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Implications of dissipativity, inverse optimal control, and stability margins for nonlinear stochastic feedback regulators

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Summary

In this paper, we derive stability margins for optimal and inverse optimal stochastic feedback regulators. Specifically, gain, sector, and disk margin guarantees are obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion with cross-weighting terms. Furthermore, using the newly developed notion of stochastic dissipativity, we derive a return difference inequality to provide connections between stochastic dissipativity and optimality of nonlinear controllers for stochastic dynamical systems. In particular, using extended Kalman-Yakubovich-Popov conditions characterizing stochastic dissipativity, we show that our optimal feedback control law satisfies a return difference inequality predicated on the infinitesimal generator of a controlled Markov diffusion process if and only if the controller is stochastically dissipative with respect to a specific quadratic supply rate.

KEYWORDS

controlled Markov diffusion processes, disk margins, inverse optimality, stochastic control, stochastic dissipativity

1 | INTRODUCTION

In a recent paper by Rajpurohit and Haddad,[1] the authors presented a framework for analyzing and designing feedback controllers for nonlinear stochastic dynamical systems. Specifically, a stochastic feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional was considered and the performance functional was evaluated in closed form as long as the nonlinear-nonquadratic cost functional considered was related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability in probability of the nonlinear closed-loop system. Furthermore, the Lyapunov function was shown to be the solution of the steady-state stochastic Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending stochastic linear-quadratic control to nonlinear-nonquadratic problems.

The approach in the work of the aforementioned authors[1] focused on the role of the Lyapunov function guaranteeing stochastic stability of the closed-loop system and its connection to the steady-state solution of the stochastic Hamilton-Jacobi-Bellman equation characterizing the optimal nonlinear feedback controller. In order to avoid the complexity in solving the stochastic steady-state, Hamilton-Jacobi-Bellman equation, we do not attempt to minimize a given *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that minimizes a *derived* cost functional that provides the flexibility in specifying the control law. This corresponds to addressing an *inverse optimal stochastic control problem*.[1-9]

The inverse optimal control design approach provides a framework for constructing the Lyapunov function for the closed-loop system that serves as an optimal value function and, as shown in Freeman and Kokotović[6] and Sepulchre et al[7] for deterministic systems, achieves desired stability margins. Specifically, nonlinear inverse optimal controllers that minimize a *meaningful* (in the terminology of the aforementioned works[6,7]) nonlinear-nonquadratic performance criterion involving a nonlinear-nonquadratic nonnegative-definite function of the state and a quadratic positive-definite function of the feedback control are shown to possess sector margin guarantees to component decoupled input nonlinearities in the conic sector $(\frac{1}{2}, \infty)$.

Using the framework developed in Rajpurohit and Haddad,[1] in this paper, we derive stability margins for optimal and inverse optimal nonlinear *stochastic* feedback regulators. Specifically, sufficient conditions for gain, sector, and disk margin guarantees are obtained for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal Hamilton-Jacobi-Bellman controllers that minimize a nonlinear-nonquadratic performance criterion *with* cross-weighting terms. In the case where the cross-weighting term in the performance criterion is deleted our results recover the gain, sector, and disk margins for the deterministic optimal control problem presented in Sepulchre et al.[7]

Alternatively, retaining the cross-terms in the performance criterion and specializing the optimal nonlinear-nonquadratic problem to a stochastic linear-quadratic problem with a multiplicative noise disturbance, our results recover the analogous gain and phase margins for the deterministic linear-quadratic optimal control problem given in Chung et al.[10] Even though the inclusion of cross-weighting terms in the performance criterion is shown to degrade gain, sector, and disk margins, the extra flexibility provided by the cross-weighting terms makes it possible to guarantee optimal and inverse optimal nonlinear controllers that may be far superior in terms of transient performance over meaningful inverse optimal controllers.

Finally, using the newly developed notion of stochastic dissipativity[11,12] for controlled Markov diffusion processes characterized via extended Kalman-Yakubovich-Popov conditions in terms of the drift and diffusion dynamics developed in Rajpurohit and Haddad,[12] we provide explicit connections between stochastic stability margins, stochastic meaningful inverse optimality, and stochastic dissipativity with respect to a specific quadratic supply rate. In particular, we derive a stochastic counterpart to the classical return difference inequality for continuous-time systems with continuously differentiable flows[3,13] for stochastic dynamical systems and provide connections between stochastic dissipativity and optimality for stochastic nonlinear controllers. In particular, we show an equivalence between stochastic dissipativity and optimality holds for stochastic dynamical systems. Specifically, we show that an optimal nonlinear feedback controller $\phi(x)$ satisfying a return difference condition predicated on the infinitesimal generator of a controlled Markov diffusion process is equivalent to the fact that the stochastic dynamical system with input u and output $y = -\phi(x)$ is stochastically dissipative with respect to a supply rate of the form $[u + y]^{\mathrm{T}}[u + y] - u^{\mathrm{T}}u$.

2 | NOTATION, DEFINITIONS, AND MATHEMATICAL PRELIMINARIES

In this section, we establish notation, definitions, and review some basic results on stability of nonlinear stochastic dynamical systems.[14-18] Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm or an induced matrix norm (depending on context), $\sigma_{\max}(\cdot)$ (respectively, $\sigma_{\min}(\cdot)$) for the maximum (respectively, minimum) singular value, $M \geq 0$ (respectively, $M > 0$) to denote that the Hermitian matrix M is nonnegative (respectively, positive) definite, A^{T} for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$.[16] Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure,[17p10] with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Wiener process $w(t)$ up to time t. We denote by $\mathcal{G}$ a stochastic dynamical system generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega : x(t, \omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all

Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here, we use the notation $x(t)$ to represent the stochastic process $x(t, \omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ over the semi-infinite parameter space $[0, \infty)$ by $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n \times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant a.s. All inequalities and equalities involving random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$-a.s. Furthermore, $\mathbb{E}[\cdot]$ and $\mathbb{E}^{x_0}[\cdot]$ denote, respectively, the expectation with respect to the probability measure $\mathbb{P}$ and with respect to the classical Wiener measure $\mathbb{P}^{x_0}$.

A stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ is called a *martingale* with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if $x(t)$ is a $\mathcal{F}_t$-measurable random vector for all $t \geq 0$, $\mathbb{E}[x(t)] < \infty$, and $x(\tau) = \mathbb{E}[x(t)|\mathcal{F}_\tau]$ for all $t \geq \tau \geq 0$, where, for a given $x \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and a σ-algebra $\mathcal{E} \subseteq \mathcal{F}$, $\mathbb{E}[x|\mathcal{E}]$ denotes conditional expectation with all moments taken under the measure $\mathbb{P}$. If we replace the equality in the aforementioned equation with "$\leq$" (respectively, "$\geq$"), then $x(\cdot)$ is a *supermartingale* (respectively, *submartingale*). A random variable $\tau : \Omega \to [0, \infty]$ is called a *stopping time* with respect to $\mathcal{F}_t$ if and only if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$, $t \geq 0$.

Finally, we write $\mathrm{tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x) \triangleq \frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$, ie, $\mathcal{H}_n \triangleq \{x : \Omega \to \mathbb{R}^n\}$. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n : x : \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n : x \overset{\text{a.s.}}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V : \mathcal{D} \to \mathbb{R}$ that are two-times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0, \tag{1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (ie, Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0))$, $t \geq t_0$, and $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ are continuous functions and satisfy $f(0) = 0$ and $D(0) = 0$. The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x : [t_0, \tau] \times \Omega \to \mathcal{D}$ is said to be a *solution* of (1) on the time interval $[t_0, \tau]$ with initial condition $x(t_0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (ie, $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$, $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\mathrm{d}s + \int_{t_0}^{t} D(x(s))\mathrm{d}w(s) \qquad \text{a.s.}, \quad t \in [t_0, \tau], \tag{2}$$

where the integrals in (2) are Itô integrals.

