (1) is *uniformly semistable* if all the equilibrium points of (1) are uniformly semistable.

Next, we present the notion of semiobservability which plays a critical role in semistability analysis of linear dynamical systems. For details, see [9].

Definition 2.2 (*[9]*). Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. The pair (A, C) is *semiobservable* if

$$\bigcap_{k=1}^{n} \mathcal{N}\left(CA^{k-1}\right) = \mathcal{N}(A). \tag{2}$$

Semiobservability is an extension of the classical notion of observability. In particular, semiobservability is an extension of zero-state observability to *equilibrium observability*. For details, see [9]. The following lemmas and propositions are needed for the main results of the paper.

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If the pair (A, C) is semiobservable, then*

$$\mathcal{N}(A) \cap \mathcal{N}(C) = \mathcal{N}(A). \tag{3}$$

Proof. Note that, by definition of semiobservability, $\mathcal{N}(A) \cap \mathcal{N}(C) \subseteq \mathcal{N}(A)$. Let $x \in \mathcal{N}(A)$. Then it follows from (2) that $Cx = 0$, and hence, $\mathcal{N}(A) \subseteq \mathcal{N}(A) \cap \mathcal{N}(C)$. Thus, (3) holds. ■

Lemma 2.2 (*[10,9]*). *Consider the switched dynamical system (1). Assume that there exists a family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$,*

$$0 = A_p^{\mathrm{T}} P_p + P_p A_p + R_p, \quad p \in \mathcal{P}, \tag{4}$$

where $R_p = C_p^{\mathrm{T}} C_p$, $C_p \in \mathbb{R}^{l\times n}$, and the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined set of symmetric, nonnegative-definite matrices $\{R_p : p \in \mathcal{P}\}$. Then the following statements hold:

(i) $\mathcal{N}(P_p) \subseteq \mathcal{N}(A_p) \subseteq \mathcal{N}(R_p)$, $p \in \mathcal{P}$.
(ii) $\mathcal{N}(A_p) \cap \mathcal{R}(A_p) = \{0\}$, $p \in \mathcal{P}$.

Proposition 2.1. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$, (4) holds, the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined set of symmetric, nonnegative-definite matrices $\{R_p : p \in \mathcal{P}\}$, and*

$$x^{\mathrm{T}}(t)(P_{\sigma(t)} + L_{\sigma(t)}^{\mathrm{T}} L_{\sigma(t)})x(t) \le x^{\mathrm{T}}(t)(P_{\sigma(t^-)} + L_{\sigma(t^-)}^{\mathrm{T}} L_{\sigma(t^-)})x(t), \quad t \ge 0, \tag{5}$$

where $L_p \triangleq I_n - A_p A_p^{\mathrm{D}}$. Then (1) is Lyapunov stable. If, in addition, $\{A_p : p \in \mathcal{P}\}$ is a compact set, then (1) is uniformly Lyapunov stable.

Proof. Let $p \in \mathcal{P}$. Since, by Lemma 2.2, $\mathcal{N}(A_p) \cap \mathcal{R}(A_p) = \{0\}$, it follows from Lemma 4.14 of [11] that A_p is group invertible. Furthermore, since $L_p^2 = L_p$, L_p is the unique $n \times n$ matrix satisfying $\mathcal{N}(L_p) = \mathcal{R}(A_p)$, $\mathcal{R}(L_p) = \mathcal{N}(A_p)$, and $L_p x = x$ for all $x \in \mathcal{N}(A_p)$.

Consider the multiple nonnegative functions

$$V_p(x) = x^{\mathrm{T}} P_p x + x^{\mathrm{T}} L_p^{\mathrm{T}} L_p x, \quad p \in \mathcal{P},\ x \in \mathbb{R}^n, \tag{6}$$

where P_p satisfies (4). If $V_p(x) = 0$ for some $x \in \mathbb{R}^n$, then $P_p x = 0$ and $L_p x = 0$. It follows from (i) of Lemma 2.2 that $x \in \mathcal{N}(A_p)$, while $L_p x = 0$ implies $x \in \mathcal{R}(A_p)$. Now, it follows from (ii) of Lemma 2.2 that $x = 0$. Hence, the family of functions $V_p(\cdot)$ are positive definite. Now, for every $x_{\mathrm{e}} \in \mathcal{E}$, consider the multiple Lyapunov function candidates $V_p(x - x_{\mathrm{e}})$, $p \in \mathcal{P}$. Note that since $A_p x_{\mathrm{e}} = 0$ for all $p \in \mathcal{P}$, it follows that $x(t) - x_{\mathrm{e}}$, $t \ge 0$, is also a solution of (1). Now, it follows from (5) that

$$V_{\sigma(t)}(x(t) - x_{\mathrm{e}}) \le V_{\sigma(t^-)}(x(t) - x_{\mathrm{e}}), \quad t \ge 0. \tag{7}$$

Next, note that

$$\begin{aligned} \dot V_{\sigma(t)}(x(t) - x_{\mathrm{e}}) &= -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R_{\sigma(t)}(x(t) - x_{\mathrm{e}}) + 2(x(t) - x_{\mathrm{e}})^{\mathrm{T}} L_{\sigma(t)}^{\mathrm{T}} L_{\sigma(t)} A_{\sigma(t)}(x(t) - x_{\mathrm{e}}) \\ &= -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R_{\sigma(t)}(x(t) - x_{\mathrm{e}}) \\ &\le 0, \quad t \ge 0. \end{aligned} \tag{8}$$

Now, it follows from Theorem 2.3 of [12] that (1) is Lyapunov stable. Finally, if $\{A_p : p \in \mathcal{P}\}$ is compact, then $\{L_p^{\mathrm{T}} L_p : p \in \mathcal{P}\}$ is compact. Hence, it follows from Theorem 3 of [5] that (1) is uniformly Lyapunov stable. ■

Proposition 2.2. *Consider the switched dynamical system (1). Assume that every point in $\mathcal{E}$ is Lyapunov stable. Furthermore, assume that for a given $\sigma(t) \in \mathcal{S}$ and $x_0 \in \mathbb{R}^q$, the trajectory of (1) satisfies $x(t) \to \mathcal{E}$ as $t \to \infty$. Then $x(t) \to z$ as $t \to \infty$, where $z \in \mathcal{E}$. Alternatively, assume that every point in $\mathcal{E}$ is uniformly Lyapunov stable and for a given $x_0 \in \mathbb{R}^q$, the trajectory of (1) satisfies $x(t) \to \mathcal{E}$ as $t \to \infty$ uniformly in $\sigma(t) \in \mathcal{S}$. Then $x(t) \to z$ as $t \to \infty$ uniformly in $\sigma(t) \in \mathcal{S}$, where $z \in \mathcal{E}$.*

