Universal Feedback Controllers and Inverse Optimality for Nonlinear Stochastic Systems

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In this paper, we develop a constructive finite time stabilizing feedback control law for stochastic dynamical systems driven by Wiener processes based on the existence of a stochastic control Lyapunov function. In addition, we present necessary and sufficient conditions for continuity of such controllers. Moreover, using stochastic control Lyapunov functions, we construct a universal inverse optimal feedback control law for nonlinear stochastic dynamical systems that possess guaranteed gain and sector margins. An illustrative numerical example involving the control of thermoacoustic instabilities in combustion processes is presented to demonstrate the efficacy of the proposed framework.
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1 Introduction

The consideration of Lyapunov functions for proving stability of feedback dynamical systems is one of the cornerstones of systems and control theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein [1] to show the existence of a feedback stabilizing controller. A constructive feedback control law based on a universal construction of smooth control Lyapunov functions was given by Sontag [2]. An extended notion of nonsmooth control Lyapunov functions as well as a universal feedback controller for discontinuous dynamical systems based on the existence of nonsmooth Lyapunov functions defined in the sense of generalized Clarke gradients and set-valued Lie derivatives was developed in Refs. [3–6].

The aforementioned results on control Lyapunov functions along with the constructive feedback control laws predicated on these generalized energy functions are developed for deterministic dynamical systems. In numerous applications, where dynamical models are used to describe the behavior of natural and engineering systems, stochastic components and random disturbances are often incorporated into the models. The stochastic aspects of the models are used to quantify system uncertainty as well as the dynamic relationships of sequences of random events between system–environment interactions. In Refs. [7–9], the authors provide Lyapunov-like techniques for stochastic stabilization. Specifically, asymptotic stability in probability of affine in the control stochastic dynamical systems using stochastic control Lyapunov functions leading to the existence of smooth, except possibly at the equilibrium point of the system, stochastically stabilizing feedback control laws are provided.

In this paper, we build on the results of Refs. [7–9] as well as on the recent stochastic finite time stabilization framework of Ref. [10] to develop a constructive universal feedback control law for stochastic finite time stabilization of stochastic dynamical systems. In addition, we present necessary and sufficient conditions for continuity of such controllers. Finally, we show that for every nonlinear stochastic dynamical system for which a stochastic control Lyapunov function can be constructed, there exists an inverse optimal feedback control law in the sense of Refs. [11] and [12] with guaranteed sector and gain margins of $(1/2, \infty)$.

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2 Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results on stability of nonlinear stochastic dynamical systems [13–15]. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm or an induced matrix norm (depending on context), $\|\cdot\|_{\rm F}$ for the Frobenius matrix norm, $A^{\rm T}$ for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [13]. Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure [14], with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Wiener process $w(t)$ up to time t. We denote by $\mathcal{G}$ a stochastic dynamical system generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x: \bar{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega: x(t,\omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here, we use the notation $x(t)$ to represent the stochastic process $x(t,\omega)$ omitting its dependence on ω.

Finally, we write ${\rm tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \partial V(x)/\partial x$ for the Fréchet derivative of V at x, $V''(x) \triangleq \partial^2 V(x)/\partial x^2$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n \triangleq \{x: \Omega \to \mathbb{R}^n: \mathbb{E}[x^{\rm T}x] < \infty\}$, where $\mathbb{E}$ denotes expectation. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n: x: \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n: x \overset{\rm a.s.}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V: \mathcal{D} \to \mathbb{R}$ that are two times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\rm a.s.}{=} x_0, \quad t \geq t_0 \tag{1}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is an $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space

$(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t)-w(t_0)), t\geq t_0$, and $f\colon \mathcal{D}\to\mathbb{R}^n$ and $D\colon \mathcal{D}\to\mathbb{R}^{n\times d}$ are continuous functions and satisfy $f(x_e)=0$ and $D(x_e)=0$ for some $x_e\in\mathcal{D}$.

Here, we assume that $f\colon \mathcal{D}\to\mathbb{R}^n$ and $D\colon \mathcal{D}\to\mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$||f(x)-f(y)||+||D(x)-D(y)||_{\mathrm{F}}\leq L||x-y||,\quad x,y\in\mathcal{D} \tag{2}$$

and the growth restriction condition

$$||f(x)||^2+||D(x)||_{\mathrm{F}}^2\leq L^2(1+||x||^2),\quad x\in\mathcal{D} \tag{3}$$

for some Lipschitz constant $L>0$, and hence, since $x(t_0)\in\mathcal{H}_n^{\mathcal{D}}$ and $x(t_0)$ is independent of $(w(t)-w(t_0)), t\geq t_0$, it follows that there exists a unique solution $x\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, where $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ denotes the set of equivalence class of measurable and square-integrable $\mathbb{R}^n$ valued random processes on $(\Omega,\mathcal{F},\mathbb{P})$ over the semi-infinite parameter space $[0,\infty)$, to Eq. (1) in the following sense. For every $x\in\mathcal{H}_n^{\mathcal{D}}\backslash\{0\}$ there exists $T_x>0$ such that if $x_1\colon[t_0,\tau_1]\times\Omega\to\mathcal{D}$ and $x_2\colon[t_0,\tau_2]\times\Omega\to\mathcal{D}$ are two solutions of (1); that is, if $x_1,x_2\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ with continuous sample paths almost surely solve (1), then $T_x\leq\min\{\tau_1,\tau_2\}$ and $\mathbb{P}(x_1(t)=x_2(t),\ t_0\leq t\leq T_x)=1$.

The following definition introduces the notions of Lyapunov and asymptotic stability in probability. Recall that an *equilibrium point* $x_e=0$ of Eq. (1) is a point such that $f(0)=0$ and $D(0)=0$. In this case, $x_e=0$ is an equilibrium point of Eq. (1) if and only if the zero solution (i.e., the zero stochastic process) $x(\cdot)\overset{\text{a.s.}}{\equiv}0$ is a solution of Eq. (1).

DEFINITION 2.1. *(i) The zero solution* $x(t)\overset{\text{a.s.}}{\equiv}0$ *to (1) is* Lyapunov stable in probability *if, for every* $\varepsilon>0$ *and* $\rho\in(0,1)$*, there exist* $\delta=\delta(\rho,\varepsilon)>0$ *such that, for all* $x_0\in\mathcal{B}_\delta(0)$ *[16]*

$$\mathbb{P}^{x_0}\left(\sup_{t\geq t_0}||x(t)||>\varepsilon\right)\leq\rho \tag{4}$$

(ii) The zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ *to (1) is* locally asymptotically stable in probability *if it is Lyapunov stable in probability and, for every* $\rho\in(0,1)$*, there exist* $\delta=\delta(\rho)>0$ *such that, for all* $x_0\in\mathcal{B}_\delta(0)$

$$\mathbb{P}^{x_0}\left(\lim_{t\to\infty}||x(t)||=0\right)\geq1-\rho \tag{5}$$

(iii) The zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ *to Eq. (1) is* globally asymptotically stable in probability *if it is Lyapunov stable in probability and, for all* $x_0\in\mathbb{R}^n$

$$\mathbb{P}^{x_0}\left(\lim_{t\to\infty}||x(t)||=0\right)=1 \tag{6}$$

Remark 2.1. A more general stochastic stability notion can also be introduced here involving stochastic stability and convergence to an invariant (stationary) distribution. In this case, state convergence is not to an equilibrium point but rather to a stationary distribution. This framework can relax the vanishing perturbation assumption $D(0)=0$ and requires a more involved analysis and synthesis framework showing stability of the underlying Markov semigroup [17].