Note that, for each fixed $t \geq t_0$, the random variable $\omega \mapsto x(t, \omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t, \omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq t_0$. A pathwise solution $t \mapsto x(t)$ of (1) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal pathwise solutions to (1) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume that (1) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* of (1) are given in Corollary 6.3.5 in the work of Arnold.[16]

Furthermore, we assume that $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\|, \quad x, y \in \mathcal{D}, \tag{3}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2), \quad x \in \mathcal{D}, \tag{4}$$

for some Lipschitz constant $L > 0$, and hence, since $x(t_0) \in \mathcal{H}_n^{\mathcal{D}}$ and $x(t_0)$ are independent of $(w(t)-w(t_0))$, $t \geq t_0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ of (1) in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}} \setminus \{0\}$, there exists $\tau_x > 0$ such that if $x_1 : [t_0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [t_0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (1); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, with continuous sample paths a.s., solve (1), and then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}(x_1(t) = x_2(t), \quad t_0 \leq t \leq \tau_x) = 1$.

A weaker sufficient condition for the existence of a unique solution to (1) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (3) without the growth condition (4) is given in Wu et al.[19] Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by (see Theorem 3.4 of Khasminskii[18] and Theorem 9.2.8 of Arnold[16])

$$\mathbb{P}(x(t) \in \mathcal{B} : x(t_0) \stackrel{\text{a.s.}}{=} x_0) = \mathbb{P}(t - t_0, x_0, 0, \mathcal{B}), \quad x_0 \in \mathbb{R}^n, \tag{5}$$

for all $t \geq t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(s, x, t, \mathcal{B})$, $t \geq s$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Finally, recall that every continuous process with Feller transition probability function is also a strong Markov process.[18p101]

Definition 1 (See Definition 7.7 of Øksendal[17]). Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then, the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq t_0$, with $x(t_0) \stackrel{\text{a.s.}}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D}. \tag{6}$$

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq t_0$, satisfies (1), then the limit in (6) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq t_0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (1) and is given by (see Theorem 7.9 of Øksendal[17])

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \operatorname{tr} D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D}. \tag{7}$$

The following definition introduces the notions of Lyapunov and asymptotic stability in probability. Recall that an *equilibrium point* $x_{\mathrm{e}} = 0$ of (1) is a point such that $f(0) = 0$ and $D(0) = 0$. In this case, $x_{\mathrm{e}} = 0$ is an equilibrium point of (1) if and only if the zero solution (ie, the zero stochastic process), $x(\cdot) \stackrel{\text{a.s.}}{=} 0$ is a solution of (1).

Definition 2 (Kushner[14]). (i) The zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ to (1) is *Lyapunov stable in probability* if, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geq t_0} \|x(t)\| > \varepsilon\right) \leq \rho. \tag{8}$$

(ii) The zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ to (1) is *locally asymptotically stable in probability* if it is Lyapunov stable in probability and, for every $\rho \in (0, 1)$, there exist $\delta = \delta(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t)\| = 0\right) \geq 1 - \rho. \tag{9}$$

(iii) The zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ to (1) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t)\| = 0\right) = 1. \tag{10}$$

Remark 1. A more general stochastic stability notion can also be introduced here involving stochastic stability and convergence to an invariant (stationary) distribution. In this case, state convergence is not to an equilibrium point but rather to a stationary distribution. This framework can relax the vanishing perturbation assumption $D(0) = 0$ and requires a more involved analysis and synthesis framework showing stability of the underlying Markov semigroup.[20]

Finally, we provide sufficient conditions for local and global asymptotic stability in probability for the nonlinear stochastic dynamical system (1).

Theorem 1 (See Theorem 5.3 and Corollary 5.1 of Khasminskii[18]). *Consider the nonlinear stochastic dynamical system (1) and assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that*

$$V(0) = 0, \tag{11}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{12}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\ D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x) \leq 0, \qquad x \in \mathcal{D}. \tag{13}$$

Then, the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv}$ to (1) is Lyapunov stable in probability. If, in addition,

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\, D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x) < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{14}$$

then the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ to (1) is asymptotically stable in probability. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ to (1) is globally asymptotically stable in probability.

3 | DISSIPATIVITY THEORY FOR STOCHASTIC SYSTEMS

In this section, we present several key results on stochastic dissipativity developed in Wu et al[11] and Rajpurohit and Haddad[12] that are necessary for the main results of this paper. Specifically, we consider nonlinear stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \quad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{15}$$

$$y(t) = H(x(t), u(t)), \tag{16}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $y(t) \in \mathcal{H}_l^Y$, $Y \subseteq \mathbb{R}^l$, $F : \mathcal{D} \times U \to \mathbb{R}^n$, $D : \mathcal{D} \times U \to \mathbb{R}^{n \times d}$, and $H : \mathcal{D} \times U \to Y$. For the dynamical system $\mathcal{G}$ given by (15) and (16) defined on the state space $\mathcal{H}_n^{\mathcal{D}}$, $\mathcal{U}$ and $\mathcal{Y}$ define an input and output space, respectively, consisting of measurable bounded $\mathcal{H}_m^U$-valued and $\mathcal{H}_l^Y$-valued stochastic processes on the semi-infinite interval $[0, \infty)$. The set $\mathcal{H}_m^U$ contains the set of input values with measurable sample paths satisfying a *nonanticipativity condition*, ie, for every $u(\cdot) \in \mathcal{U}$ and $t \in [t_0, \infty)$, $u(t) \in \mathcal{H}_m^U$, and for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau), w(\tau), \tau \leq s$, and $x(t_0)$. The set $\mathcal{H}_l^Y$ contains the set of output values, ie, for every $y(\cdot) \in \mathcal{Y}$ and $t \in [0, \infty)$, $y(t) \in \mathcal{H}_l^Y$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator, ie, if $u(\cdot) \in \mathcal{U}$ (respectively, $y(\cdot) \in \mathcal{Y}$), then the function defined by $u_T \triangleq u(t+T)$ (respectively, $y_T \triangleq y(t+T)$) is contained in $\mathcal{U}$ (respectively, $\mathcal{Y}$) for all $T \geq 0$.

Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$, we assume that the conditions for existence and uniqueness of solutions are satisfied, ie, $u(\cdot)$ satisfies sufficient regularity conditions such that the system (15) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (15) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ such that $u(\cdot)$ is nonanticipative and takes values in a compact metrizable set $\mathcal{U}$. Furthermore, we assume the uniform Lipschitz continuity and growth conditions (3) and (4) hold for the controlled drift and diffusion terms $F(x, u)$ and $D(x, u)$ uniformly in u. In this case, it follows from Theorem 2.2.4 of Arapostathis et al[21] that there exists a pathwise unique solution to (15) in $(\Omega, \{\mathcal{F}_{t \geq t_0}\}, \mathbb{P}^{x_0})$.

For the stochastic dynamical system $\mathcal{G}$ given by (15) and (16), a function $r : \mathcal{H}_m^U \times \mathcal{H}_l^Y \to \mathcal{H}_1$ such that $r(0, 0) \stackrel{\text{a.s.}}{=} 0$ is called a *supply rate* if $r(u(t), y(t))$ is locally Lebesgue integrable for all input-output pairs satisfying (15) and (16), ie, for all input-output pairs $u(\cdot) \in \mathcal{U}$ and $y(\cdot) \in \mathcal{Y}$ satisfying (15) and (16), $r(\cdot, \cdot)$ satisfies $\mathbb{E}[\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s] < \infty$, $t_1, t_2 \geq 0$.

Definition 3. A stochastic dynamical system $\mathcal{G}$ of the form (15) and (16) is *stochastically dissipative with respect to the supply rate* $r(u, y)$ if there exists a measurable and nonnegative function $V_{\mathrm{s}} : \mathcal{D} \to \mathbb{R}$, called a *storage function* for $\mathcal{G}$, such that $V_s(0) = 0$ and $V_s(x(t)) - \int_{t_0}^{t} r(u(s), y(s))\mathrm{d}s$, $t \geq t_0$, is a $\mathcal{F}_t$-supermartingale for all $t_0, t \geq 0$, where $x(t)$, $t \geq t_0$, is the solution of (15) with $u(\cdot) \in \mathcal{U}$; or, equivalently,

$$\mathbb{E}[V_s(x(t))|\mathcal{F}_{\tau_0}] \leq V_s(x(\tau_0)) + \mathbb{E}\left[\int_{\tau_0}^{\tau} r(u(s), y(s))\mathrm{d}s \middle| \mathcal{F}_{\tau_0}\right], \quad \tau \overset{\text{a.s.}}{\geq} \tau_0, \tag{17}$$

where τ and τ_0 are finite $\mathcal{F}_t$-stopping times.

