benefits and costs to influence the rate allocation determined by an
uninformed queue manager.

The model of an M/M/1 queue is at the core of many models of
a data communications network. In such networks, it is highly
desirable for a controller to adjust the flow of throughput as a
function of the impact such adjustments have on the users. Typically,
there is a significant degree of communication between the network
administration and the users to determine an “ideal” or efficient
use of the common resource. For example, communication occurs
via regular e-mail or by way of a users’ committee that meets
periodically and reallocations are made—the iterations of the process
in this paper is a stylized model of such periodic communication and
adjustment. The problem of manipulability through the reporting of
false information becomes a crucial issue when such communication
takes place. The approach to this problem in [1] is based on
the planning algorithms for public goods provision developed by
Malinvaud [12] and Drèze and de la Vallée Poussin [13].

The current paper is motivated by [1]. The incentive compatible
flow control algorithm given therein requires a subsidization of the
users by the queue manager to ensure manipulation-proofness. We
have argued that such subsidization renders the algorithm impractical
and theoretically inconsistent. We have proposed a budget balanced
flow control algorithm which is also incentive compatible.

It is assumed, however, that the users are myopic optimizers and
maximize their utilities at each instant of the procedure rather than
considering the entire time horizon. Allowing for far-sighted users
leads to impossibility theorems (originating with [14]) which propose
that incentive compatibility, budget balance and far-sightedness are
incompatible requirements. In Chakravorti [15], we address this issue
by weakening the incentive requirement in the context of flow control
in a computer network.

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Optimal Dynamic Output Feedback for Nonzero
Set Point Regulation: The Discrete-Time Case

Wassim M. Haddad and Raymond Moser

Abstract—Standard discrete-time LQG theory is generalized to a
regulation problem involving a priori specified nonzero set points for
the state and control variables and nonzero-mean disturbances. The
optimal control law consists of a closed-loop component feeding back
measurements and a constant open-loop component which accounts for
the nonzero set point and the nonzero mean disturbance. For generality,
the results are obtained for the problem of fixed-order (i.e., full-
and reduced-order) dynamic compensation. It is shown that the closed-loop
controller can be designed independently of the open-loop control.

I. INTRODUCTION

As discussed by Haddad and Bernstein [1], the standard discrete-
time quadratic performance criterion expresses the desire to maintain
the state and the control variables in the neighborhood of the origin.
As is well known [2, pp. 504-509], if regulation is desired about
nonzero state and control offsets, then the nonzero set point problem
presents no additional difficulty so long as the state set points can
be translated to the origin and standard regulation theory can be
applied. A closer inspection, however, reveals that such a translation
may be either suboptimal or impossible. Specifically, the offset in
the control may correspond to an unacceptably high level of control
effort. Additionally, if the number of state variables with specified
nonzero set points is greater than the number of controls, then an
offset in the control to the origin does not exist.

Motivated by the work of Artstein and Leizarowitz [3], [4] on
the more general problem of periodic and nonperiodic tracking, the
continuous-time nonzero set point problem via static output feedback
was addressed by Bernstein and Haddad [5] while extensions to
fixed-order (i.e., full- and reduced-order) dynamic compensation
were reported in [6]. Haddad and Bernstein [1] also considered the
discrete-time nonzero set point problem via static output feedback.
The goal of the present paper is to extend the results given in
[1] to the case of fixed-order dynamic compensation. As in the
continuous-time case [6], the solution we obtain has the satisfying

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feature that the closed-loop dynamic-feedback compensation gains are independent of the open-loop control components which arise from the state and control set points. Thus if the state set point is changed during operation, then only the open-loop control components require updating. Consequently, there is no need to recalculate the closed-loop gains by solving Riccati equations in real time. The overall theory thus permits the treatment of step commands within standard discrete-time linear quadratic Gaussian (LQG) theory. Finally, for completeness we also present the full-state feedback nonzero set point regulation problem and apply the result to an angular position servo control problem.