Next, we provide sufficient conditions for local and global asymptotic stability in probability for the nonlinear stochastic dynamical system (1). First, however, recall that the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t\geq0$, with $x(0)\overset{\text{a.s.}}{\equiv}x_0$, is defined by

$$\mathcal{L}V(x_0)\triangleq\lim_{t\to0^+}\frac{\mathbb{E}^{x_0}[V(x(t))]-V(x_0)}{t},\quad x_0\in\mathcal{D} \tag{7}$$

where $\mathbb{E}^{x_0}$ denotes the expectation with respect to the transition probability measure $\mathbb{P}^{x_0}(x(t)\in\mathcal{D})\triangleq\mathbb{P}(t_0,x_0,t,\mathcal{D})$ [14]. If $V\in\mathrm{C}^2$ and has a compact support, and $x(t)$, $t\geq0$, satisfies Eq. (1), then the limit in Eq. (7) exists for all $x\in\mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t\geq0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (1) and is given by [14]

$$\mathcal{L}V(x)\triangleq\frac{\partial V(x)}{\partial x}f(x)+\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)\frac{\partial^2V(x)}{\partial x^2}D(x),\quad x\in\mathcal{D} \tag{8}$$

THEOREM 2.1. *Consider the nonlinear stochastic dynamical system (1) and assume that there exists a two times continuously differentiable function* $V\colon\mathcal{D}\to\mathbb{R}$ *such that [15]*

$$V(0)=0 \tag{9}$$

$$V(x)>0,\quad x\in\mathcal{D},\quad x\neq0 \tag{10}$$

$$\frac{\partial V(x)}{\partial x}f(x)+\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)\frac{\partial^2V(x)}{\partial x^2}D(x)\leq0,\quad x\in\mathcal{D} \tag{11}$$

Then, the zero solution $x(t)\overset{\text{a.s.}}{\equiv}$ *to Eq. (1) is Lyapunov stable in probability. If, in addition*

$$\frac{\partial V(x)}{\partial x}f(x)+\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)\frac{\partial^2V(x)}{\partial x^2}D(x)<0,\quad x\in\mathcal{D},\quad x\neq0 \tag{12}$$

then the zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ *to Eq. (1) is asymptotically stable in probability. Moreover, if* $\mathcal{D}=\mathbb{R}^n$ *and* $V(\cdot)$ *is radially unbounded, then the zero solution* $x(t)\overset{\text{a.s.}}{\equiv}0$ *to Eq. (1) is globally asymptotically stable in probability.*

Next, we present the notion of stochastic finite time stability involving finite time almost sure convergence along with stochastic Lyapunov stability. To present this notion, we need some additional notation and definitions. The measurable map $s\colon[0,\tau_x)\times\mathcal{D}\times\Omega\to\mathcal{D}$ denotes the *dynamic* or *flow* of the stochastic dynamical system (1) and, for all $t,\tau\in[0,\tau_x)$, satisfies the *cocycle* property $s(\tau,s(t,x),\omega)=s(t+\tau,x,\omega)$ and the *identity* (on $\mathcal{D}$) property $s(0,x,\omega)=x$ for all $x\in\mathcal{D}$ and $\omega\in\Omega$. The measurable map $s_t\triangleq s(t,\cdot,\omega)\colon\mathcal{D}\to\mathcal{D}$ is continuously differentiable for all $t\in[0,\tau_x)$ outside a $\mathbb{P}$-nullset and the sample path trajectory $s^x\triangleq s(\cdot,x,\omega)\colon[0,\tau_x)\to\mathcal{D}$ is continuous in $\mathcal{D}$ for all $t\in[0,\tau_x)$. Thus, for every $x\in\mathcal{D}$, there exists a trajectory of measures defined for all $t\in[0,\tau_x)$ satisfying the dynamical processes (1) with initial condition $x(0)\overset{\text{a.s.}}{\equiv}x_0$. For simplicity of exposition we write $s(t,x)$ for $s(t,x,\omega)$ omitting its dependence on ω.

For the results in the paper involving finite time stability, we assume that the uniform Lipschitz continuity condition (2) and the growth condition (3) are satisfied for all $x,\ y\in\mathcal{D}\backslash\{0\}$. Furthermore, we assume that for every initial condition $x_0\in\mathcal{D}\backslash\{0\}$, (1) has a unique solution in forward time. Analogous assumptions are made for the controlled problem.

DEFINITION 2.2. *The zero solution* $x(t)\overset{\text{a.s.}}{\equiv}0$ *to (1) is* (globally) stochastically finite-time stable *if there exists an operator* $T\colon\mathcal{H}_n\to\mathcal{H}_1^{[0,\infty)}$*, called the* stochastic settling-time operator*, such that the following statements hold [10].*

(i) Finite-time convergence in probability. *For every* $x(0)\in\mathcal{H}_n$*,* $s^{x(0)}(t)$ *is defined on* $[0,T(x(0)))$*,* $s^{x(0)}(t)\in\mathcal{H}_n$ *for all* $t\in[0,T(x(0)))$*, and*

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(x(0))}||s^{x(0)}(t)||=0\right)=1$$

(ii) Lyapunov stability in probability. *For every* $\varepsilon>0$,

$$\lim_{x_0\to0}\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(x(0)))}||s^{x(0)}(t)||>\varepsilon\right)=0$$

Equivalently, for every $\varepsilon > 0$ *and* $\rho \in (0,1)$, *there exist* $\delta = \delta(\varepsilon,\rho) > 0$ *such that, for all* $x_0 \in \mathcal{B}_\delta(0)$, $\mathbb{P}^{x_0}(\sup_{t\in[0,T(x(0)))} ||s^{x(0)}(t)|| > \varepsilon) \le \rho$.

(iii) Finiteness of the stochastic settling-time operator. *For every* $x \in \mathcal{H}_n$ *the stochastic settling-time operator* $T(x)$ *exists and is finite with probability one, that is,* $\mathbb{E}^x[T(x)] < \infty$.

It is easy to see from Definition 2.2 that

$$T(x(0)) = \inf\{t \in \bar{\mathbb{R}}_+ : s(t,x(0)) = 0\}, \quad x(0) \in \mathcal{H}_n^{\mathbb{R}^n}$$

Proposition 2.1. *Suppose the origin is a stochastically finite time stable equilibrium of (1) and let* $T: \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty]}$ *be the stochastic finite time operator. Then, the following statements hold:*

(i) If $\tau \ge 0$ *and* $x(0) \in \mathcal{H}_n$, *then* $T(s(\tau,x(0))) \overset{\text{a.s.}}{\equiv} \max\{T(x(0)) - \tau, 0\}$.

(ii) $T(\cdot)$ *is sample continuous on* $\mathcal{H}_n$ *if and only if* $T(\cdot)$ *is sample continuous at 0.*

Proof. The proof is a direct consequence of Proposition 3.2 given in Ref. [10] and, hence, is omitted. ■

Next, we present a sufficient condition for global stochastic finite time stability.