Definition 4. A nonlinear stochastic dynamical system $\mathcal{G}$ is *completely stochastically reachable* if, for all $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, there exist a finite random variable $\tau_{\mathcal{B}_\varepsilon(x_0)} \overset{\text{a.s.}}{\geq} 0$, called the *first hitting time*, defined by $\tau_{\mathcal{B}_\varepsilon(x_0)}(\omega) \triangleq \inf\{t \geq 0 : x(t,\omega) \in \mathcal{B}_\varepsilon(x_0)\}$, and a square integrable input $u(t)$ defined on $[0, \tau_{\mathcal{B}_\varepsilon(x_0)}]$ such that the state $x(t)$, $t \geq 0$, can be driven from $x(0) \overset{\text{a.s.}}{=} 0$ to $x(\tau_{\mathcal{B}_\varepsilon(x_0)})$ and $\mathbb{E}[\tau_{x_0}] < \infty$, where $\tau_{x_0} \triangleq \sup_{\varepsilon>0}\tau_{\mathcal{B}_\varepsilon(x_0)}$ and the supremum is taken pointwise. A nonlinear stochastic dynamical system $\mathcal{G}$ is *zero-state observable* if $u(t) \overset{\text{a.s.}}{\equiv} 0$ and $y(t) \overset{\text{a.s.}}{\equiv} 0$ implies $x(t) \overset{\text{a.s.}}{\equiv} 0$.

If $V_s(\cdot)$ is two-times continuously differentiable, then an equivalent statement for the stochastic dissipativeness of $\mathcal{G}$ with respect to the supply rate $r(u, y)$ can be characterized by the infinitesimal generator $\mathcal{L}$.

Proposition 1 (Wu et al[11] and Rajpurohit and Haddad[12]). *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (15) and (16). If $V_s : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is two-times continuously differentiable and has a compact support, then $\mathcal{G}$ is stochastically dissipative with respect to supply rate $r(\cdot,\cdot)$ if and only if*

$$\begin{aligned}\mathcal{L}V_s(x) &\triangleq \frac{\partial V(x)}{\partial x}F(x,u) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x,u)\frac{\partial^2 V(x)}{\partial x^2}D(x,u)\\ &\leq r(u, H(x,u)), \quad (x,u) \in \mathcal{D} \times U.\end{aligned} \tag{18}$$

Next, we show that stochastic dissipativeness of nonlinear affine stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{19}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{20}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$ is an open set with $0 \in U$, $y(t) \in \mathcal{H}_l^Y$, $Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $D : \mathcal{D} \to \mathbb{R}^{n\times d}$, $h : \mathcal{D} \to \mathbb{R}^l$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$, can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$. We assume that $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuously differentiable mappings and $\mathcal{G}$ has at least one equilibrium so that, without loss of generality, $f(0) = 0$, $D(0) = 0$, and $h(0) = 0$. Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$, we assume that the required properties for the existence and uniqueness of solutions in forward time are satisfied.

For the following result, we consider the special case of dissipative systems with quadratic supply rates.[22-25] Specifically, we set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$; let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l\times m}$ be given, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices, and assume $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$, so that, as shown by Theorem 3.2 in Rajpurohit and Haddad,[12] all storage functions for $\mathcal{G}$ are positive definite. Moreover, we assume that there exists a two-times continuously differentiable storage function $V_s(x)$, $x \in \mathbb{R}^n$, for the stochastic dynamical system $\mathcal{G}$.

Theorem 2 (Rajpurohit and Haddad[12]). *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u, y) = y^TQy + 2y^TSu + u^TRu$ if and only if there exist functions $V_s : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_s(\cdot)$ is two-times continuously differentiable and positive definite, $V_s(0) = 0$, and, for all $x \in \mathbb{R}^n$,*

$$0 = V_s'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V_s''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{21}$$

$$0 = \frac{1}{2}V_s'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{22}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x). \tag{23}$$

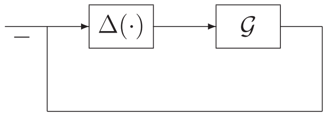

FIGURE 1 Multiplicative input uncertainty of $\mathcal{G}$ and input operator $\Delta(\cdot)$

If, alternatively,

$$\mathcal{N}(x) \triangleq R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) > 0, \quad x \in \mathbb{R}^n,$$

then $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u,y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exists a two-times continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_s(\cdot)$ is positive definite, $V_s(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 \geq V_{\mathrm{s}}'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + \left[\frac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x)+S)\right] \cdot \mathcal{N}^{-1}(x)\left[\frac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x)+S)\right]^{\mathrm{T}}. \tag{24}$$

4 | STABILITY MARGINS FOR STOCHASTIC FEEDBACK REGULATORS

To develop relative stability margins for nonlinear stochastic regulators, consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{25}$$

$$y(t) = -\phi(x(t)), \tag{26}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $D : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfies $D(0) = 0$, and $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is an admissible feedback controller such that $\mathcal{G}$ is globally asymptotically stable in probability with $u = -y$. Furthermore, we assume that $\mathcal{G}$ is zero-state observable.

Next, we define the relative stability margins for $\mathcal{G}$ given by (25) and (26). Specifically, let $u_{\mathrm{c}} \triangleq -y$, $y_{\mathrm{c}} \triangleq u$, and consider the negative feedback interconnection $u = \Delta(-y)$ of $\mathcal{G}$ and $\Delta(\cdot)$ given in Figure 1, where $\Delta(\cdot)$ is either a linear operator $\Delta(u_{\mathrm{c}}) = \Delta u_{\mathrm{c}}$, a nonlinear static operator $\Delta(u_{\mathrm{c}}) = \sigma(u_{\mathrm{c}})$, or a nonlinear dynamic operator $\Delta(\cdot)$ with input u_{c} and output y_{c}. Furthermore, we assume that, in the nominal case, $\Delta(\cdot) = I(\cdot)$ so that the nominal closed-loop system is globally asymptotically stable in probability.

Definition 5. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26) is said to have a *gain margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_{\mathrm{c}}) = \Delta u_{\mathrm{c}}$ is globally asymptotically stable in probability for all $\Delta = \mathrm{diag}[k_1, \ldots, k_m]$, where $k_i \in (\alpha, \beta)$, $i = 1, \ldots, m$.

Definition 6. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26) is said to have a *sector margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_{\mathrm{c}}) = \sigma(u_{\mathrm{c}})$ is globally asymptotically stable in probability for all nonlinearities $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ such that $\sigma(0) = 0$, $\sigma(u_{\mathrm{c}}) = [\sigma_1(u_{\mathrm{c}1}), \ldots, \sigma_m(u_{\mathrm{c}m})]^{\mathrm{T}}$, and $\alpha u_{\mathrm{c}i}^2 < \sigma_i(u_{\mathrm{c}i})u_{\mathrm{c}i} < \beta u_{\mathrm{c}i}^2$, for all $u_{\mathrm{c}i} \neq 0$, $i = 1, \ldots, m$.

Definition 7. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26) is said to have a *disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally asymptotically stable in probability for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state observable and stochastically dissipative with respect to the supply rate $r(u_{\mathrm{c}}, y_{\mathrm{c}}) = u_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}} - \frac{1}{\hat{\alpha}+\hat{\beta}}y_{\mathrm{c}}^{\mathrm{T}}y_{\mathrm{c}} - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}}u_{\mathrm{c}}^{\mathrm{T}}u_{\mathrm{c}}$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Definition 8. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26) is said to have a *structured disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally asymptotically stable in probability for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is zero-state observable, $\Delta(u_{\mathrm{c}}) = \mathrm{diag}[\delta_1(u_{\mathrm{c}1}), \ldots, \delta_m(u_{\mathrm{c}m})]$, and $\delta_i(\cdot)$, $i = 1, \ldots, m$, is stochastically dissipative with respect to the supply rate $r(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = u_{\mathrm{c}i}y_{\mathrm{c}i} - \frac{1}{\hat{\alpha}+\hat{\beta}}y_{\mathrm{c}i}^2 - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}}u_{\mathrm{c}i}^2$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Remark 2. Note that, if $\mathcal{G}$ has a disk margin (α, β), then $\mathcal{G}$ has gain and sector margins (α, β).