Proof. Let $x_{\mathrm{e}} \in \mathcal{E}$. Choosing x_0 sufficiently close to x_{e}, it follows from Lyapunov stability of x_{e} that the trajectories of (1) starting sufficiently close to x_{e} are bounded, and hence, there exists an increasing sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} x(t_i)$ exists. Next, since $x(t) \to \mathcal{E}$ as $t \to \infty$, it follows that $\lim_{i\to\infty} x(t_i) \in \mathcal{E}$. Let $z \triangleq \lim_{i\to\infty} x(t_i) \in \mathcal{E}$. We show that $\lim_{t\to\infty} x(t) = z$. Note that, by assumption, $z \in \mathcal{E}$ is a Lyapunov stable equilibrium point. Let $\varepsilon > 0$ and note that since z is Lyapunov stable, it follows that there exists $\delta > 0$ such that $x(t) \in \mathcal{B}_\varepsilon(z)$ for all $x_0 \in \mathcal{B}_\delta(z)$ and $t \ge 0$. Next, since

$z = \lim_{i\to\infty} x(t_i)$, it follows that there exists $k \geq 1$ such that $x(t_k) \in \mathcal{B}_\delta(z)$. We claim that $x(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq t_k$. Suppose, *ad absurdum*, $x(t) \notin \mathcal{B}_\varepsilon(z)$ for some $t \geq t_k$. Then, by continuity of $x(\cdot)$, there exists $\tau_i > t_i$ such that $x(\tau_i) \notin \mathcal{B}_\varepsilon(z)$ for every $i \geq k$. Namely, there exists a divergent sequence $\{\tau_i\}_{i=1}^\infty$ such that $x(\tau_i) \notin \mathcal{B}_\varepsilon(z)$ for all $\tau_i > t_k$. This contradicts the Lyapunov stability of z. Since ε is arbitrary, it follows that $z = \lim_{t\to\infty} x(t)$. The proof of the second assertion is similar and, hence, is omitted. ■

Lemma 2.3. *Let $A \in \mathbb{R}^{n\times n}$. Assume that there exists a symmetric, nonnegative-definite matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$0 = A^{\mathrm{T}}P + PA + R, \tag{9}$$

where $R = C^{\mathrm{T}}C$, $C \in \mathbb{R}^{l\times n}$, and the pair (A, C) is semiobservable. Then $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$ *and, if* $0 \in \operatorname{spec}(A)$, *then 0 is semisimple. Alternatively, assume that there exists a symmetric, positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that (9) holds and*

$$\operatorname{rank}\begin{bmatrix} A - \jmath\omega I_n \\ C \end{bmatrix} = n \tag{10}$$

for every nonzero $\omega \in \mathbb{R}$. Then $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$ *and, if* $0 \in \operatorname{spec}(A)$, *then 0 is semisimple.*

Proof. Consider the dynamical system $\mathcal{G}$ given by $\dot{x} = Ax$. Then it follows from Theorem 2.2 of [9] that $\mathcal{G}$ is semistable. Note that $\mathcal{G}$ is semistable if and only if the matrix A is semistable. Hence, it follows from (ii) of Definition 11.7.1 of [13] that $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$ and, if $0 \in \operatorname{spec}(A)$, then 0 is semisimple. The second assertion is a direct consequence of Corollary 11.8.1 of [13]. ■

Lemma 2.4. *Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If $\operatorname{rank} A < n$ and the pair (A, C) is semiobservable, then there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that*

$$S^{-1}AS = \begin{bmatrix} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}, \qquad CS = \begin{bmatrix} \hat{C}_1 & 0_{l\times 1} \end{bmatrix}, \tag{11}$$

where $\hat{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\hat{C}_1 \in \mathbb{R}^{l\times(n-1)}$. Furthermore, if $\operatorname{rank} A = n - 1$ and the pair (A, C) is semiobservable, then there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & 0_{(n-r-1)\times r} & 0_{(n-r-1)\times 1} \\ A_{21} & A_{22} & 0_{r\times 1} \\ A_{31} & A_{32} & 0_{1\times 1} \end{bmatrix}, \qquad CT = \begin{bmatrix} C_1 & 0_{l\times(r+1)} \end{bmatrix}, \tag{12}$$

where the pair (A_{11}, C_1) is observable, A_{22} is asymptotically stable, $A_{11} \in \mathbb{R}^{(n-r-1)\times(n-r-1)}$, $A_{21} \in \mathbb{R}^{r\times(n-r-1)}$, $A_{22} \in \mathbb{R}^{r\times r}$, $A_{31} \in \mathbb{R}^{1\times(n-r-1)}$, $A_{32} \in \mathbb{R}^{1\times r}$, $[A_{31}, A_{32}] = [0_{1\times(n-3)}, 1, 0_{1\times 1}]U^{-1}$, $U \in \mathbb{R}^{(n-1)\times(n-1)}$ is nonsingular, and $C_1 \in \mathbb{R}^{l\times(n-r-1)}$.

Proof. Since $\operatorname{rank} A < n$, it follows that 0 is an eigenvalue of A. Now, since the pair (A, C) is semiobservable, it follows from Lemma 2.1 that $\mathcal{N}(A) \cap \mathcal{N}(C) = \mathcal{N}(A)$, that is,

$$\mathcal{N}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) = \mathcal{N}(A). \tag{13}$$

Next, it follows from the real Jordan decomposition (Theorem 5.3.5 of [13]) that there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that

$$S^{-1}AS = \begin{bmatrix} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}, \tag{14}$$

where $\hat{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$. Note that $\mathcal{N}(AS) = \mathcal{N}(S^{-1}AS)$ and

$$\mathcal{N}\left(\begin{bmatrix} AS \\ CS \end{bmatrix}\right) = \mathcal{N}(AS). \tag{15}$$

Hence,

$$\mathcal{N}\left(\begin{matrix} S\begin{bmatrix} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix} \\ \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \end{matrix}\right) = \mathcal{N}\left(\begin{bmatrix} \hat{A}_{11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}\right), \tag{16}$$

where $[\hat{C}_1, \hat{C}_2] = CS$. Now, it follows from (16) that $\hat{C}_2 = 0_{l\times 1}$, which implies that (11) holds.