Theorem 2.2. *Consider the nonlinear stochastic dynamical system* $\mathcal{G}$ *given by Eq. (1) with* $\mathcal{D} = \mathbb{R}^n$. *If there exist a radially unbounded positive definite function* $V: \mathbb{R}^n \to \mathbb{R}_+$ *and a function* $\eta: \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $V(0) = 0$, $V(x)$ *is two times continuously differentiable for all* $x \in \mathbb{R}^n$, $\eta(\cdot)$ *is continuously differentiable, and, for all* $x \in \mathbb{R}^n$

$$V'(x)f(x) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x)V''(x)D(x) \le -\eta(V(x)) \tag{13}$$

$$\int_0^\varepsilon \frac{\mathrm{d}v}{\eta(v)} < \infty, \quad \varepsilon \in [0,\infty) \tag{14}$$

$$\eta'(v) > 0, \quad v \ge 0 \tag{15}$$

then $\mathcal{G}$ *is globally stochastically finite time stable. Moreover, there exists a settling-time operator* $T: \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$ *such that*

$$\mathbb{E}^{x_0}[T(x_0)] \le \int_0^{V(x_0)} \frac{\mathrm{d}v}{\eta(v)}, \quad x_0 \in \mathbb{R}^n \tag{16}$$

Proof. Let $\varepsilon > 0$ and $\rho > 0$, and define $\mathcal{D}_{\varepsilon,\rho} \triangleq \{x \in \mathcal{B}_\varepsilon(0): V(x) < \alpha(\varepsilon)\rho\}$, where $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function. Since $V(\cdot)$ is continuous and $V(0) = 0$, it follows that $\mathcal{D}_{\varepsilon,\rho}$ is nonempty and there exists $\delta = \delta(\varepsilon,\rho) > 0$ such that $V(x) < \alpha(\varepsilon)\rho$, $x \in \mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0) \subseteq \mathcal{D}_{\varepsilon,\rho}$. Next, $V(x(t))$ is a (positive) supermartingale [15], and hence, for every $x(0) \in \mathcal{H}_n^{\mathcal{B}_\delta(0)} \subseteq \mathcal{H}_n^{\mathcal{D}_\rho}$, it follows from $\alpha(||x||) \le V(x)$, $x \in \mathbb{R}^n$, and the extended version of the Markov inequality for monotonically increasing functions [18] that

$$\begin{aligned}
\mathbb{P}^{x_0}\left(\sup_{t\ge0} ||x(t)|| > \varepsilon\right) &\le \sup_{t\ge0} \frac{\mathbb{E}^{x_0}[\alpha(||x(t)||)]}{\alpha(\varepsilon)} \\
&\le \sup_{t\ge0} \frac{\mathbb{E}^{x_0}[V(x(t))]}{\alpha(\varepsilon)} \\
&\le \frac{\mathbb{E}^{x_0}[V(x(0))]}{\alpha(\varepsilon)} \\
&\le \rho
\end{aligned}$$

which proves Lyapunov stability in probability. ■

To prove global asymptotic stability in probability, it follows from Eq. (13) and Ref. [19] that $\lim_{t\to\infty} \eta(V(x(t)) \overset{\text{a.s.}}{\equiv} 0$, which, since $\eta: \mathbb{R}_+ \to \mathbb{R}_+$, further implies that $\lim_{t\to\infty} V(x(t)) \overset{\text{a.s.}}{=} 0$. Now, it follows from $\alpha(||x||) \le V(x)$, $x \in \mathbb{R}^n$, that

$$\lim_{t\to\infty} \alpha(||x(t)||) \le \lim_{t\to\infty} V(x(t)) \overset{\text{a.s.}}{=} 0$$

which implies $\mathbb{P}^{x_0}(\lim_{t\to\infty} ||x(t)|| = 0) = 1$ for all $x_0 \in \mathbb{R}^n$. Hence, $\mathcal{G}$ is globally asymptotically stable in probability and the stochastic settling-time operator $T(x(0)) \le \infty$ almost surely.

Next, we show that $T(x(0))$ is finite with probability one and satisfies (16), and hence, $\mathbb{E}^{x_0}[T(x(0))] < \infty$. Define

$$T_0 \triangleq T(x(0)), \quad \alpha(V) \triangleq \int_0^V \frac{\mathrm{d}v}{\eta(v)}, \quad V \in \bar{\mathbb{R}}_+$$

Now, using Itô's (chain rule) formula, the stochastic differential of $V(x(t))$ along the system trajectories $x(t)$, $t \ge 0$, is given by

$$\mathrm{d}V(x(t)) = \mathcal{L}V(x(t))\mathrm{d}t + \frac{\partial V}{\partial x}D(x(t))\mathrm{d}w(t)$$

Next, using Eq. (13), it follows that

$$\begin{aligned}
\int_0^{T_0} \mathrm{d}\tau &= \int_0^{T_0} \frac{\eta(V(x(\tau)))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\
&\le \int_0^{T_0} -\frac{\mathcal{L}V(x(\tau))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\
&\le \int_0^{T_0} -\frac{\mathrm{d}V(x(t))}{\eta(V(x(\tau)))} + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) \\
&= \int_0^{T_0} -\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau)
\end{aligned} \tag{17}$$

Once again, using Itô's (chain rule) formula, it follows that

$$\begin{aligned}
&\mathrm{d}\alpha(V(x(t))) \\
&= \left[\frac{\partial\alpha(V(x))}{\partial x}f(x(t)) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x(t))\frac{\partial^2\alpha(V(x))}{\partial x^2}D(x(t))\right]\mathrm{d}t + \frac{\partial\alpha(V(x))}{\partial x}\mathrm{d}w(t) \\
&= \left[\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x(t))\frac{\partial}{\partial x}\left(\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\right)D(x(t))\right]\mathrm{d}t + \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\mathrm{d}w(t) \\
&= \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\left[\left(\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x(t))\frac{\partial^2(V(x))}{\partial x^2}D(x(t))\right)\mathrm{d}t + \frac{\partial V(x)}{\partial x}\mathrm{d}w(t)\right] + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x(t))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t \\
&= \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x(t))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t
\end{aligned} \tag{18}$$

Hence, it follows from Eqs. (17) and (15) that

$$\begin{aligned}\int_0^{T_0}\mathrm{d}\tau &\le \int_0^{T_0} -\mathrm{d}\alpha(V(x(\tau))) + \int_0^{T_0}\frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) + \int_0^{T_0}\frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x(\tau))\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(\tau))\mathrm{d}\tau \\ &= \alpha(V(x(0))) - \alpha(V(x(T_0))) + \int_0^{T_0}\frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) - \int_0^{T_0}\frac{\eta'(V)}{\eta^2(V)}\frac{1}{2}\mathrm{tr}\left(\frac{\partial V(x)}{\partial x}D^{\mathrm{T}}(x(\tau))\right)^{\mathrm{T}}\left(\frac{\partial V(x)}{\partial x}D(x(\tau))\right)\mathrm{d}\tau \\ &\le \int_0^{V(x(0))}\frac{\mathrm{d}v}{\eta(V)} - \int_0^{V(x(T_0))}\frac{\mathrm{d}v}{\eta(V)} + \int_0^{T_0}\frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau)\end{aligned} \tag{19}$$

Taking the expectation on both sides of Eq. (19) and using the fact that $x(0)\overset{\text{a.s.}}{=}x_0$ and $x(T_0)\overset{\text{a.s.}}{\equiv}0$ yields

$$\mathbb{E}^{x_0}\left[\int_0^{T_0}\mathrm{d}\tau\right] = \mathbb{E}^{x_0}[T(x_0)] \le \int_0^{V(x_0)}\frac{\mathrm{d}v}{\eta(V)} \tag{20}$$

which implies Eq. (16).