5 | NONLINEAR-NONQUADRATIC OPTIMAL REGULATORS FOR STOCHASTIC DYNAMICAL SYSTEMS

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear-nonquadratic cost functional. In particular, consider the controlled nonlinear stochastic dynamical system (15), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ such that $u(t) \in \mathcal{H}_m^U$ for almost all $t \geq t_0$ and $u(\cdot)$ is nonanticipative and takes values in a given compact, metrizable set $\mathcal{U}$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies (15), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, the *closed-loop system* (15) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)))\mathrm{d}t + D(x(t), \phi(x(t)))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0. \tag{27}$$

Next, we present a main theorem for stochastic stabilization characterizing feedback controllers that guarantee local and global closed-loop stability in probability and minimize a nonlinear-nonquadratic performance measure. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and, for every $\rho \in (0, 1)$, define the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0, \rho) \triangleq \Big\{ u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (15) is such that } \mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) \geq 1 - \rho, \\ \text{where } \mathfrak{B}_{x_0}^{u(\cdot)} \triangleq \Big\{ x(\{t \geq t_0\}, \omega) : \lim_{t\to\infty} \|x(t, \omega)\| = 0, \ \omega \in \Omega \Big\} \Big\}.$$

Furthermore, define the indicator function of the set $\mathfrak{B}_{x_0}^{u(\cdot)}$ by

$$\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}, \\ 0, & \text{otherwise.} \end{cases}$$

The set $\mathfrak{B}_{x_0}^{u(\cdot)}$ denotes the set of all controlled sample paths of (15) for which $\lim_{t\to\infty}\|x(t, \omega)\| = 0$ and $x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}$, $\omega \in \Omega$. Since in *local* stochastic stability theory there exists a probability of less than or equal to ρ that the system solution $x(t, \omega)$ leaves the subset $\mathcal{B}_\varepsilon(0)$ for every $x_0 \in \mathcal{B}_\delta(0)$, ie, the probability of escape is continuous at $x_0 = 0$ with small deviations from the equilibrium implying a small probability of escape, the set $\mathfrak{B}_{x_0}^{u(\cdot)}$ and $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^{u(\cdot)})$ are necessary for defining a well-posed cost functional for the optimal control problem formulation given in Theorem 3.

Theorem 3 (Rajpurohit and Haddad[1]). *Consider the nonlinear stochastic controlled dynamical system (15) with performance measure*

$$J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right], \tag{28}$$

where $u(\cdot)$ is an admissible control and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)$ denotes the indicator function of the set $\mathfrak{B}_{x_0}^{u(\cdot)}$. Assume that there exist a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$V(0) = 0, \tag{29}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{30}$$

$$\phi(0) = 0, \tag{31}$$

$$V'(x)F(x,\phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{32}$$

$$H(x,\phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{33}$$

$$H(x,u) \geq 0, \qquad x \in \mathcal{D}, \qquad u \in U, \tag{34}$$

where

$$H(x,u) \triangleq L(x,u) + V'(x)F(x,u) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x,u)V''(x)D(x,u). \tag{35}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, *the zero solution* $x(t) \overset{\text{a.s.}}{\equiv} 0$ *of the closed-loop system (27) is locally asymptotically stable in probability and, for every* $\rho \in (0,1)$, *there exist* $\delta = \delta(\rho)$ *and* $\mathfrak{B}_{x_0}^{\phi(x(\cdot))}$ *with* $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}) \geq 1-\rho$ *such that, for all* $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$,

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = V(x_0). \tag{36}$$

In addition, if $x_0 \in \mathcal{B}_\delta(0)$, *then the feedback control* $u(\cdot) = \phi(x(\cdot))$ *minimizes (28) in the sense that*

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = \min_{u(\cdot)\in\mathcal{S}(x_0,\rho)} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right). \tag{37}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, *and* $V(x) \to \infty$ *as* $\|x\| \to \infty$, *then the zero solution* $x(t) \overset{\text{a.s.}}{\equiv} 0$ *of the closed-loop system (27) is globally asymptotically stable in probability and (37) holds with* $\rho = 0$ *and* $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}) = 1$, $x_0 \in \mathbb{R}^n$.

It is important to note here that, in the case where the optimal feedback control $\phi(\cdot)$ guarantees global asymptotic stability in probability, $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^{\phi(\cdot)}) = 1$, and hence, $\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. Moreover, all the admissible controls $u(\cdot)$ that guarantee global attraction in probability also satisfy $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^{u(\cdot)}) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\rho = 0$ and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$\begin{aligned} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathrm{d}t\right] \end{aligned} \tag{38}$$

and

$$\begin{aligned} J\left(x_0, \phi(\cdot), \mathfrak{B}_{x_0}^{\phi(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),\phi(x(t)))\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega)\mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),\phi(x(t)))\mathrm{d}t\right]. \end{aligned} \tag{39}$$

Thus, in the remainder of this paper, we omit the dependence on $\mathfrak{B}_{x_0}^{\phi(\cdot)}$ and $\mathfrak{B}_{x_0}^{u(\cdot)}$ in the cost functional and we write $\mathcal{S}(x_0)$ for $\mathcal{S}(x_0,\rho)$ for all the results concerning globally stabilizing controllers in probability.

Next, we specialize Theorem 3 to affine stochastic dynamical systems. Specifically, we construct nonlinear feedback controllers using an optimal control framework that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the infinitesimal generator is negative along the closed-loop system sample trajectories while providing sufficient conditions for the existence of stochastically asymptotically stabilizing solutions to the stochastic Hamilton-Jacobi-Bellman equation. Thus, these results provide a family of globally stochastically stabilizing controllers parameterized by the cost functional that is minimized.

The controllers obtained next are predicated on an inverse optimal stochastic control problem.[1-9] Consider the nonlinear affine stochastic dynamical system given by (25) with performance integrands $L(x,u)$ of the form

$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u, \tag{40}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}, L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, where $R_2(x) > 0, x \in \mathbb{R}^n$, so that

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right]. \tag{41}$$

Theorem 4 (Rajpurohit and Haddad[1]). *Consider the nonlinear controlled affine stochastic dynamical system (25) with performance measure (41). Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that*

$$V(0) = 0, \tag{42}$$

$$L_2(0) = 0, \tag{43}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{44}$$

$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \frac{1}{2}tr\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{45}$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{46}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}}, \tag{47}$$

and the performance measure (41), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x), \tag{48}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{49}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{50}$$

Note that (45) is equivalent to

$$\mathcal{L}V(x) \triangleq V'(x)[f(x) + G(x)\phi(x)] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{51}$$

with $\phi(x)$ given by (47). Furthermore, conditions (42), (44), and (51) ensure that $V(\cdot)$ is a Lyapunov function for the closed-loop system (46). As discussed in Rajpurohit and Haddad,[1] it is important to recognize that the function $L_2(x)$, which appears in the integrand of the performance measure (40), is an arbitrary function of $x \in \mathbb{R}^n$ subject to conditions (43) and (45). Thus, $L_2(x)$ provides flexibility in choosing the control law.

With $L_1(x)$ given by (48) and $\phi(x)$ given by (47), $L(x, u)$ can be expressed as

$$\begin{aligned} L(x, u) &= u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) + L_2(x)(u - \phi(x)) - V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \left[u + \frac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x)\left[u + \frac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] - V'(x)[f(x) + G(x)\phi(x)] \\ &\quad - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) - \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x). \end{aligned} \tag{52}$$

Since $R_2(x) > 0$, $x \in \mathbb{R}^n$, the first term on the right-hand side of (52) is nonnegative, while (51) implies that the second, third, and fourth terms collectively are nonnegative. Thus, it follows that

$$L(x,u) \geq -\frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{53}$$

which shows that $L(x, u)$ may be negative. As a result, there may exist a control input u for which the performance measure $J(x_0, u)$ is negative. However, if the control u is a regulation controller, ie, $u \in \mathcal{S}(x_0)$, then it follows from (49) and (50) that

$$J(x_0, u(\cdot)) \geq V(x_0) \geq 0, \qquad x_0 \in \mathbb{R}^n, \qquad u(\cdot) \in \mathcal{S}(x_0). \tag{54}$$

Furthermore, in this case, substituting $u = \phi(x)$ into (52) yields

$$L(x, \phi(x)) = -V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x), \tag{55}$$

which, by (51), is positive.

6 | GAIN, SECTOR, AND DISK MARGINS OF NONLINEAR-NONQUADRATIC OPTIMAL REGULATORS FOR STOCHASTIC DYNAMICAL SYSTEMS

In this section, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion for stochastic dynamical systems. Specifically, sufficient conditions that guarantee gain, sector, and disk margins are given in terms of the state, control, and cross-weighting nonlinear-nonquadratic weighting functions.