To show the second assertion, consider the pair $(\hat{A}_{11}, \hat{C}_1)$. Then it follows from the Kalman decomposition (Proposition 12.9.11 of [13]) that there exists an invertible matrix $U \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$U^{-1}\hat{A}_{11}U = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad \hat{C}_1 U = \begin{bmatrix} C_1 & 0 \end{bmatrix}. \tag{17}$$

Now, with

$$T \triangleq S \begin{bmatrix} U & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{bmatrix} \tag{18}$$

and $[A_{31}, A_{32}] \triangleq [0_{1\times(n-3)}, 1, 0]U^{-1}$, it follows that (12) holds. ■

3. Semistability of switched linear systems

In this section, we present several sufficient conditions for semistability of switched linear systems.

Theorem 3.1. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, nonnegative-definite matrices such that, for every $\sigma \in \mathcal{S}$, (4) and (5) hold, and the pair (A_p, C_p) is semiobservable for every $p \in \mathcal{P}$ and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. Then the following statements hold:*

(i) *If $\mathcal{S} \subset \mathcal{S}_p[\tau, T]$ for some $\tau > 0$, $0 < T < \infty$, and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (1) is uniformly semistable.*

(ii) *If $\mathcal{S} \subset \bigcup_{\tau>0,0<T\leq\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (1) is semistable.*

Proof. (i) It follows from Proposition 2.1 that (1) is uniformly Lyapunov stable. To show uniform semistability, it follows from Proposition 2.2 that we need to show $x(t) \to \mathcal{E}$ as $t \to \infty$ uniformly in σ. Let $\sigma \in \mathcal{S}$, let $x(t)$, $t \geq 0$, be a solution to (1), and let $\mathcal{T} \triangleq \{t_1, \tau_1, t_2, \tau_2, \ldots, t_k, \tau_k\} \subset (0, t)$ be an increasing sequence of time instants in the interval $(0, t)$ such that the lengths of the intervals $[t_i, \tau_i)$ are no smaller than τ on which $\sigma = p_i$ and the intervals between these have length no larger than T, that is, $\tau_i \geq t_i + \tau$ for $i \in \{1, 2, \ldots, k\}$, $t_{i+1} \leq \tau_i + T$ for $i \in \{1, 2, \ldots, k-1\}$, $t \leq \tau_k + T$, and $t_1 \leq T$. Next, it follows from Lemma 2.3 and Assumption 1 that $\mathrm{spec}(A_p) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and 0 is semisimple for every $p \in \mathcal{P}$. Now, it follows from Lemma 2.4 that there exists an invertible matrix $S_p \in \mathbb{R}^{n\times n}$ such that, with $[x_a^{\mathrm{T}}, x_s]^{\mathrm{T}} = S_p x$, (1) can be transformed into the form

$$\begin{bmatrix} \dot{x}_a \\ \dot{x}_s \end{bmatrix} = \begin{bmatrix} \hat{A}_{p11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_a \\ x_s \end{bmatrix}, \qquad y = \begin{bmatrix} \hat{C}_{p1} & 0_{l\times 1} \end{bmatrix} \begin{bmatrix} x_a \\ x_s \end{bmatrix}, \tag{19}$$

where $x_a \in \mathbb{R}^{n-1}$, $x_s \in \mathbb{R}$, and $\hat{A}_{p11}$ is asymptotically stable. Since $\hat{A}_{p11}$ is asymptotically stable, it follows that $\|e^{\hat{A}_{p11}t}\| < 1$ for every $t > 0$ and $p \in \mathcal{P}$.

Let $\mathcal{J}$ be the set of all sequences $p_1, p_2, \ldots, p_q \in \mathcal{P}$ with length of at most $\lceil T/\tau \rceil$, where $\lceil\cdot\rceil$ is a ceiling function defined by $\lceil x \rceil \triangleq \min\{n \in \mathbb{Z} : x \leq n\}$. Define

$$\mu \triangleq \max_{\tau_1\in[\tau,\tau+T]} \max_{\tau_2\in[\tau,\tau+T]} \cdots \max_{\tau_q\in[\tau,\tau+T]} \max_{\mathcal{J}} \|e^{\hat{A}_{p_q11}\tau_q} \cdots e^{\hat{A}_{p_211}\tau_2} e^{\hat{A}_{p_111}\tau_1}\|. \tag{20}$$

Note that $\mathcal{J}$ is a finite set and $[\tau, \tau + T]$ is compact. Hence, it follows that

$$\mu \leq \max_{\mathcal{J}} \prod_{i=1}^{q} \max_{\tau_i\in[\tau,\tau+T]} \|e^{\hat{A}_{p_i11}\tau_i}\| < 1. \tag{21}$$

Next, it follows from (20) that

$$\|e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}\| \leq \mu, \quad i \in \{1, 2, \ldots, k\}. \tag{22}$$

Let $\Phi_\sigma(t, s)$ denote the state transition matrix of $\dot{x}_a = \hat{A}_{\sigma 11}x_a$ and note that

$$\Phi_\sigma(t, 0) = \Phi_\sigma(t, t_k)\Phi_\sigma(t_k, t_{k-1}) \cdots \Phi_\sigma(t_1, 0), \quad t > 0. \tag{23}$$

If $t < T + \tau$, then $\mathcal{T} = \emptyset$. Hence, for $t \geq T + \tau$, it follows that $\Phi_\sigma(t_{i+1}, t_i) = e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}$, $i \in \{1, 2, \ldots, k-1\}$. Hence, it follows from (22) and (23) that

$$\|\Phi_\sigma(t, 0)\| \leq \|\Phi_\sigma(t, t_k)\| \cdot \|\Phi_\sigma(t_k, t_{k-1})\| \cdots \|\Phi_\sigma(t_1, 0)\| \leq \mu^k. \tag{24}$$

Since $x_a(t) = \Phi_\sigma(t, 0)x_a(0)$ and $0 < \mu < 1$, it follows from (24) that $\lim_{t\to\infty} x_a(t) = 0$. Furthermore, since $t_1 \leq T$, and μ and k are independent of the switching signal σ, it follows that $x(t) \to 0$ as $t \to \infty$ uniformly in σ.