Remark 2.2. If $\eta(V)=cV^{\theta}$, where $c>0$ and $\theta\in(0,1)$, then $\eta(\cdot)$ satisfies Eqs. (14) and (15). In this case, Eq. (16) becomes

$$\mathbb{E}^{x_0}[T(x(0))] \le \frac{V(x_0)^{1-\theta}}{c(1-\theta)}$$

For deterministic dynamical systems, this specialization recovers the finite time stability results given in Ref. [20].

Finally, we consider the controlled nonlinear stochastic dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\mathrm{d}t + D(x(t))\mathrm{d}w(t),\quad x(0)\overset{\text{a.s.}}{=}x_0,\quad t\ge 0 \tag{21}$$

$$y(t) = -\phi(x(t)) \tag{22}$$

where $\phi:\mathbb{R}^n\to\mathbb{R}^m$, with a nonlinear-nonquadratic performance criterion

$$J(x_0,u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^{\infty}\left[L_1(x(t)) + u^{\mathrm{T}}(t)R_2(x(t))u(t)\right]\mathrm{d}t\right] \tag{23}$$

where $L_1:\mathbb{R}^n\to\mathbb{R}$ and $R_2:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ are such that $L_1(x)\ge 0$, $x\in\mathbb{R}^n$, and $R_2(x)>0$, $x\in\mathbb{R}^n$. In this case, the optimal nonlinear feedback controller $u=\phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (23) is given by the following result. For the statement of this result, define the set of stochastic regulation controllers given by

$$\begin{aligned}\mathcal{S}(x_0) \triangleq \Big\{&u(\cdot):u(\cdot)\text{ is admissible and } x(\cdot)\text{ given by (21) is such} \\ &\text{that } \mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)=1, x_0\in\mathbb{R}^n,\text{ where } \mathfrak{B}_{x_0}^{u(\cdot)} \\ &\triangleq \left\{x(\{t\ge t_0\},\omega): \lim_{t\to\infty}||x(t,\omega)||=0,\ \omega\in\Omega\right\}\Big\}\end{aligned}$$

THEOREM 2.3. *Consider the nonlinear stochastic dynamical system (21) with performance functional (23) with* $L_1(x)\ge 0$, $x\in\mathbb{R}^n$. *Assume that there exists a two times continuously differentiable function* $V:\mathbb{R}^n\to\mathbb{R}$ *such that*

$$V(0)=0 \tag{24}$$

$$V(x)>0,\quad x\in\mathbb{R}^n,\quad x\ne 0 \tag{25}$$

$$\begin{aligned}0 &= L_1(x) + V'(x)f(x) + \frac{1}{2}\mathrm{tr}\,D^{\mathrm{T}}(x)V''(x)D(x) \\ &\quad -\frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x),\quad x\in\mathbb{R}^n\end{aligned} \tag{26}$$

and

$$V(x)\to\infty \text{ as } ||x||\to\infty \tag{27}$$

Furthermore, assume that the systems (23) and (22) are zero-state observable with $y=L_1(x)$. *Then, the zero solution* $x(t)\overset{\text{a.s.}}{\equiv}0$ *of the closed-loop system*

$$\begin{aligned}\mathrm{d}x(t) &= [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t \\ &\quad + D(x(t))\mathrm{d}w(t),\quad x(0)\overset{\text{a.s.}}{=}x_0,\quad t\ge 0\end{aligned} \tag{28}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) \tag{29}$$

and the performance functional (22) is minimized in the sense that

$$J(x_0,\phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0,u(\cdot)),\quad x_0\in\mathbb{R}^n \tag{30}$$

Finally,

$$J(x_0,\phi(x(\cdot))) = V(x_0),\quad x_0\in\mathbb{R}^n \tag{31}$$

Proof. The proof is similar to the proof of Theorem 8.3 for the deterministic optimal control problem given in Ref. [21]. ■

Finally, we provide sector and gain margins for the nonlinear stochastic dynamical system $\mathcal{G}$ given by Eqs. (21) and (22). For the statement of the next theorem, recall the definitions of gain and sector margins for $\mathcal{G}$ given in Ref. [21].

THEOREM 2.4. *Consider that the nonlinear stochastic dynamical system* $\mathcal{G}$ *given by (21) and (22) where* $\phi(x)$ *is a stabilizing feedback control law given by (29) and where* $V(x)$, $x\in\mathbb{R}^n$, *satisfies (26). Furthermore, assume* $R_2(x)=\mathrm{diag}[r_1(x),\ldots,r_m(x)]$, *where* $r_i:\mathbb{R}^n\to\mathbb{R}$, $r_i(x)>0$, $i=1,\ldots,m$, *and* $L_1(x)\ge 0$, $x\in\mathbb{R}^n$. *Then, the nonlinear dynamical system* $\mathcal{G}$ *has a sector (and, hence, gain) margin* $(1/2,\infty)$.

Proof. The result is a direct consequence of Theorem 6.4 of Ref. [12]. ■

3 Stochastic Control Lyapunov Functions

In this section, we consider a feedback control problem and introduce the notion of *stochastic control Lyapunov functions*. Furthermore, using the concept of stochastic control Lyapunov functions, we provide necessary and sufficient conditions for stochastic nonlinear system stabilization.

Consider the nonlinear stochastic controlled dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0 \tag{32}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $F: \mathcal{D} \times U \to \mathbb{R}^n$, and $D: \mathcal{D} \times U \to \mathbb{R}^{n \times d}$. Here, we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that Eq. (32) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in Eq. (32) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq t_0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(t_0)$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume that $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (2) and (3) hold for the controlled drift and diffusion terms $F(x, u)$ and $D(x, u)$ uniformly in u. In this case, it follows from Theorem 2.2.4 of Ref. [22] that there exists a pathwise unique solution to Eq. (32) in $\left(\Omega, \{\mathcal{F}_{t \geq t_0}\}, \mathbb{P}^{x_0}\right)$.

A measurable function $\phi: \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies Eq. (32), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, the *closed-loop system* (32) has the form

$$\begin{aligned}\mathrm{d}x(t) &= F(x(t), \phi(x(t)) \\ &\quad + D(x(t), \phi(x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0\end{aligned} \tag{33}$$

The following two definitions are required for stating the results of this section.

Definition 3.1. *Let $\phi: \mathcal{D} \to U$ be a measurable mapping on $\mathcal{D} \backslash \{0\}$ with $\phi(0) = 0$. Then, (32) is* stochastically feedback asymptotically stabilizable *if the zero solution $x(t) \overset{\text{a.s.}}{=} 0$ of the closed-loop system (33) is stochastically asymptotically stable.*

Definition 3.2. *Consider the controlled nonlinear stochastic dynamical system given by (32). A two times continuously differentiable positive-definite function $V: \mathcal{D} \to \mathbb{R}$ satisfying [8]*

$$\inf_{u \in U}\left[V'(x)F(x,u) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x,u)V''(x)D(x,u)\right] < 0, \quad x \in \mathcal{D}, \quad x \neq 0 \tag{34}$$

is called a stochastic control Lyapunov function.