In particular, we consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{56}$$

$$y(t) = -\phi(x(t)), \tag{57}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^m$, with a nonlinear-nonquadratic performance criterion

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right], \tag{58}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are given such that $R_2(x) > 0$, $x \in \mathbb{R}^n$, and $L_2(0) = 0$. In this case, the optimal nonlinear feedback controller $u = \phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (58) is given by the following result.

Theorem 5. *Consider the nonlinear stochastic dynamical system (56) and (57) with performance functional (58). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$V(0) = 0, \tag{59}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{60}$$

$$L_2(0) = 0, \tag{61}$$

$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \frac{1}{2}tr\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{62}$$

$$0 = L_1(x) + V'(x)f(x) + \frac{1}{2}tr\, D^{\mathrm{T}}(x)V''(x)D(x) - \frac{1}{4}\left[V'(x)G(x) + L_2(x)\right] \cdot R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}}, \quad x \in \mathbb{R}^n, \tag{63}$$

and

$$V(x) \to \infty \ as \ \|x\| \to \infty. \tag{64}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ *of the closed-loop system*

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{65}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}}, \tag{66}$$

and the performance functional (58) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{67}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{68}$$

Proof. The proof is identical to the proof of Theorem 4 given in Rajpurohit and Haddad.[1] □

The following key lemma is needed for developing the main result of this section.

Lemma 1. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies*

$$0 = V'(x)f(x) + L_1(x) - \frac{1}{4}\left[V'(x)G(x) + L_2(x)\right] R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x). \tag{69}$$

Furthermore, suppose there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and

$$(1-\theta^2)L_1(x) - \frac{1}{4}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \geq 0, \qquad x \in \mathbb{R}^n. \tag{70}$$

Then, for all $u(\cdot) \in \mathcal{U}$ and $t_1, t_2 \geq 0$, $t_1 < t_2$, the solution $x(t)$, $t \geq 0$, to (25) and (26) satisfies

$$\mathcal{L}V(x) \leq [u+y]^{\mathrm{T}}R_2(x)[u+y] - \theta^2 u^{\mathrm{T}}R_2(x)u, \tag{71}$$

which implies

$$\mathbb{E}\left[V(x(t_2))|\mathcal{F}_{t_1}\right] \leq V(x(t_1)) + \mathbb{E}\left[\int_{t_1}^{t_2} \left([u(s)+y(s)]^{\mathrm{T}}R_2(x(s))[u(s)+y(s)] - \theta^2 u^{\mathrm{T}}(s)R_2(x(s))u(s)\right)\mathrm{d}s \,\middle|\, \mathcal{F}_{t_1}\right]. \tag{72}$$

Proof. Note that it follows from (69) and (70) that, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\begin{aligned}
\theta^2 u^{\mathrm{T}} R_2(x) u &\le \theta^2 u^{\mathrm{T}} R_2(x) u + \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right] \\
&\quad \cdot R_2(x) \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right]^{\mathrm{T}} \\
&= u^{\mathrm{T}} R_2(x) u + \frac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) + L_2(x) u \\
&\le u^{\mathrm{T}} R_2(x) u + L_2(x) u + L_1(x) \\
&= u^{\mathrm{T}} R_2(x) u + L_2(x) u - V'(x) f(x) + \phi^{\mathrm{T}}(x) R_2(x) \phi(x) - \frac{1}{2} \mathrm{tr}\, D^{\mathrm{T}}(x) V''(x) D(x) \\
&= [u+y]^{\mathrm{T}} R_2(x) [u+y] - V'(x)[f(x) + G(x)u] - \frac{1}{2} \mathrm{tr}\, D^{\mathrm{T}}(x) V''(x) D(x),
\end{aligned}$$

which implies that, for all $u(\cdot) \in \mathcal{U}$,

$$\theta^2 u^{\mathrm{T}} R_2(x) u \le [u+y]^{\mathrm{T}} R_2(x) [u+y] - \mathcal{L}V(x). \tag{73}$$

Now, using Dynkin's formula (see Theorem 7.12 of Øksendal[17]),

$$\mathbb{E}\left[V(x(t_2)) | \mathcal{F}_{t_1}\right] \le V(x(t_1)) + \mathbb{E}\left[\int_{t_1}^{t_2} \left([u(s)+y(s)]^{\mathrm{T}} R_2(x(s)) [u(s)+y(s)] - \theta^2 u^{\mathrm{T}}(s) R_2(x(s)) u(s)\right) \mathrm{d}s \middle| \mathcal{F}_{t_1}\right], \tag{74}$$

is immediate. □

Next, we present disk margins for the nonlinear-nonquadratic optimal regulator given by Theorem 4. First, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is a constant diagonal matrix.

Theorem 6. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (48). If $R_2(x) \equiv \mathrm{diag}[r_1, \ldots, r_m]$, where $r_i > 0$, $i = 1, \ldots, m$, and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (70) is satisfied; then, the nonlinear stochastic dynamical system $\mathcal{G}$ has a structured disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$. If, in addition, $R_2(x) \equiv I$ and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (70) is satisfied, then the nonlinear stochastic dynamical system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.*

Proof. Note that, for all $u(\cdot) \in \mathcal{U}$, it follows from Lemma 1 that the solution $x(t)$, $t \ge 0$, to (25) satisfies

$$\mathcal{L}V(x) \le [u+y]^{\mathrm{T}} R_2 [u+y] - \theta^2 u^{\mathrm{T}} R_2 u. \tag{75}$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2} V(x)$, it follows from Proposition 1 that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = u^{\mathrm{T}} R_2 y + \frac{1-\theta^2}{2} u^{\mathrm{T}} R_2 u + y^{\mathrm{T}} R_2 y$. Now, the result is a direct consequence of Definitions 7 and 8 with $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$. □

Example 1. Consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x_1(t) = -x_1(t) + x_1(t) x_2^2(t) + g_1 x_1(t) \mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \ge 0, \tag{76}$$

$$\mathrm{d}x_2(t) = -x_2(t) + x_1(t) u(t) + g_2 x_2(t) \mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{77}$$

where $g_1 < \sqrt{2}$ and $g_2 < \sqrt{2}$, with performance functional

$$J(x_{10}, x_{20}, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left[\left(2 - g_1^2\right) x_1^2(t) + \left(2 - g_2^2\right) x_2^2(t) + \frac{1}{2} u^2(t)\right] \mathrm{d}t\right]. \tag{78}$$

To design an optimal control law $\phi(x_1, x_2)$ that minimizes (78), we use Theorem 5 with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [-x_1 + x_1 x_2^2, -x_2]^{\mathrm{T}}$, $G(x) = [0, x_1]^{\mathrm{T}}$, $D(x) = [g_1 x_1,\ g_2 x_2]^{\mathrm{T}}$, $L_1(x) = (2 - g_1^2)x_1^2 + (2 - g_2^2)x_2^2$, $L_2(x) = 0$, and $R_2(x) = \frac{1}{2}$. In particular, it follows from (63) that

$$0 = V'(x)\begin{bmatrix} -x_1 + x_1x_2^2 \\ -x_2 \end{bmatrix} - \frac{1}{2}V'(x)\begin{bmatrix} 0 & 0 \\ 0 & x_1^2 \end{bmatrix}V'^{\mathrm{T}}(x) + \frac{1}{2}\mathrm{tr}[g_1x_1\ g_2x_2]V''(x)\begin{bmatrix} g_1x_1 \\ g_2x_2 \end{bmatrix} + \left(2 - g_1^2\right)x_1^2 + \left(2 - g_2^2\right)x_2^2, \tag{79}$$

which implies that $V'(x) = [2x_1, 2x_2]$. Furthermore, since $V(0) = 0$, $V(x) = x_1^2 + x_2^2$. Hence, the optimal feedback control law is given by $\phi(x) = -\frac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) = -2x_1x_2$.