Next, note that $\dot{x}_s(t) = [0_{1\times(n-3)}, 1, 0]x_a(t)$, $t \geq 0$. Hence, $x_s(t)$ is continuously differentiable and $\lim_{t\to\infty} \dot{x}_s(t) = 0$ uniformly in σ. Thus, for every $h > 0$,

$$|x_s(t+h) - x_s(t)| \leq h|\dot{x}(\xi)|, \quad t < \xi < t+h, \tag{25}$$

which implies that $\lim_{t\to\infty} |x_s(t+h) - x_s(t)| = 0$ uniformly in σ, and hence, $\lim_{t\to\infty} x_s(t)$ exists. Let $\lim_{t\to\infty} x_s(t) = \alpha_s \in \mathbb{R}$. Now, since

$$x(t_i + h_i) - x(t_i) = S_{\sigma(t_i)} \begin{bmatrix} x_a(t_i + h_i) - x_a(t_i) \\ x_s(t_i + h_i) - x_s(t_i) \end{bmatrix}, \tag{26}$$

where $0 < h_i < t_{i+1} - t_i$, $i \in \overline{\mathbb{Z}}_+$, and $\{S_p : p \in \mathcal{P}\}$ is compact, it follows that $\lim_{i\to\infty} \|x(t_i + h_i) - x(t_i)\| = 0$. Furthermore, since for $i \in \overline{\mathbb{Z}}_+$,

$$x(t_{i+1}^-) - x(t_i) = S_{\sigma(t_{i+1}^-)} \begin{bmatrix} x_a(t_{i+1}^-) \\ x_s(t_{i+1}^-) \end{bmatrix} - S_{\sigma(t_i)} \begin{bmatrix} x_a(t_i) \\ x_s(t_i) \end{bmatrix} = S_{\sigma(t_i)} \begin{bmatrix} x_a(t_{i+1}) - x_a(t_i) \\ x_s(t_{i+1}) - x_s(t_i) \end{bmatrix},$$

it follows that $\lim_{i\to\infty} \|x(t_{i+1}) - x(t_i)\| = 0$. Hence, for every $t \geq 0$ and $h > 0$, it follows that

$$x(t+h) - x(t) = x(t+h) - x(t_{i+j}) + \sum_{k=0}^{j-1} x(t_{i+k}) - x(t_{i+k-1}) + x(t_{i-1}) - x(t), \tag{27}$$

where $t_{i-1} < t \leq t_i < t_{i+1} < \cdots < t_{i+j} < t+h \leq t_{i+j+1}$. Hence,

$$\|x(t+h) - x(t)\| \leq \|x(t+h) - x(t_{i+j})\| + \sum_{k=0}^{j-1} \|x(t_{i+k}) - x(t_{i+k-1})\| + \|x(t) - x(t_{i-1})\|,$$

which implies that $\lim_{t\to\infty} \|x(t+h) - x(t)\| = 0$, and hence, $\lim_{t\to\infty} x(t)$ exists. Let $\lim_{t\to\infty} x(t) = \beta \in \mathbb{R}^n$. Note that this convergence is also uniform in σ.

Define $z_\sigma \triangleq S_\sigma^{-1}[0_{1\times(n-1)}, \alpha_s]^{\mathrm{T}}$. Then $x(t) - z_{\sigma(t)} = S_{\sigma(t)}^{-1}[x_a^{\mathrm{T}}(t), x_s(t) - \alpha_s]^{\mathrm{T}}$. Since the set $\{S_p^{-1} : p \in \mathcal{P}\}$ is compact, it follows that there exists $b > 0$ such that $\|S_p^{-1}\| \leq b$ for all $p \in \mathcal{P}$. Hence,

$$\|x(t) - z_{\sigma(t)}\| \leq b \left\| \begin{bmatrix} x_a^{\mathrm{T}}(t) \\ x_s(t) - \alpha_s \end{bmatrix} \right\|, \quad t \geq 0, \tag{28}$$

which implies that $\lim_{t\to\infty} \|\beta - z_{\sigma(t)}\| = 0$. Hence, $\lim_{t\to\infty} z_{\sigma(t)} = \beta$. Note that $z_\sigma \in \mathcal{N}(A_\sigma)$ for every $\sigma \in \mathcal{S}$. Now, it follows from $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, that $\beta \in \bigcap_{i=0}^{\infty} \mathcal{N}(A_{\sigma(t_i)}) = \bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$, uniformly in σ. Finally, it follows from Proposition 2.2 that (1) is uniformly semistable.

(ii) It follows from Proposition 2.1 that (1) is Lyapunov stable. To show semistability, it follows from Proposition 2.2 that we need to show $x(t) \to \mathcal{E}$ as $t \to \infty$. Let $\sigma \in \mathcal{S}$ and let $x(t)$, $t \geq 0$, be a solution to (1). Then $\sigma \in \mathcal{S}_p[\tau, T]$ for some $\tau > 0$ and $T \leq \infty$. However, τ and T are not uniform over all switching signals $\sigma(\cdot)$. If $T = \infty$, then it follows that there exists a switching time instant $t_m < \infty$ such that $x(t)$ is continuously differentiable for all $t > t_m$. In this case, it follows from Lemma 2.3 that $x(t) \to \mathcal{E}$ as $t \to \infty$.

Now we consider the case where $T < \infty$. Let $\mathcal{T} \triangleq \{t_1, \tau_1, t_2, \tau_2, \ldots, t_k, \tau_k\} \subset (0, t)$ be as defined in (i). Next, it follows from Lemma 2.4 that there exists an invertible matrix $T_p \in \mathbb{R}^{n\times n}$ such that with $[x_o^{\mathrm{T}}, x_u^{\mathrm{T}}, x_s]^{\mathrm{T}} = T_p x$, (1) can be transformed into the form

$$\begin{bmatrix} \dot{x}_o \\ \dot{x}_u \\ \dot{x}_s \end{bmatrix} = \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} & 0_{(n-r-1)\times 1} \\ A_{p21} & A_{p22} & 0_{r\times 1} \\ A_{p31} & A_{p32} & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_o \\ x_u \\ x_s \end{bmatrix}, \qquad y = \begin{bmatrix} C_{p1} & 0_{l\times(r+1)} \end{bmatrix} \begin{bmatrix} x_o \\ x_u \\ x_s \end{bmatrix}, \tag{29}$$

where $x_o \in \mathbb{R}^{n-r-1}$, $x_u \in \mathbb{R}^r$, $x_s \in \mathbb{R}$, $y \in \mathbb{R}^l$, the pair (A_{p11}, C_{p1}) is observable, and A_{p22} is asymptotically stable. Since (A_{p11}, C_{p1}) is observable, it follows from Lemma 1 of [14] that for $\lambda, \delta > 0$ there exists a matrix $K_p \in \mathbb{R}^{(n-r-1)\times l}$ such that $\|e^{(A_{p11}+K_pC_{p1})t}\| \leq \delta e^{-\lambda(t-\tau)}$, $t \geq \tau$, $p \in \mathcal{P}$.