Note that if Eq. (34) holds, then there exists a feedback control law $\phi: \mathcal{D} \to U$ such that $V'(x)F(x, \phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x, \phi(x)) V''(x)D(x, \phi(x)) < 0$, $x \in \mathcal{D}, x \neq 0$, and hence, Theorem 2.1 implies that if there exists a stochastic control Lyapunov function for the nonlinear stochastic dynamical system (32), then there exists a feedback control law $\phi(x)$ such that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop nonlinear stochastic dynamical system (32) is stochastically asymptotically stable. Conversely, if there exists a feedback control law $u = \phi(x)$ such that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the nonlinear stochastic dynamical system (32) is stochastically asymptotically stable and $D(x)$, $x \in \mathbb{R}^n$, satisfies a nondegeneracy condition, then it follows from Theorem 3.2 of Ref. [23] that there exists a two times continuously differentiable positive-definite function $V: \mathcal{D} \to \mathbb{R}$ such that $V'(x)F(x, \phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x, \phi(x))V''(x)D(x, \phi(x)) < 0$, $x \in \mathcal{D}$, $x \neq 0$, or, equivalently, there exists a stochastic control Lyapunov function for the nonlinear stochastic dynamical system (32). Hence, a given nonlinear stochastic dynamical system of the form (32) is stochastically feedback asymptotically stabilizable if and only if there exists a stochastic control Lyapunov function satisfying Eq. (34). Finally, in the case where $\mathcal{D} = \mathbb{R}^n$ and $U = \mathbb{R}^m$, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to Eq. (32) is globally stochastically asymptotically stabilizable if and only if $V(x) \to \infty$ as $\|x\| \to \infty$.

Next, we consider the special case of nonlinear stochastic affine systems in the control and construct state feedback controllers that globally stochastically asymptotically stabilize the zero solution of the nonlinear stochastic dynamical system under the assumption that the system has a radially unbounded stochastic control Lyapunov function. Specifically, we consider nonlinear stochastic affine systems of the form

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0 \tag{35}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G: \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $D: \mathbb{R}^n \to \mathbb{R}^d$, and $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are continuous functions.

Theorem 3.1. *Consider the controlled nonlinear stochastic dynamical system given by Eq. (35). Then, a two times continuously differentiable positive-definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ is a stochastic control Lyapunov function of Eq. (35) if and only if*

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{R} \tag{36}$$

where $\mathcal{R} \triangleq \{x \in \mathbb{R}^n, x \neq 0: \ V'(x)G(x) = 0\}$.

Proof. The proof is a direct consequence of the definition of a stochastic control Lyapunov function by noting that for systems of the form (35)

$$\inf_{u \in \mathbb{R}^m}\left[V'(x)[f(x) + G(x)u] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x)\right] = -\infty, \quad x \notin \mathcal{R}, \quad x \neq 0$$

Hence, Eq. (34) is equivalent to Eq. (36), which proves the result. ∎

It follows from Theorem 3.1 that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of a nonlinear stochastic affine system of the form (35) is globally stochastically feedback asymptotically stabilizable if and only if there exists a two times continuously differentiable positive-definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ satisfying Eq. (36). Hence, Theorem 3.1 provides necessary and sufficient conditions for nonlinear stochastic system stabilization.

Next, using Theorem 3.1, we *construct* an explicit feedback control law that is a function of the stochastic control Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \dfrac{(\alpha(x) + \xi(x)) + \sqrt{(\alpha(x) + \xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0 \end{cases} \tag{37}$$

where $\alpha(x) \triangleq V'(x)f(x)$, $\beta(x) \triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $\xi(x) \triangleq \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x)$, and $c_0 \geq 0$. In this case, the stochastic control Lyapunov function $V(\cdot)$ of Eq. (35) is a Lyapunov function for the closed-loop system (35) with $u = \phi(x)$, where $\phi(x)$ is given by Eq. (37). In particular, the infinitesimal generator $\mathcal{L}V(\cdot)$ of the nonlinear stochastic dynamical system (35) with $u = \phi(x)$ given by Eq. (37) is given by

$$\begin{aligned}\mathcal{L}V(x) &\triangleq V'(x)[f(x) + G(x)\phi(x)] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \alpha(x) + \beta^{\mathrm{T}}(x)\phi(x) + \xi(x) \\ &= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}, & \beta(x) \neq 0, \\ \alpha(x) + \xi(x), & \beta(x) = 0, \end{cases} \\ &< 0, \quad x \in \mathbb{R}^n, \quad x \neq 0 \end{aligned} \tag{38}$$

which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (35) guaranteeing global stochastic asymptotic stability with $u = \phi(x)$ given by Eq. (37).

Remark 3.1. Note that the concept of a stochastic control Lyapunov function involving differentiability of higher order than one for $V(\cdot)$ along with the constructive feedback control law (37) based on the stochastic control Lyapunov function generalizes Sontag's universal feedback control formula for deterministic systems to stochastic dynamical systems. In particular, setting the diffusion term $D(x) \equiv 0$ in Eq. (37), one recovers the standard universal feedback control formula as given in Ref. [2].

Since $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are smooth, it follows that $\alpha(x)$, $\beta(x)$, and $\xi(x)$, $x \in \mathbb{R}^n$, are smooth functions, and hence, $\phi(x)$ given by Eq. (37) is smooth for all $x \in \mathbb{R}^n$ if either $\beta(x) \neq 0$ or $\alpha(x) + \xi(x) < 0$. Hence, the feedback control law given by Eq. (37) is smooth everywhere except for the origin. The following result provides necessary and sufficient conditions under which the feedback control law given by Eq. (37) is guaranteed to be continuous and Lipschitz continuous at the origin in addition to being smooth everywhere else.

THEOREM 3.2. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (35) with a radially unbounded stochastic control Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$. Then, the following statements hold:*

(*i*) *The control law $\phi(x)$ given by (37) is continuous at $x = 0$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < ||x|| < \delta$, there exists $u \in \mathbb{R}^m$ such that $||u|| < \varepsilon$ and $\alpha(x) + \beta^{\mathrm{T}}(x)u + \xi(x) < 0$.*

(*ii*) *There exists a stabilizing control law $\hat{\phi}(x)$ such that $\alpha(x) + \beta^{\mathrm{T}}(x)\hat{\phi}(x) + \xi(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and $\hat{\phi}(x)$ is Lipschitz continuous at $x = 0$ if and only if the control law $\phi(x)$ given by Eq. (37) is Lipschitz continuous at $x = 0$.*