Finally, note that (62) implies

$$\begin{aligned}\mathcal{L}V(x) &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}\begin{bmatrix} -x_1 + x_1x_2^2 \\ -x_2 - 2x_1^2x_2 \end{bmatrix} + g_1^2x_1^2 + g_2^2x_2^2 \\ &= -\left(2 - g_1^2\right)x_1^2 - \left(2 - g_2^2\right)x_2^2 - 2x_1^2x_2^2 < 0,\end{aligned} \tag{80}$$

for all $(x_1, x_2) \neq (0, 0)$, and hence, $\phi(x_1, x_2) = -2x_1x_2$ is a global stabilizer for (76) and (77). Now, with $L_1(x) > 0$ and $L_2(x) = 0$, (70) is always satisfied with $\theta \in (0, 1)$. Therefore, the largest value that θ can attain such that (70) holds is $\theta_{\max} = 1$, which leads to a disk margin of $(\frac{1}{2}, \infty)$.

Next, we consider the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is not a diagonal constant matrix. For the following result, define

$$\bar{\gamma} \triangleq \sup_{x\in\mathbb{R}^n} \sigma_{\max}(R_2(x)), \qquad \underline{\gamma} \triangleq \inf_{x\in\mathbb{R}^n} \sigma_{\min}(R_2(x)), \tag{81}$$

where $R_2(x)$ is such that $\bar{\gamma} < \infty$ and $\underline{\gamma} > 0$.

Theorem 7. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (48). If there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and (70) is satisfied, then the nonlinear stochastic system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where $\eta \triangleq \sqrt{\underline{\gamma}/\bar{\gamma}}$.*

Proof. Note that, for all $u(\cdot) \in \mathcal{U}$, it follows from Lemma 1 that the solution $x(t)$, $t \geq 0$, to (25) satisfies

$$\begin{aligned}\mathcal{L}V(x) &\leq [u + y]^{\mathrm{T}}R_2(x)[u + y] - \theta^2u^{\mathrm{T}}R_2(x)u \\ &\leq \bar{\gamma}[u + y]^{\mathrm{T}}[u + y] - \underline{\gamma}\theta^2u^{\mathrm{T}}u.\end{aligned} \tag{82}$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2\underline{\gamma}}V(x)$, it follows from Proposition 1 that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = u^{\mathrm{T}}y + \frac{1-\eta^2\theta^2}{2}u^{\mathrm{T}}u + y^{\mathrm{T}}y$. Now, the result is a direct consequence of Definition 7 with $\alpha = \frac{1}{1+\eta\theta}$ and $\beta = \frac{1}{1-\eta\theta}$. □

Next, we provide an alternative result that guarantees sector and gain margins for the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal.

Theorem 8. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (48). Furthermore, let $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$. If $\mathcal{G}$ is zero-state observable and there exists $\theta \in \mathbb{R}$ such that $0 < \theta < 1$ and*

$$(1 - \theta^2)L_1(x) - \frac{1}{4}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \geq 0, \qquad x \in \mathbb{R}^n, \tag{83}$$

then the nonlinear stochastic dynamical system $\mathcal{G}$ has a sector (and, hence, gain) margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.

Proof. Let $\Delta(-y) = \sigma(-y)$, where $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity such that $\sigma(0) = 0$, $\sigma(v) = [\sigma_1(v_1), \ldots, \sigma_m(v_m)]^{\mathrm{T}}$, and $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, for all $v_i \neq 0$, $i = 1, \ldots, m$, where $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$; or, equivalently, $(\sigma_i(v_i) - \alpha v_i)(\sigma_i(v_i) - \beta v_i) < 0$, for all $v_i \neq 0$, $i = 1, \ldots, m$. In this case, the closed-loop system (25) and (26) with $u = \sigma(-y)$ is given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\sigma(\phi(x(t)))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{84}$$

Next, consider the Lyapunov function candidate $V(x)$, $x \in \mathbb{R}^n$, satisfying (48) and let $\mathcal{L}V(x)$ denote the Lyapunov infinitesimal generator of the closed-loop system (84). Now, it follows from (48) and (83) that

$$\begin{aligned}
\mathcal{L}V(x) &= V'(x)f(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\
&\leq V'(x)f(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) + L_1(x) \\
&\quad - \frac{1}{4(1-\theta^2)}L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) \\
&\quad + (1-\theta^2)\left[\sigma(\phi(x)) + \frac{1}{2(1-\theta^2)}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x) \\
&\quad \cdot \left[\sigma(\phi(x)) + \frac{1}{2(1-\theta^2)}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] \\
&= V'(x)f(x) + L_1(x) + V'(x)G(x)\sigma(\phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\
&\quad + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x))R_2(x)\sigma(\phi(x)) + L_2(x)\sigma(\phi(x)) \\
&= \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - 2\phi^{\mathrm{T}}(x)R_2(x)\sigma(\phi(x)) \\
&\quad + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x))R_2(x)\sigma(\phi(x)) \\
&= \sum_{i=1}^{m} r_i(x)\left(\frac{1}{\beta}\sigma_i(-y_i) + y_i\right)\left(\frac{1}{\alpha}\sigma_i(-y_i) + y_i\right) \\
&= \frac{1}{\alpha\beta}\sum_{i=1}^{m} r_i(x)\,(\sigma_i(-y_i) + \alpha y_i)\,(\sigma_i(-y_i) + \beta y_i) \\
&\leq 0, \quad x \in \mathbb{R}^n,
\end{aligned}$$

which, by Theorem 1, implies that the closed-loop system (84) is Lyapunov stable in probability.

Next, it follows from Corollary 4.1 of Mao[26] that $\mathcal{L}V(x) \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$, and note that $\mathcal{L}V(x) = 0$ if and only if $y = 0$. Now, since $\mathcal{G}$ is zero-state observable, it follows that $x(t) \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$. Thus, since $V_{\mathrm{s}}(\cdot)$ is radially unbounded, the closed-loop system (84) is globally asymptotically stable in probability for all $\sigma(\cdot)$ such that $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, $v_i \neq 0$, $i = 1, \ldots, m$, which implies that the nonlinear stochastic system $\mathcal{G}$ given by (25) and (26) has sector (and, hence, gain) margins (α, β). □

Note that, in the case where $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal, Theorem 8 guarantees larger gain and sector margins to the gain and sector margin guarantees provided by Theorem 7. However, Theorem 8 does not provide disk margin guarantees.

7 | INVERSE OPTIMALITY OF NONLINEAR STOCHASTIC FEEDBACK REGULATORS

In this section, we give sufficient conditions that guarantee that a given nonlinear feedback controller has prespecified disk, sector, and gain margins.

Proposition 2. *Let $\theta \in (0, 1)$ and let $R_2 \in \mathbb{R}^{m \times m}$ be a positive-definite matrix. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law. Then, there exist* *functions* $V : \mathbb{R}^n \to \mathbb{R}$, $L_1 : \mathbb{R}^n \to \mathbb{R}$, *and* $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ *such that* $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}}$, $V(\cdot)$ *is two-times continuously differentiable,* $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, *and, for all* $x \in \mathbb{R}^n$,

$$0 = V'(x)f(x) + L_1(x) - \frac{1}{4}\left[V'(x)G(x) + L_2(x)\right]R_2^{-1}\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}} + \frac{1}{2}tr\,D^{\mathrm{T}}(x)V''(x)D(x), \tag{85}$$

$$0 \leq (1-\theta^2)L_1(x) - \frac{1}{4}L_2(x)R_2^{-1}L_2^{\mathrm{T}}(x), \tag{86}$$

if and only if, for all $u(\cdot) \in \mathcal{U}$, *there exists* $V : \mathbb{R}^n \to \mathbb{R}$ *such that* $V(0) = 0$, $V(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, *and the solution* $x(t)$, $t \geq 0$, *to (25) satisfies*

$$\mathcal{L}V(x) \leq [u+y]^{\mathrm{T}}R_2(x)[u+y] - \theta^2 u^{\mathrm{T}}R_2(x)u. \tag{87}$$

Proof. If there exist functions $V : \mathbb{R}^n \to \mathbb{R}$, $L_1 : \mathbb{R}^n \to \mathbb{R}$, and $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}}$ and (85) and (86) are satisfied, then it follows from Lemma 1 that (87) is satisfied. Conversely, if for $u(\cdot) \in \mathcal{U}$ the solution $x(t)$, $t \geq 0$, to (25) satisfies (87), then with $Q = R_2$, $S = R_2$, and $R = (1-\theta^2)R_2$, it follows from (24) of Theorem 2 that

$$\begin{aligned} 0 \geq\ & V'(x)f(x) + \frac{1}{2}\mathrm{tr}\,D^{\mathrm{T}}(x)V''(x)D(x) - \phi^{\mathrm{T}}(x)R_2\phi(x) + \frac{1}{4(1-\theta^2)}\left[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)\right] \\ & \cdot R_2^{-1}\left[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)\right]^{\mathrm{T}}, \quad x \in \mathbb{R}^n. \end{aligned}$$

The result now follows with $L_1(x) = -V'(x)f(x) + \phi^{\mathrm{T}}(x)R_2\phi(x) - \frac{1}{2}\mathrm{tr}\,D^{\mathrm{T}}(x)V''(x)D(x)$ and $L_2(x) = -[2\phi^{\mathrm{T}}(x)R_2 + V'(x)G(x)]$. □

Note that, if (85) and (86) are satisfied, then it follows from Theorem 4 that the feedback control law $\phi(x) = -\frac{1}{2}R_2^{-1}[V'(x)G(x) + L_2(x)]^{\mathrm{T}}$ minimizes the cost functional (41). Hence, Proposition 2 provides necessary and sufficient conditions for optimality of a given stochastically stabilizing feedback control law with prespecified disk margin guarantees.