Now, consider $\dot{x}_o = (A_{\sigma 11} + K_\sigma C_{\sigma 1})x_o - K_\sigma y$. First, we show that $\int_0^\infty \|y(t)\|^2 \mathrm{d}t < \infty$. Note that it follows from (8) that $\dot{V}_{\sigma(t)}(x(t)) = -x^{\mathrm{T}}(t)C_{\sigma(t)}^{\mathrm{T}}C_{\sigma(t)}x(t) = -\|y(t)\|^2$. Hence, $\int_0^\infty \|y(t)\|^2 \mathrm{d}t \leq V_{\sigma(0)}(x(0)) < \infty$. Next, note that

$$x_o(t) = e^{(A_{p11}+K_pC_{p1})t}x_o(\tau_k) - \int_{\tau_k}^{t} e^{(A_{p11}+K_pC_{p1})(t-s)}K_p y(s)\mathrm{d}s, \quad t \in [\tau_k, t_{k+1}). \tag{30}$$

Hence, for every $t \in [\tau_k, t_{k+1})$, it follows from the Cauchy–Schwarz inequality that

$$\|x_o(t)\| \leq \delta e^{-\lambda(t-\tau)}\|x_o(\tau_k)\| + \alpha \left(\int_{\tau_k}^{t} \|y(s)\|^2 \mathrm{d}s \right)^{1/2}, \tag{31}$$

where $\alpha \triangleq (\int_0^\infty \|e^{(A_{\sigma 11}+K_\sigma C_{\sigma 1})s}K_\sigma\|^2 ds)^{1/2} < \infty$ since $\{A_p : p \in \mathcal{P}\}$ and $\{C_p : p \in \mathcal{P}\}$ are compact. Since (1) is Lyapunov stable, $\|x_o(t)\|$, $t \geq 0$, is bounded.

Next, we show that $\lim_{t\to\infty} x_o(t) = 0$. Suppose, *ad absurdum*, $x_o(t) \not\to 0$ as $t \to \infty$. Then $\lim_{t\to\infty} x_o(t) = \nu \neq 0$ or $\liminf_{t\to\infty} x_o(t) \neq \limsup_{t\to\infty} x_o(t)$. Note that τ_k was chosen so that $\tau_k \to \infty$ as $t \to \infty$. Since $\int_0^\infty \|y(t)\|^2 dt < \infty$, it follows that $\lim_{\tau_k\to\infty} \int_{t_k}^\infty \|y(t)\|^2 dt = 0$. Hence, $\lim_{t\to\infty} \int_{\tau_k}^t \|y(s)\|^2 ds = 0$. Thus, if $\lim_{t\to\infty} x_o(t) = \nu \neq 0$, then by taking the limit on both sides of (31), it follows that $\|\nu\| \leq \delta\|\nu\|$, which is a contradiction since δ is arbitrary. Next, let $a \triangleq \liminf_{t\to\infty} \|x_o(t)\|$ and $b \triangleq \limsup_{t\to\infty} \|x_o(t)\|$ and note that $0 \leq a < b < \infty$. Choose an unbounded sequence $\{\eta_n\}_{n=1}^\infty$ with $\tau_k \leq \eta_{n_k} < t_{k+1}$ so that $\limsup_{n\to\infty} \|x_o(\eta_n)\| = b$. By taking $t = \eta_{n_k}$ in (31) and $n_k \to \infty$, it follows that $b \leq \delta b$, which is a contradiction since δ is arbitrary. Thus, $\lim_{t\to\infty} x_o(t) = 0$.

Next, since $U_p^{-1}[0, x_u^T]^T$ belongs to the unobservable subspace of the pair $(\hat{A}_{p11}, \hat{C}_{p1})$, where $U_p \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the Kalman transformation matrix of the pair $(\hat{A}_{p11}, \hat{C}_{p1})$, and $\hat{A}_{p11}$ and $\hat{C}_{p1}$ are given by (19), it follows that $U_p^{-1}[0, x_u^T]^T$ belongs to the smallest subspace $\mathcal{M}$ that is $\hat{A}_{p11}$-invariant[1] for all $p \in \mathcal{P}$ and contains the unobservable subspaces of all pairs $(\hat{A}_{p11}, \hat{C}_{p1})$, $p \in \mathcal{P}$. Since $\hat{A}_{p11}$ is a full rank matrix, it follows that $\mathcal{M} = \{0\}$. Hence, $\lim_{t\to\infty} x_u(t) = 0$.

Note that

$$\begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)} \tag{32}$$

is a full rank matrix and $[A_{p31}, A_{p32}] \in \mathbb{R}^{1\times(n-1)}$. Then it follows that there exists $g_p \in \mathbb{R}^{1\times(n-1)}$ such that

$$[A_{p31}, A_{p32}] = g_p \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix}. \tag{33}$$

Hence,

$$\dot{x}_s(t) = [A_{p31}, A_{p32}] \begin{bmatrix} x_o(t) \\ x_u(t) \end{bmatrix} = g_p \begin{bmatrix} A_{p11} & 0_{(n-r-1)\times r} \\ A_{p21} & A_{p22} \end{bmatrix} \begin{bmatrix} x_o(t) \\ x_u(t) \end{bmatrix} = g_p \begin{bmatrix} \dot{x}_o(t) \\ \dot{x}_u(t) \end{bmatrix}. \tag{34}$$

Now, it follows that

$$x_s(t_i + h_i) - x_s(t_i) = g_{\sigma(t_i)} \begin{bmatrix} x_o(t_i + h_i) - x_o(t_i) \\ x_u(t_i + h_i) - x_u(t_i) \end{bmatrix}, \quad 0 < h_i \leq t_{i+1} - t_i,\ i \in \overline{\mathbb{Z}}_+, \tag{35}$$

which implies that $\lim_{i\to\infty} |x_s(t_i + h_i) - x_s(t_i)| = 0$. Using similar arguments as in the proof of (i), it follows that $\lim_{t\to\infty} |x(t+h) - x(t)| = 0$ for $h > 0$, and hence, $\lim_{t\to\infty} x_s(t)$ exists. The rest of the proof is similar to the proof of (i). ■

Next, we present a stronger result for ensuring semistability for the switched linear system (1).

Theorem 3.2. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $\sigma \in \mathcal{S}$, (4) holds and*

$$x^T(t)P_{\sigma(t)}x(t) \leq x^T(t)P_{\sigma(t^-)}x(t), \quad t \geq 0, \tag{36}$$

for every $p \in \mathcal{P}$ and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Assume that $\{A_p : p \in \mathcal{P}\}$ is compact and $\operatorname{rank} A_p < n$ for every $p \in \mathcal{P}$. Furthermore, assume that there exists an invertible matrix $S_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{P}$, such that (1) can be transformed into (19). If $\mathcal{S} \subset \bigcup_{\tau>0,0<T<\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (1) is semistable.