Proof. Necessity of (i) is trivial with $u = \phi(x)$. Conversely, assume that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < ||x|| < \delta$, there exists $u \in \mathbb{R}^m$ such that $||u|| < \varepsilon$ and $\alpha(x) + \beta^{\mathrm{T}}(x)u + \xi(x) < 0$. In this case, since $||u|| < \varepsilon$ it follows from the Cauchy-Schwarz inequality that $\alpha(x) + \xi(x) < \varepsilon\|\beta(x)\|$. Furthermore, since $V(\cdot)$ is two-times continuously differentiable and $G(\cdot)$ is continuous it follows that there exists $\hat{\delta} > 0$ such that for all $0 < ||x|| < \hat{\delta}$, $\|\beta(x)\| < \varepsilon$. Hence, for all $0 < ||x|| < \delta_{\min}$, where $\delta_{\min} \triangleq \min\{\delta, \hat{\delta}\}$, it follows that $\alpha(x) + \xi(x) < \varepsilon\|\beta(x)\|$ and $\|\beta(x)\| < \varepsilon$. Furthermore, if $\beta(x) = 0$, then $\|\phi(x)\| = 0$, and if $\beta(x) \neq 0$, then it follows from Eq. (37) that

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| \\ &\quad + \frac{|\alpha(x) + \xi(x) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}|}{\|\beta(x)\|} \\ &\leq \frac{2(\alpha(x)+\xi(x)) + (c_0+1)\|\beta(x)\|^2}{\|\beta(x)\|} \\ &\leq (c_0+3)\varepsilon, \quad 0 < ||x|| < \delta_{\min}, \quad \alpha(x) + \xi(x) > 0\end{aligned}$$

and

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| \\ &\quad + \frac{(\alpha(x) + \xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\|\beta(x)\|} \\ &\leq c_0\|\beta(x)\| + \frac{\beta^{\mathrm{T}}(x)\beta(x)}{\|\beta(x)\|} \\ &= (c_0+1)\|\beta(x)\| < (c_0+1)\varepsilon, \qquad 0 < ||x|| < \delta_{\min}, \\ &\quad \alpha(x) + \xi(x) \leq 0\end{aligned}$$

Hence, it follows that for every $\hat{\varepsilon} \triangleq (c_0+3)\varepsilon > 0$, there exists $\delta_{\min} > 0$ such that for all $||x|| < \delta_{\min}$, $\|\phi(x)\| < \hat{\varepsilon}$, which implies that $\phi(\cdot)$ is continuous at the origin.

Next, to show necessity of (ii), assume that there exists a stabilizing control $\hat{\phi}(x)$ such that $\alpha(x) + \beta^{\mathrm{T}}(x)\hat{\phi}(x) + \xi(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, and $\hat{\phi}(x)$ is Lipschitz continuous at $x = 0$ with a Lipschitz constant $\hat{L}$; that is, there exists $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(0)$, $\|\hat{\phi}(x)\| \leq \hat{L}||x||$. Now, since $V(\cdot)$ is continuous and $V'(0) = 0$, it follows that there exists $K > 0$ such that $\|\beta(x)\| \leq K||x||$, $x \in \mathcal{B}_\delta(0)$. Hence

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| \\ &\quad + \frac{|(\alpha(x) + \xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}|}{\|\beta(x)\|} \\ &\leq \frac{2(\alpha(x)+\xi(x)) + (c_0+1)\|\beta(x)\|^2}{\|\beta(x)\|} \\ &\leq \left(2\hat{L} + (c_0+1)K\right)||x||, \quad x \in \mathcal{B}_\delta(0), \quad \alpha(x) + \xi(x) > 0\end{aligned}$$

and

$$\begin{aligned}\|\phi(x)\| &\leq c_0\|\beta(x)\| \\ &\quad + \frac{(\alpha(x) + \xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\|\beta(x)\|} \\ &\leq c_0\|\beta(x)\| + \frac{\beta^{\mathrm{T}}(x)\beta(x)}{\|\beta(x)\|} = (c_0+1)\|\beta(x)\| \\ &< (c_0+1)K||x||, \quad x \in \mathcal{B}_\delta(0), \quad \alpha(x) + \xi(x) \leq 0\end{aligned}$$

which implies that for all $x \in \mathcal{B}_\delta(0)$, $\|\phi(x)\| \leq L||x||$, where $L \triangleq 2\hat{L} + (c_0+1)K$, and hence, $\phi(\cdot)$ is Lipschitz continuous.

Finally, sufficiency of (ii) follows immediately with $\hat{\phi}(x) = \phi(x)$. ∎

Next, we present sufficient conditions for stochastic finite time stabilization using a control Lyapunov function involving a scalar differential inequality.

THEOREM 3.3. *Consider the nonlinear stochastic dynamical system (35). Assume that there exists a two-times continuously differentiable function $V:\mathcal{D}\to\bar{\mathbb{R}}_+$ such that $V(\cdot)$ is positive definite and*

$$V'(x)f(x)+\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)V''(x)D(x)\le -c(V(x))^{\alpha},\quad x\in\mathcal{R} \tag{39}$$

where $c>0$, $\alpha\in(0,1)$, and $\mathcal{R}\triangleq\{x\in\mathbb{R}^n, x\neq 0:\ V'(x)G(x)=0\}$. Then, the nonlinear stochastic dynamical system (35) with the feedback controller $u=\phi(x)$, $x\in\mathbb{R}^n$, given by

$$\phi(x)=\begin{cases}-\left(c_0+\dfrac{\left(\alpha(x)+\xi(x)+c(V(x))^{\alpha}\right)+\sqrt{\left(\alpha(x)+\xi(x)+c(V(x))^{\alpha}\right)^2+\left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x)\neq 0,\\ 0, & \beta(x)=0\end{cases} \tag{40}$$

where $c_0>0$, $\alpha(x)\triangleq V'(x)f(x)$, $x\in\mathbb{R}^n$, $\beta(x)\triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $x\in\mathbb{R}^n$, and $\xi(x)\triangleq(1/2)D^{\mathrm{T}}(x)V''(x)D(x)$, $x\in\mathbb{R}^n$, is stochastically finite time stable and there exists a stochastic settling time operator $T:\mathcal{H}_m\to\mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}\left[T(x_0)\right]\le\frac{1}{c(1-\alpha)}\left(V(x_0)\right)^{1-\alpha},\quad x_0\in\mathbb{R}^n \tag{41}$$

Furthermore, $V(\cdot)$ is a stochastic control Lyapunov function.

Proof. The infinitesimal generator $\mathcal{L}V(\cdot)$ of the closed-loop stochastic dynamical system (35), with $u=\phi(x)$, $x\in\mathbb{R}^n$, given by Eq. (40), is given by

$$\begin{aligned}\mathcal{L}V(x)&=V'(x)f(x)+V'(x)G(x)\phi(x)+\frac{1}{2}D^{\mathrm{T}}(x)V''(x)D(x)\\&=\alpha(x)+\beta^{\mathrm{T}}(x)\phi(x)+\xi(x)\\&=\begin{cases}-c_0\beta^{\mathrm{T}}(x)\beta(x)-\sqrt{\left(\alpha(x)+\xi(x)+c(V(x))^{\alpha}\right)^2+\left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}-c(V(x))^{\alpha}, & \beta(x)\neq 0,\\ \alpha(x)+\xi(x), & \beta(x)=0,\end{cases}\\&<-c(V(x))^{\alpha},\quad x\in\mathbb{R}^n\end{aligned} \tag{42}$$