The following result presents specific disk margin guarantees for inverse optimal controllers.

Theorem 9. *Let* $\theta \in (0,1)$ *be given. Consider the nonlinear stochastic dynamical system* $\mathcal{G}$ *given by (25) and (26), where* $\phi(x)$ *is a stochastically stabilizing feedback control law. Assume that there exist functions* $V : \mathbb{R}^n \to \mathbb{R}$ *and* $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ *such that* $V(\cdot)$ *is two-times continuously differentiable,* $R_2(x) > 0$, $x \in \mathbb{R}^n$, *and*

$$V(0) = 0, \tag{88}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{89}$$

$$V'(x)\left[f(x) + G(x)\phi(x)\right] + \frac{1}{2}\mathrm{tr}\,D^{\mathrm{T}}(x)V''(x)D(x) < 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{90}$$

$$\begin{aligned} & V'(x)f(x) + \frac{1}{2}\mathrm{tr}\,D^{\mathrm{T}}(x)V''(x)D(x) - \phi^{\mathrm{T}}(x)R_2^{-1}(x)\phi(x) + \frac{1}{1-\theta^2}\left(\phi^{\mathrm{T}}(x) + \frac{1}{2}V'(x)G(x)\cdot R_2^{-1}(x)\right) \\ & R_2(x)\Big(\phi^{\mathrm{T}}(x) + \frac{1}{2}V'(x)G(x)R_2^{-1}(x)\Big)^{\mathrm{T}} \leq 0, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{91}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{92}$$

Then, the nonlinear stochastic dynamical system $\mathcal{G}$ *has a disk margin* $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, *where* $\eta = \sqrt{\underline{\gamma}/\overline{\gamma}}$ *and* $\underline{\gamma}$ *and* $\overline{\gamma}$ *are given by (81). Furthermore, with the feedback control law* $\phi(x)$, *the performance functional*

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left[-V'(x(t))(f(x(t)) + G(x(t))u(t)) + (\phi(x(t)) - u(t))^{\mathrm{T}}R_2(x(t))(\phi(x(t)) - u(t))\right]\mathrm{d}t\right] \tag{93}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{94}$$

Proof. The result is a direct consequence of Theorems 4 and 7 with $L_1(x) = -V'(x)f(x) + \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x)$ and $L_2(x) = -(2\phi^{\mathrm{T}}(x)R_2(x) + V'(x)G(x))$. Specifically, in this case, all the conditions of Theorem 4 are trivially satisfied. Furthermore, note that (91) is equivalent to (70). The result is now immediate. □

The next result provides sufficient conditions that guarantee that a given nonlinear feedback controller has prespecified gain and sector margins.

Theorem 10. *Let $\theta \in (0, 1)$ be given. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law. Assume there exist functions $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$, and $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(\cdot)$ is two-times continuously differentiable and satisfies (88)–(92). Then, the nonlinear stochastic dynamical system $\mathcal{G}$ has a disk margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$. Furthermore, with the feedback control law $\phi(x)$, the performance functional (93) is minimized in the sense that*

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{95}$$

Proof. The result is a direct consequence of Theorems 4 and 8 with the proof being identical to the proof of Theorem 9. □

8 | LINEAR-QUADRATIC OPTIMAL STOCHASTIC REGULATORS

In this section, we specialize Theorems 6 and 7 to the case of linear stochastic systems with multiplicative disturbance noise. Specifically, consider the stabilizable stochastic system given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + xg^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{96}$$

$$y(t) = -Kx(t), \tag{97}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $K \in \mathbb{R}^{m\times n}$, and $g \in \mathbb{R}^d$, and assume that (A, K) is detectable and the system (96) and (97) is asymptotically stable in probability with the feedback $u = -y$; or, equivalently, $\tilde{A}+BK$ is Hurwitz, where $\tilde{A} = A+\frac{1}{2}\|g\|^2 I_n$. Furthermore, assume that K is an optimal regulator that minimizes the quadratic performance functional given by

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left[x^{\mathrm{T}}(t)R_1x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2u(t)\right]\mathrm{d}t\right], \tag{98}$$

where $R_1 \in \mathbb{R}^{n\times n}$, $R_{12} \in \mathbb{R}^{n\times m}$, and $R_2 \in \mathbb{R}^{m\times m}$ are such that $R_2 > 0$, $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and (A, R_1) is observable. In this case, it follows from Theorem 4 with $f(x) = Ax$, $G(x) = B$, $L_1(x) = x^{\mathrm{T}}R_1x$, $L_2(x) = 2x^{\mathrm{T}}R_{12}$, $R_2(x) = R_2$, $\phi(x) = Kx$, and $V(x) = x^{\mathrm{T}}Px$ that the optimal control law K is given by $K = -R_2^{-1}(B^{\mathrm{T}}P + R_{12})$, where $P > 0$ is the solution to the algebraic regulator Riccati equation given by

$$0 = \left(\tilde{A} - BR_2^{-1}R_{12}^{\mathrm{T}}\right)^{\mathrm{T}}P + P\left(\tilde{A} - BR_2^{-1}R_{12}^{\mathrm{T}}\right) + R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} - PBR_2^{-1}B^{\mathrm{T}}P. \tag{99}$$

The following results provide guarantees of disk, sector, and gain margins for the system (96) and (97).

Corollary 1. *Consider the stochastic dynamical system with multiplicative noise given by (96) and (97) and with performance functional (98), and let $\sigma_{\max}^2(R_{12}) < \sigma_{\min}(R_1)\sigma_{\min}(R_2)$. Then, with $K = -R_2^{-1}(B^{\mathrm{T}}P + R_{12})$, where $P > 0$ satisfies (99), the system (96) and (97) has disk margin (and, hence, sector and gain margins) $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where*

$$\eta = \frac{\sigma_{\min}(R_2)}{\sigma_{\max}(R_2)}, \qquad \theta = \left(1 - \frac{\sigma_{\max}^2(R_{12})}{\sigma_{\min}(R_1)\sigma_{\min}(R_2)}\right)^{1/2}. \tag{100}$$

Proof. The result is a direct consequence of Theorem 7 with $f(x) = Ax$, $G(x) = B$, $\phi(x) = Kx$, $V(x) = x^{\mathrm{T}}Px$, $L_1(x) = x^{\mathrm{T}}R_1x$, and $L_2(x) = 2x^{\mathrm{T}}R_{12}$. Specifically, note that (99) is equivalent to (48). Now, with θ given by (100), it follows that $(1 - \theta^2)R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and hence, (83) is satisfied so that all the conditions of Theorem 7 are satisfied. □

Corollary 2. *Consider the stochastic dynamical system with multiplicative noise given by (96) and (97) and with performance functional (98), and let* $\sigma_{\max}^2(R_{12}) < \sigma_{\min}(R_1)\sigma_{\min}(R_2)$*, where* R_2 *is diagonal. Then, with* $K = -R_2^{-1}(B^{\mathrm{T}}P + R_{12})$*, where* $P > 0$ *satisfies (99), the system (96) and (97) has structured disk margin (and, hence, gain and sector) margin* $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$*, where*

$$\theta = \left(1 - \frac{\sigma_{\max}^2(R_{12})}{\sigma_{\min}(R_1)\sigma_{\min}(R_2)}\right)^{1/2}. \tag{101}$$