Proof. The proof is similar to the proof of (ii) of Theorem 3.1 and, hence, is omitted. ■

The next result uses the geometric (rank) condition given in Lemma 2.3 to develop a sufficient condition for semistability.

Theorem 3.3. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $\sigma \in \mathcal{S}$, (4) and (36) hold, and*

$$\operatorname{rank} \begin{bmatrix} A_p - \jmath\omega I_n \\ C_p \end{bmatrix} = n \tag{37}$$

for every nonzero $\omega \in \mathbb{R}$ and every $p \in \mathcal{P}$, and for an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. Then the following statements hold:

(i) *If $\mathcal{S} \subset \mathcal{S}_p[\tau, T]$ for some $\tau > 0$, $0 < T < \infty$, and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (1) is uniformly semistable.*
(ii) *If $\mathcal{S} \subset \bigcup_{\tau>0,0<T<\infty} \mathcal{S}_p[\tau, T]$ and $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]} \mathcal{N}(A_{\sigma(s)})$, $t \geq 0$, then (1) is semistable.*

[1] Given a matrix $A \in \mathbb{R}^{n\times n}$, a subspace $\mathcal{M}$ of $\mathbb{R}^n$ is A-*invariant* if and only if the state of $\dot{x}(t) = Ax(t)$ starting at time τ is such that $x(\tau) \in \mathcal{M}$, then $x(t) \in \mathcal{M}$ for all $t \geq \tau$.

Proof. The proofs of Lyapunov stability and uniform Lyapunov stability are similar to the proof of Proposition 2.1 by considering the family of Lyapunov functions $V_p(x) = x^{\mathrm{T}}P_px$. Next, it follows from Lemma 2.3 and Assumption 1 that $\mathrm{spec}(A_p) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\} \cup \{0\}$ and 0 is semisimple for every $p \in \mathcal{P}$. Now, the proofs of (i) and (ii) are similar to the proofs of (i) and (ii) of Theorem 3.1, respectively. ■

Finally, we develop sufficient conditions for semistability of switched linear systems involving less restrictive hypothesis than those assumed in Theorems 3.1 and 3.3.

Theorem 3.4. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $p \in \mathcal{P}$ and $\sigma \in \mathcal{S}$, (4) and (36) hold, and there exists an infinite sequence of nonempty, bounded, nonoverlapping time-intervals $[t_{i_j}, t_{i_{j+k_j}})$, $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, where t_k denotes the switching time instant, such that the switching times t_k satisfy $t_{k+1} - t_k \geq \tau > 0$, $k \in \overline{\mathbb{Z}}_+$, $t_0 \triangleq 0$, with the property that across each such interval,*

$$\mathrm{rank}\begin{bmatrix} A_{\sigma(t_{i_{j+\ell}})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_{j+\ell}})} \end{bmatrix} = n \tag{38}$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, and an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. If $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i} \mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, then (1) is semistable.

Proof. The proof of Lyapunov stability is similar to the proof of Proposition 2.1 by considering the family of Lyapunov functions $V_p(x) = x^{\mathrm{T}}P_px$. Since A_σ is Lyapunov stable for $\sigma \in \mathcal{S}$, it follows from (i) of Definition 11.7.1 of [13] that $\mathrm{spec}(A_\sigma) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq 0\}$ and, if $\lambda \in \mathrm{spec}(A_\sigma)$ and $\mathrm{Re}\,\lambda = 0$, then λ is semisimple. Since, by assumption, $\bigcap_{p\in\mathcal{P}} \mathcal{N}(A_p) - \{0\} \neq \emptyset$, it follows that there exists $z \in \mathbb{R}^n$, $z \neq 0$, such that $A_{\sigma(t)}z = 0$ for all $t \geq 0$, which further implies that 0 is a common eigenvalue of $A_{\sigma(t)}$ for all $t \geq 0$. Hence, $0 \in \mathrm{spec}(A_\sigma)$ and 0 is semisimple. Then, using similar arguments as in the proof of Lemma 2.4, it follows that for every $\sigma \in \mathcal{S}$ there exists an invertible matrix $S_\sigma \in \mathbb{R}^{n\times n}$ such that the matrix A_σ can be transformed into the form

$$S_\sigma^{-1}A_\sigma S_\sigma = \begin{bmatrix} \hat{A}_{\sigma 11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix}, \tag{39}$$

where $\hat{A}_{\sigma 11} \in \mathbb{R}^{(n-1)\times(n-1)}$ is Lyapunov stable. Furthermore, since

$$\mathrm{rank}\begin{bmatrix} A_{\sigma(t_{i_{j+\ell}})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_{j+\ell}})} \end{bmatrix} = n \tag{40}$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, it follows from Lemma 2.3 that $\hat{A}_{\sigma(t_{i_{j+\ell}})11} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\ell = 0, 1, \ldots, k_j - 1$, is asymptotically stable. Since $\hat{A}_{\sigma 11}$ is Lyapunov stable, it follows from Proposition 11.2.3 of [13] that

$$\|e^{\hat{A}_{\sigma(t_i)11}(t_{i+1}-t_i)}\| \leq 1, \quad i \in \overline{\mathbb{Z}}_+. \tag{41}$$

Moreover, since $\hat{A}_{\sigma(t_{i_{j+\ell}})11} \in \mathbb{R}$ is asymptotically stable, it follows that $\|e^{\hat{A}_{\sigma(t_{i_{j+\ell}})11}t}\| < 1$ for every $t > 0$ and $\ell = 0, 1, \ldots, k_j - 1$.

Consider the switched dynamical system given by

$$\begin{bmatrix} \dot{x}_{\mathrm{a}}(t) \\ \dot{x}_{\mathrm{s}}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{\sigma(t)11} & 0_{(n-1)\times 1} \\ [0_{1\times(n-3)}, 1, 0_{1\times 1}] & 0_{1\times 1} \end{bmatrix} \begin{bmatrix} x_{\mathrm{a}}(t) \\ x_{\mathrm{s}}(t) \end{bmatrix}, \qquad \begin{bmatrix} x_{\mathrm{a}}(0) \\ x_{\mathrm{s}}(0) \end{bmatrix} = S_{\sigma(0)}x(0), \quad t \geq 0. \tag{42}$$

Clearly, $[x_{\mathrm{a}}^{\mathrm{T}}(t), x_{\mathrm{s}}(t)]^{\mathrm{T}} = S_{\sigma(t)}x(t)$, where $x(t)$ denotes the solution of (1). By assumption there exists a finite upper bound T on the lengths of the intervals $[t_{i_j}, t_{i_{j+k_j}})$ across which

$$\mathrm{rank}\begin{bmatrix} A_{\sigma(t_{i_{j+\ell}})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_{j+\ell}})} \end{bmatrix} = n \tag{43}$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$. Since $t_{i+1} - t_i \geq \tau$, $i \geq 0$, it follows that $k_j \leq \lceil T/\tau \rceil$, $j \geq 1$.