Now, it follows from Theorem 2.2 with $\eta(V)=cV^{\theta}$ that the zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ to Eq. (35) is stochastically finite time stable with the stochastic settling time $\mathbb{E}^{x_0}\left[T(x_0)\right]\le\left(1/c(1-\alpha)\right)\left(V(x_0)\right)^{1-\alpha}$, $x_0\in\mathbb{R}^n$. In this case, it follows from Definition 3.2 that $V(x)$, $x\in\mathbb{R}^n$, is a stochastic control Lyapunov function. ■

Since $f(\cdot)$, $G(\cdot)$, and $D(\cdot)$ are continuous and $V(\cdot)$ is two times continuously differentiable, it follows that $\alpha(x)$, $\beta(x)$, and $\xi(x)$, $x\in\mathbb{R}^n$, are continuous functions, and hence, $\phi(x)$ given by (40) is continuous for all $x\in\mathbb{R}^n$ if either $\beta(x)\neq 0$ or $\alpha(x)+\xi(x)+c(V(x))^{\alpha}<0$ for all $x\in\mathbb{R}^n$. Hence, the feedback control law given by Eq. (40) is continuous everywhere except for the origin. However, as shown in Theorem 3.2, the feedback control law $\phi(x)$ given in Eq. (40) is continuous on $\mathbb{R}^n$ if and only if, for every $\varepsilon>0$, there exists $\delta>0$ such that for all $0<||x||<\delta$, there exists $u\in\mathbb{R}^m$ such that $||u||<\varepsilon$ and $\alpha(x)+\beta^{\mathrm{T}}(x)u+\xi(x)+c(V(x))^{\alpha}<0$.

4 Meaningful Inverse Optimality and Control Lyapunov Functions

In this section, we show that given a stochastic control Lyapunov function for a controlled nonlinear stochastic dynamical system, the feedback control law given by Eq. (37) guarantees sector and gain margins of $(1/2,\infty)$.

THEOREM 4.1. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by Eq. (21) and let the two times continuously differentiable positive-definite, radially unbounded function $V:\mathbb{R}^n\to\mathbb{R}$ be a stochastic control Lyapunov function of Eq. (21), that is,*

$$V'(x)f(x)+\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)V''(x)D(x)<0,\quad x\in\mathcal{R}, \tag{43}$$

where $\mathcal{R}\triangleq\{x\in\mathbb{R}^n:\ x\neq 0,\ V'(x)G(x)=0\}$. Then, with the feedback stabilizing control law given by

$$\phi(x)=\begin{cases}-\left(c_0+\dfrac{\left(\alpha(x)+\xi(x)\right)+\sqrt{\left(\alpha(x)+\xi(x)\right)^2+\left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x)\neq 0,\\ 0, & \beta(x)=0,\end{cases} \tag{44}$$

where $\alpha(x)\triangleq V'(x)f(x)$, $\beta(x)\triangleq G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)$, $\xi(x)=\frac{1}{2}\operatorname{tr}D^{\mathrm{T}}(x)V''(x)D(x)$, and $c_0>0$, the nonlinear stochastic dynamical system $\mathcal{G}$ given by Eqs. (21) and (22) has a sector (and, hence, gain) margin $(1/2,\infty)$. Furthermore, with the feedback control law $u=\phi(x)$ the performance functional

$$J(x_0,u(\cdot))=\mathbb{E}^{x_0}\left[\int_0^{\infty}\left[\alpha(x(t))+\xi(x(t))-\frac{\gamma(x(t))}{2}\beta^{\mathrm{T}}(x(t))\beta(x(t))+\frac{1}{2\gamma(x(t))}u^{\mathrm{T}}(t)u(t)\right]\mathrm{d}t\right], \tag{45}$$

where

$$\gamma(x) \triangleq \begin{cases} \left(c_0 + \dfrac{(\alpha(x)+\xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2}}{\beta^{\mathrm{T}}(x)\beta(x)} \right), & \beta(x) \neq 0, \\ c_0, & \beta(x) = 0 \end{cases} \tag{46}$$

is minimized in the sense that

$$J\left(x_0, \phi(x(\cdot))\right) = \min_{u \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n \tag{47}$$

Proof. The result is a direct consequence of Theorems 2.3 and 2.4 with $R_2(x) = \left(1/2\gamma(x)\right)I_m$ and $L_1(x) = -(\alpha(x) + \xi(x)) + \left(\gamma(x)/2\right)\beta^{\mathrm{T}}(x)\beta(x)$. Specifically, it follows from Eq. (46) that $R_2(x) > 0$, $x \in \mathbb{R}^n$, and

$$\begin{aligned} L_1(x) &= -(\alpha(x) + \xi(x)) + \frac{\gamma(x)}{2}\beta^{\mathrm{T}}(x)\beta(x) \\ &= \begin{cases} \dfrac{1}{2}\left(c_0\beta^{\mathrm{T}}(x)\beta(x) - (\alpha(x)+\xi(x)) + \sqrt{(\alpha(x)+\xi(x))^2 + \left(\beta^{\mathrm{T}}(x)\beta(x)\right)^2} \right), & \beta(x) \neq 0, \\ -(\alpha(x)+\xi(x)), & \beta(x) = 0 \end{cases} \end{aligned} \tag{48}$$

Now, it follows from Eq. (48) that $L_1(x) \geq 0$, $\beta(x) \neq 0$, and, since $V(\cdot)$ is a stochastic control Lyapunov function of Eq. (21), it follows from Theorem 3.1 that $L_1(x) = -(\alpha(x) + \xi(x)) \geq 0$ for all $x \in \mathcal{R} = \{x \in \mathbb{R}^n\colon x \neq 0,\ \beta(x) = 0\}$. Hence, Eq. (48) yields $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, so that all conditions of Theorem 2.4 are satisfied. ■

Theorem 4.1 shows that given a nonlinear stochastic dynamical system for which a stochastic control Lyapunov function can be constructed, the feedback control law given by Eq. (44) is inverse optimal with respect to a meaningful cost functional and has a sector (and, hence, gain) margin $(1/2, \infty)$.

Remark 4.1. Using the stochastic finite time optimal feedback control framework developed in Ref. [10], the stochastic finite time controller (40) can also be shown to be inverse optimal with respect to a *meaningful* (in the terminology of Ref. [11]) nonlinear-nonquadratic performance functional with guaranteed sector and gain margins. However, due to the space limitations, we do not present this result here.

5 Illustrative Numerical Example

Our example considers control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [24–27]. The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations, which can severely impact operating conditions and system performance. These pressure oscillations, known as *thermoacoustic instabilities*, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting. Hence, the need for active control to mitigate combustion-induced pressure instabilities is critical.