Proof. The result is a direct consequence of Theorem 6 with $f(x) = Ax$, $G(x) = B$, $\phi(x) = Kx$, $V(x) = x^{\mathrm{T}}Px$, $L_1(x) = x^{\mathrm{T}}R_1x$, and $L_2(x) = 2x^{\mathrm{T}}R_{12}$. Specifically, note that (99) is equivalent to (48). Now, with θ given by (101), it follows that $(1-\theta^2)R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$, and hence, (83) is satisfied so that all the conditions of Theorem 6 are satisfied. □

The gain margins obtained in Corollary 2 are precisely the gain margins given in Chung et al[10] for deterministic linear-quadratic optimal regulators with cross-weighting terms in the performance criterion. Furthermore, since Corollary 2 guarantees structured disk margins of $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$, it follows that the system has a phase margin ϕ given by

$$\cos(\phi) = 1 - \frac{\theta^2}{2}, \tag{102}$$

or, equivalently,

$$\sin\left(\frac{\phi}{2}\right) = \frac{\theta}{2}. \tag{103}$$

In the case where $R_{12} = 0$, it follows from (101) that $\theta = 1$, and hence, Corollary 2 guarantees a phase margin of ±60° in each input-output channel. In addition, requiring that $R_1 \geq 0$, it follows from Corollary 2 that the system given by (96) and (97) has a gain and sector margin of $(\frac{1}{2}, \infty)$.

9 | STABILITY MARGINS, MEANINGFUL INVERSE OPTIMALITY, AND STOCHASTIC DISSIPATIVITY

In this section, we specialize the results of Section 4 to the case where $L(x,u)$ is nonnegative for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$. In the terminology of Sepulchre et al[7] and Freeman and Kokotović,[9] this corresponds to a *meaningful cost functional*. Here, we assume $L_2(x) \equiv 0$ and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$. In this case, we establish connections between stochastic dissipativity and optimality for nonlinear stochastic controllers. The first result specializes Theorem 4 to the case in which $L_2(x) \equiv 0$.

Theorem 11. *Consider the nonlinear stochastic dynamical system (25) with performance functional (41) with* $L_2(x) \equiv 0$ *and* $L_1(x) \geq 0$, $x \in \mathbb{R}^n$*. Assume there exists a two-times continuously differentiable function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$V(0) = 0, \tag{104}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{105}$$

$$0 = L_1(x) + V'(x)f(x) + \frac{1}{2}tr\, D^{\mathrm{T}}(x)V''(x)D(x) - \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \quad x \in \mathbb{R}^n, \tag{106}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{107}$$

Furthermore, assume that the system (26) and (41) is zero-state observable with $y = L_1(x)$*. Then, the zero solution* $x(t) \overset{\text{a.s.}}{\equiv} 0$ *of the closed-loop system*

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{108}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{109}$$

and the performance functional (41) is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{110}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{111}$$

Proof. The proof is similar to the proof of Theorem 4. □

Next, we show that, for a given nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), there exists an equivalence between optimality and stochastic dissipativity. For the following result, we assume that, for a given nonlinear stochastic system (25), if there exists a feedback control law $\phi(x)$ that minimizes the performance functional (41) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then there exists a two-times continuously differentiable positive-definite function $V(x)$, $x \in \mathbb{R}^n$, such that (106) is satisfied.

Theorem 12. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26). The feedback control law $u = \phi(x)$ is optimal with respect to a performance functional (40) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, if and only if the nonlinear stochastic system $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^T y + 2u^T y$ and has a two-times continuously differentiable positive-definite radially unbounded storage function $V(x)$, $x \in \mathbb{R}^n$.*

Proof. If the control law $\phi(x)$ is optimal with respect to a performance functional (40) with $R_2(x) \equiv I$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0, x \in \mathbb{R}^n$, then, by assumption, there exists a two-times continuously differentiable positive-definite function $V(x)$ such that (106) is satisfied. Hence, it follows from Proposition 2 that the solution $x(t)$, $t \geq 0$, to (25) satisfies

$$\mathcal{L}V(x) \leq [u + y]^{\mathrm{T}}R_2(x)[u + y] - \theta^2 u^{\mathrm{T}}R_2(x)u, \tag{112}$$

which implies, by Proposition 1, that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$.

Conversely, if $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a two-times continuously differentiable positive-definite storage function, then, with $h(x) = -\phi(x)$, $J(x) \equiv 0$, $Q = I$, $R = 0$, and $S = 2I$, it follows from Theorem 2 that there exists a function $\ell : \mathbb{R}^n \to \mathbb{R}^p$ such that $\phi(x) = -\frac{1}{2}G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$ and, for all $x \in \mathbb{R}^n$,

$$0 = V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) - \frac{1}{4}V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\ell(x).$$

Now, the result follows from Theorem 11 with $L_1(x) = \ell^{\mathrm{T}}(x)\ell(x)$. □

Example 2. Consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = -x(t) + u(t) + gx(t)\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{113}$$

where $g < \sqrt{3}$, with performance functional

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left[(3 - g^2)x^2(t) + u^2(t)\right]\mathrm{d}t\right]. \tag{114}$$

To design an optimal control law $\phi(x)$ that minimizes (114), we use Theorem 5 with $f(x) = -x$, $G(x) = 1$, $D(x) = gx$, $L_1(x) = (3 - g^2)x^2$, $L_2(x) = 0$, and $R_2(x) = 1$. Now, note that (63) holds with $V(x) = x^2$. Therefore, the optimal control law is given by

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}} = -x. \tag{115}$$

Now, from Proposition 2, since (85) and (86) hold with $\theta = 1$, we have

$$\mathcal{L}V(x) \le (u+y)^2 - u^2 = y^2 + 2uy, \tag{116}$$

where $y = -\phi(x) = x$, which implies, by Proposition 1, that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u,y) = y^2 + 2uy$.

The next result gives disk and structured disk margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26).

Corollary 3. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (41), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) with $L_2(x) \equiv 0$ and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (48). Furthermore, assume $R_2(x) = \mathrm{diag}[r_1, \ldots, r_m]$, where $r_i > 0$, $i = 1, \ldots, m$, and $L_1(x) \ge 0$, $x \in \mathbb{R}^n$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ has a structured disk margin $(\frac{1}{2}, \infty)$. If, in addition, $R_2(x) \equiv I_m$, then the nonlinear stochastic system $\mathcal{G}$ has a disk margin $(\frac{1}{2}, \infty)$*

Proof. The result is a direct consequence of Theorem 6. Specifically, if $L_1(x) \ge 0$, $x \in \mathbb{R}^n$, and $L_2(x) \equiv 0$, then (70) is trivially satisfied for all $\theta \in (0,1)$. Now, the result follows immediately by letting $\theta \to 1$. □

Finally, we provide sector and gain margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26).

Corollary 4. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25) and (26), where $\phi(x)$ is a stochastically stabilizing feedback control law given by (47) with $L_2(x) \equiv 0$ and where $V(x)$, $x \in \mathbb{R}^n$, satisfies (48). Furthermore, assume $R_2(x) = \mathrm{diag}[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$, and $L_1(x) \ge 0$, $x \in \mathbb{R}^n$. Then, the nonlinear stochastic dynamical system $\mathcal{G}$ has a sector (and, hence, gain) margin $(\frac{1}{2}, \infty)$.*

Proof. The result is a direct consequence of Theorem 8. Specifically, if $L_1(x) \ge 0$, $x \in \mathbb{R}^n$, and $L_2(x) \equiv 0$, then (70) is trivially satisfied for all $\theta \in (0,1)$. Now, the result follows immediately by letting $\theta \to 1$. □

10 | CONCLUSION

In this paper, we have used the notions of stochastic stability and stochastic dissipativity theory to develop sufficient conditions for gain, sector, and disk margin guarantees for nonlinear stochastic dynamical systems controlled by nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion. Using these results, connections between stochastic dissipativity and optimality of nonlinear stochastic systems were established. These results provide a generalization of the deterministic meaningful inverse optimal nonlinear regulator stability margins and the classical linear-quadratic optimal regulator gain and phase margins to stochastic nonlinear feedback regulators.

Extensions of this framework for exploring connections between optimal finite-time stabilization[27,28] and finite-time stabilization[29] for stochastic dynamical systems are currently under development. The proposed framework can also allow us to further explore connections with stochastic inverse optimal control, stochastic dissipativity, and stability margins for finite-time stabilizing regulators that minimize a derived cost functional involving subquadratic terms.

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