Let $\mathcal{J}$ be the set of all sequences $p_1, p_2, \ldots, p_q \in \mathcal{P}$ with length of at most $\lceil T/\tau \rceil$ for which

$$\mathrm{rank}\begin{bmatrix} A_{\sigma(t_{i_{j+\ell}})} - \jmath\omega_\ell I_n \\ C_{\sigma(t_{i_{j+\ell}})} \end{bmatrix} = n \tag{44}$$

for all nonzero $\omega_\ell \in \mathbb{R}$ and every $\ell = 0, 1, \ldots, k_j - 1$, and define

$$\mu \triangleq \max_{\tau_1\in[\tau,T]} \max_{\tau_2\in[\tau,T]} \cdots \max_{\tau_q\in[\tau,T]} \max_{\mathcal{J}} \|\mathrm{e}^{\hat{A}_{p_q 11}} \cdots \mathrm{e}^{\hat{A}_{p_2 11}\tau_2}\mathrm{e}^{\hat{A}_{p_1 11}\tau_1}\|. \tag{45}$$

Note that since $\mathcal{J}$ is a finite set and $[\tau, T]$ is compact, it follows that

$$\mu \le \max_{\mathcal{J}} \prod_{i=1}^{q} \max_{\tau_i\in[\tau,T]} \|\mathrm{e}^{\hat{A}_{p_i 11}\tau_i}\| < 1. \tag{46}$$

Next, it follows from (45) that

$$\|\mathrm{e}^{\hat{A}_{\sigma(t_{i_j+k_j-1})}(t_{i_j+k_j}-t_{i_j+k_j-1})} \cdots \mathrm{e}^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})}\mathrm{e}^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})}\| \le \mu, \quad j \ge 1. \tag{47}$$

Furthermore, note that

$$\begin{aligned}
&\mathrm{e}^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots \mathrm{e}^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})}\mathrm{e}^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})} \\
&\quad = \left(\mathrm{e}^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots \mathrm{e}^{\hat{A}_{\sigma(t_{i_j+k_j})}(t_{i_j+k_j+1}-t_{i_j+k_j})}\right) \\
&\qquad \times \left(\mathrm{e}^{\hat{A}_{\sigma(t_{i_j+k_j-1})}(t_{i_j+k_j}-t_{i_j+k_j-1})} \cdots \mathrm{e}^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})}\mathrm{e}^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})}\right).
\end{aligned} \tag{48}$$

Then it follows from (41) and (47) that

$$\|\mathrm{e}^{\hat{A}_{\sigma(t_{i_{j+1}-1})}(t_{i_{j+1}}-t_{i_{j+1}-1})} \cdots \mathrm{e}^{\hat{A}_{\sigma(t_{i_j+1})}(t_{i_j+2}-t_{i_j+1})}\mathrm{e}^{\hat{A}_{\sigma(t_{i_j})}(t_{i_j+1}-t_{i_j})}\| \le \mu, \quad j \ge 1. \tag{49}$$

Now, it follows from (49) that

$$\|x_{\mathrm{a}}(t_{i_{j+1}})\| \le \mu\|x_{\mathrm{a}}(t_{i_j})\|, \quad j \ge 1. \tag{50}$$

Hence, $\|x_{\mathrm{a}}(t_{i_j})\| \le \mu^{j-1}\|x_{\mathrm{a}}(t_{i_1})\|$, which implies that $\lim_{t\to\infty} x_{\mathrm{a}}(t) = 0$. Furthermore, note that $\dot{x}_{\mathrm{s}}(t) = [0_{1\times(n-3)}, 1, 0]x_{\mathrm{a}}(t)$, $t \ge 0$. Hence, $x_{\mathrm{s}}(\cdot)$ is continuously differentiable and $\lim_{t\to\infty} \dot{x}_{\mathrm{s}}(t) = 0$. Thus, for every $h > 0$,

$$|x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| \le h|\dot{x}(\xi)|, \quad t < \xi < t+h, \tag{51}$$

which implies that $\lim_{t\to\infty} |x_{\mathrm{s}}(t+h) - x_{\mathrm{s}}(t)| = 0$, and hence, $\lim_{t\to\infty} x_{\mathrm{s}}(t)$ exists. Let $\lim_{t\to\infty} x_{\mathrm{s}}(t) = \alpha_{\mathrm{s}} \in \mathbb{R}$.

Next, since

$$x(t_i + h_i) - x(t_i) = S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i+h_i) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i+h_i) - x_{\mathrm{s}}(t_i) \end{bmatrix}, \tag{52}$$

where $0 < h_i < t_{i+1} - t_i$, $i \in \overline{\mathbb{Z}}_+$, and $\{S_p : p \in \mathcal{P}\}$ is compact, it follows that $\lim_{i\to\infty} \|x(t_i + h_i) - x(t_i)\| = 0$. Furthermore, since

$$x(t_{i+1}^-) - x(t_i) = S_{\sigma(t_{i+1}^-)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}^-) \\ x_{\mathrm{s}}(t_{i+1}^-) \end{bmatrix} - S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_i) \end{bmatrix} = S_{\sigma(t_i)} \begin{bmatrix} x_{\mathrm{a}}(t_{i+1}) - x_{\mathrm{a}}(t_i) \\ x_{\mathrm{s}}(t_{i+1}) - x_{\mathrm{s}}(t_i) \end{bmatrix}, \tag{53}$$

$i \in \overline{\mathbb{Z}}_+$, it follows that $\lim_{i\to\infty} \|x(t_{i+1}) - x(t_i)\| = 0$. Hence, for every $t \ge 0$ and $h > 0$, it follows that

$$x(t+h) - x(t) = x(t+h) - x(t_{i+j}) + \sum_{k=0}^{j-1} x(t_{i+k}) - x(t_{i+k-1}) + x(t_{i-1}) - x(t), \tag{54}$$

where $t_{i-1} < t \le t_i < t_{i+1} < \cdots < t_{i+j} < t+h \le t_{i+j+1}$. Hence,

$$\|x(t+h) - x(t)\| \le \|x(t+h) - x(t_{i+j})\| + \sum_{k=0}^{j-1} \|x(t_{i+k}) - x(t_{i+k-1})\| + \|x(t) - x(t_{i-1})\|,$$

which implies that $\lim_{t\to\infty} \|x(t+h) - x(t)\| = 0$, and hence, $\lim_{t\to\infty} x(t)$ exists. Let $\lim_{t\to\infty} x(t) = \beta \in \mathbb{R}^n$.