In this section, we design a finite time stabilizing controller for a two-mode, nonlinear time-averaged combustion model with nonlinearities present due to the second-order gas dynamics. This model is developed in Ref. [24] and is given by

$$\begin{aligned} \mathrm{d}x_1(t) &= [\alpha_1 x_1(t) + \theta_1 x_2(t) - \beta(x_1(t)x_3(t) + x_2(t)x_4(t)) + u_1(t)]\mathrm{d}t \\ &\quad + \sigma_1 x_1(t)\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0 \end{aligned} \tag{49}$$

$$\begin{aligned} \mathrm{d}x_2(t) &= [-\theta_1 x_1(t) + \alpha_1 x_2(t) + \beta(x_2(t)x_3(t) - x_1(t)x_4(t)) + u_2(t)]\mathrm{d}t \\ &\quad + \sigma_2 x_2(t)\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20} \end{aligned} \tag{50}$$

$$\begin{aligned} \mathrm{d}x_3(t) &= \left[\alpha_2 x_3(t) + \theta_2 x_4(t) + \beta\left(x_1^2(t) - x_2^2(t)\right) + u_3(t)\right]\mathrm{d}t \\ &\quad + \sigma_3 x_3(t)\mathrm{d}w(t), \quad x_3(0) \overset{\text{a.s.}}{=} x_{30} \end{aligned} \tag{51}$$

$$\begin{aligned} \mathrm{d}x_4(t) &= [-\theta_2 x_3(t) + \alpha_2 x_4(t) + 2\beta x_1(t)x_2(t) + u_4(t)]\mathrm{d}t \\ &\quad + \sigma_4 x_4(t)\mathrm{d}w(t), \quad x_4(0) \overset{\text{a.s.}}{=} x_{40} \end{aligned} \tag{52}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ represent growth/decay constants, $\theta_1, \theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = \left((\gamma+1)/8\gamma\right)\omega_1$, where γ denotes the ratio of specific heats, ω_1 is the frequency of the fundamental mode, σ_1, σ_2, σ_3, and $\sigma_4 \in \mathbb{R}$ represent augmentation factors of the variance of the state dependent stochastic disturbance, and u_i, $i = 1, \ldots, 4$, are control input signals. For the data parameters $\alpha_1 = 5$, $\alpha_2 = -55$, $\theta_1 = 4$, $\theta_2 = 32$, $\gamma = 1.4$, $\omega_1 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$, and $x_0 = [2, 3, 1, 1]^{\mathrm{T}}$, the open-loop (i.e., $u_i(t) \equiv 0, i = 1, \ldots, 4$) dynamics (49)–(52) result in sustained oscillations.

To stabilize this system in finite time, we design a feedback control law given by Eq. (40), where $V(x) = (1/2)x^{\mathrm{T}}x$, $x \in \mathbb{R}^4$, $c = 1$, $c_0 = 1$, $\alpha = (3/4)$. In this case, $V'(x) = x^{\mathrm{T}}$, $G(x) = I_4$, and hence, $\mathcal{R} = \{x \in \mathbb{R}^4, x \neq 0\colon x^{\mathrm{T}} = 0\} = \varnothing$. Thus, condition (39)

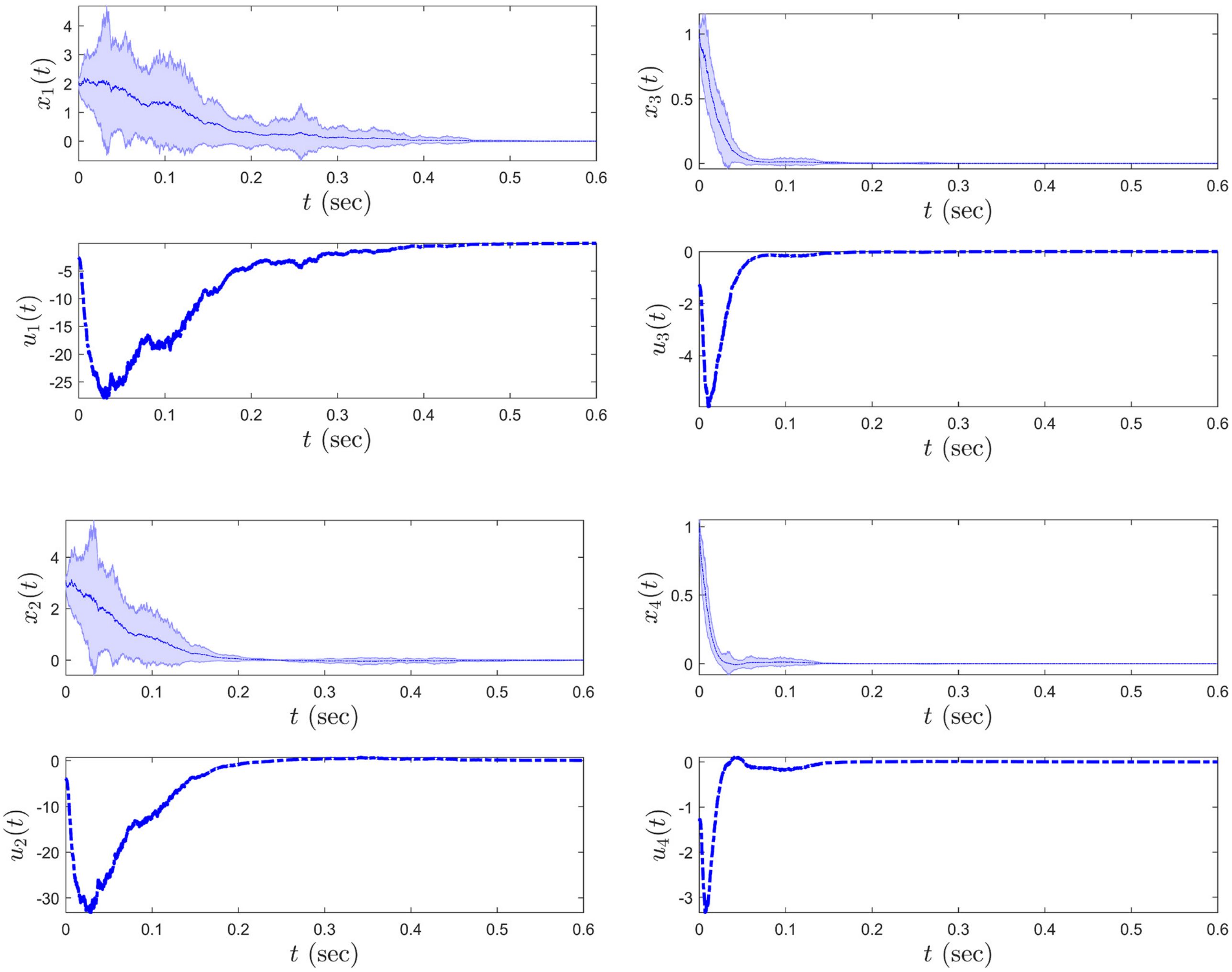

Fig. 1 A sample trajectory along with the sample standard deviation of the closed-loop system trajectories versus time. The control profile is plotted as the mean of the 30 sample runs (see color figure online).

is trivially satisfied and it follows from Theorem 3.3 that the closed-loop system (49)–(52) with the feedback control law (40) is finite time stable with $\mathbb{E}^{x_0}[T(x_0)] \leq 6.6195$. Figure 1 shows a sample trajectory along with the standard deviation of the state trajectories for $x_0 = [2, 3, 1, 1]^{\mathrm{T}}$ of the controlled system versus time along with the mean control signal versus time for 30 sample paths.

6 Conclusion

In this paper, we developed a constructive universal finite time stabilizing feedback control law for stochastic dynamical systems driven by Wiener processes based on the existence of a stochastic control Lyapunov function. Furthermore, the proposed control framework was used to construct stabilizing controllers for nonlinear stochastic dynamical systems with robustness guarantees against multiplicative input uncertainty. In future research, we will establish connections between the recently developed notion of stochastic dissipativity [28] and optimality [11] of the proposed stochastic finite time feedback control law.

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