Define $z_\sigma \triangleq S_\sigma^{-1}[0_{1\times(n-1)}, \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Then $x(t) - z_{\sigma(t)} = S_{\sigma(t)}^{-1}[x_{\mathrm{a}}^{\mathrm{T}}(t), x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}}]^{\mathrm{T}}$. Since the set $\{S_p^{-1} : p \in \mathcal{P}\}$ is compact, it follows that there exists $b > 0$ such that $\|S_p^{-1}\| \le b$ for all $p \in \mathcal{P}$. Hence,

$$\|x(t) - z_{\sigma(t)}\| \le b \left\| \begin{bmatrix} x_{\mathrm{a}}^{\mathrm{T}}(t) \\ x_{\mathrm{s}}(t) - \alpha_{\mathrm{s}} \end{bmatrix} \right\|, \quad t \ge 0, \tag{55}$$

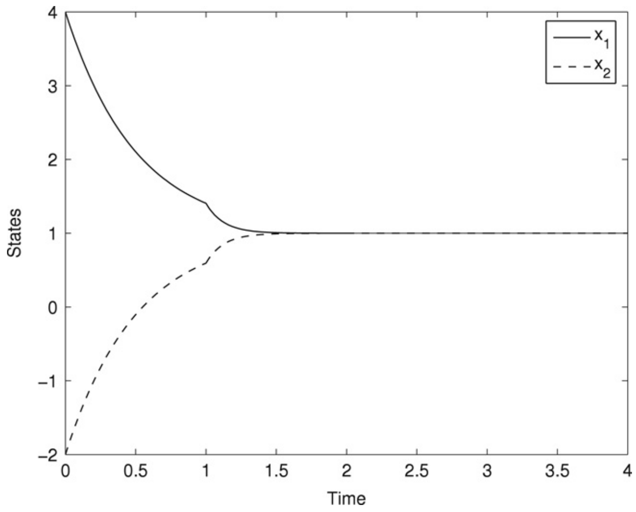

Fig. 1. State trajectories versus time.

which implies that $\lim_{t\to\infty}\|\beta - z_{\sigma(t)}\| = 0$. Hence, $\lim_{t\to\infty} z_{\sigma(t)} = \beta$. Note that $z_\sigma \in \mathcal{N}(A_\sigma)$ for every $\sigma \in \mathcal{S}$. Now, it follows from $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i}\mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, that $\beta \in \bigcap_{i=0}^{\infty}\mathcal{N}(A_{\sigma(t_i)}) = \bigcap_{p\in\mathcal{P}}\mathcal{N}(A_p) = \mathcal{E}$. Hence, $x(t) \to \mathcal{E}$ as $t \to \infty$. Finally, it follows from Proposition 2.2 that (1) is semistable. ■

Theorem 3.5. *Consider the switched dynamical system (1). Assume that there exists a compact family $\{P_p : p \in \mathcal{P}\}$ of symmetric, positive-definite matrices such that, for every $p \in \mathcal{P}$ and $\sigma \in \mathcal{S}$, (4) and (36) hold, and there exists an infinite sequence of nonempty, bounded, nonoverlapping time-intervals $[t_{i_j}, t_{i_j+k_j})$, $i \in \mathbb{Z}_+$, $j \in \mathbb{Z}_+$, where t_k denotes switching time instants, such that the switching times t_k satisfy $t_{k+1} - t_k \geq \tau > 0$, $k \in \overline{\mathbb{Z}}_+$, $t_0 \triangleq 0$, with the property that across each such interval the pair $(A_{\sigma(t_{i_j+\ell})}, C_{\sigma(t_{i_j+\ell})})$ is semiobservable for every $\ell = 0, 1, \ldots, k_j - 1$ and an appropriately defined compact set of matrices $\{C_p : p \in \mathcal{P}\}$. Furthermore, assume that $\{A_p : p \in \mathcal{P}\}$ is compact. If $\mathcal{N}(A_{\sigma(t_i)}) \subseteq \bigcap_{l=0}^{i}\mathcal{N}(A_{\sigma(t_l)})$, $i \in \overline{\mathbb{Z}}_+$, then (1) is semistable.*

Proof. Note that since the pair $(A_{\sigma(t_{i_j+\ell})}, C_{\sigma(t_{i_j+\ell})})$ is semiobservable for every $\ell = 0, 1, \ldots, k_j - 1$, it follows from Lemma 2.3 that $\hat{A}_{\sigma(t_{i_j+\ell})}$ in (39) is asymptotically stable. Now, the rest of the proof is similar to the proof of Theorem 3.4. ■

4. Illustrative numerical example

In this section, we present an example to demonstrate the proposed approach. Specifically, consider the two-agent network consensus problem given by the switched linear system (1) where A_p is given by [1]

$$A_p = p^2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{56}$$

where $p \in \mathcal{P} = \{1, 2\}$. Note that $\bigcap_{p\in\mathcal{P}}\mathcal{N}(A_p) - \{0\} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \neq 0, \alpha \in \mathbb{R}\}$ and $\mathcal{E} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. Let

$$C_p = p[1, -1]. \tag{57}$$

It is easy to verify that (4) holds with

$$P_p = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{58}$$

Furthermore, note that $\mathcal{N}(C_p) \cap \mathcal{N}(A_p) = \mathcal{N}(A_p)$, and hence, the pair (A_p, C_p) is semiobservable.

Next, let $V(x) = \frac{1}{2}x^{\mathrm{T}}x$. Then $\dot{V}(x) = 0$ for all $x \in \mathbb{R}^2$, and hence, (36) holds. Clearly, $\mathcal{N}(A_{\sigma(t)}) \subseteq \bigcap_{s\in[0,t]}\mathcal{N}(A_{\sigma(s)})$. Thus, if we take the switching time instant to be $t_k = k$, $k \in \mathbb{Z}_+$, then it follows from Theorem 3.1 that (1) is semistable. Fig. 1 shows the state trajectories versus time for $x(0) = [4, -2]$.

5. Conclusion

This paper extends the notions of uniform asymptotic stability of switched linear systems to uniform semistability of switched linear systems. In particular, semistability and uniform semistability are established using multiple Lyapunov functions. In Part II of this paper [8] we extend this theory to nonlinear switched systems.

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