Notation and Definitions

\( R, R^{\times n}, R^{\times m}, E \) real numbers, \( r \times s \) real matrices, \( R^{\times 1} \), expectation.

\( I_n, (\cdot)^T, (\cdot)^\# \) \( n \times n \) identity, transpose, group generalized inverse.

\( tr Z \) trace of square matrix \( Z \).

\( \text{asymptotically stable matrix} \) with eigenvalues in the open unit disk.

\( n, n_c, m, l, p, q, \) positive integers.

\( r \) \( n \times n \) identity matrix with eigenvalues in the open unit disk.

\( x, y, z, \alpha, \gamma, \delta \) \( n \)-dimensional vectors.

\( A, B, C, D \) \( n \times n, n \times m, l \times n, l \times m \) matrices.

\( L_1, L_2 \) \( q \times n, r \times m \) matrices.

\( \gamma_1, \gamma_2 \) \( n \)-dimensional constant disturbance vectors.

\( \alpha, \alpha_c \) \( m \)-dimensional control vectors.

\( \hat{\gamma}, \gamma, \hat{\alpha} \) \( [\gamma_1, \gamma_2, \alpha_1, \alpha_2] \).

\( w_1(k), w_2(k) \) \( n \)-dimensional zero-mean white noise processes.

\( V_1, V_2 \) \( n \times l \) cross covariance of \( w_1, w_2 \).

\( x(k), \hat{y}(k) \) \( \gamma \times r \) and \( r \times r \) state and control weightings \( R_1 \geq 0, R_2 \geq 0, L_1^T R_2 L_2 > 0 \).

\( R_1, R_2 \) \( q \times q \) cross weighting \( L_1^T R_1 L_2 - L_1^T R_2 L_2 L_2^T R_1 L_1 \geq 0 \).

\( \tilde{A}, \tilde{B} \) \( \begin{bmatrix} A & B C_c \cr B_c & A_c + B_c D C_c \end{bmatrix} \begin{bmatrix} B & 0 \cr B_c & D \end{bmatrix} \).

\( m, m_c \) \( n, n_c \)-dimensional vectors.

\( \hat{m} \) \( \begin{bmatrix} m \\ m_c \end{bmatrix} \).

\( \tilde{R}_1 \) \( \begin{bmatrix} L_1^T R_1 L_2 - L_1^T R_2 L_2 L_2^T R_1 L_1 & 0 \\ 0 & 0 \end{bmatrix} \).

\( \tilde{R}_{12}, \tilde{R}_{22}, \tilde{R}_{12}^T \) \( \begin{bmatrix} C^T \tilde{L}_2 \tilde{R}_2^T & \tilde{L}_2^T \tilde{R}_2 L_2 C_c \cr \tilde{L}_2^T \tilde{R}_2 L_2 L_2^T & 0 \end{bmatrix} \).

\( \tilde{A}, \tilde{S} \) \( \begin{bmatrix} \tilde{L}_2^T \tilde{R}_2 & \tilde{L}_2^T R_2 \cr \tilde{L}_2^T R_2 & \tilde{L}_2^T R_2 \end{bmatrix} \).

For arbitrary \( n \times n \), \( P, Q \) define

\[ Q_0 = AQCT + V_1, \quad P_0 = BT PA + L_2^T R_2 L_1, \]

\[ V_{20} = CQCT + V_2, \quad R_{02} = B^{T}PB + L_2^T R_2 L_2. \]

II. DYNAMIC COMPENSATION FOR NONZERO SET POINT REGULATION
Nonzero Set Point Problem: Given the \( n \)-th-order stabilizable and detectable plant

\[ x(k+1) = Ax(k) + Bu(k) + w_1(k) + \gamma_1, \quad k = 0, 1, 2, \cdots \quad (1) \]

design a fixed-order dynamic compensator

\[ x_o(k+1) = Ax_o(k) + B \hat{y}(k) + \alpha_c, \quad (3) \]

decreasing the steady-state performance criterion

\[ J(A_c, B_c, C_c, \alpha_c) = \lim_{k \to \infty} E[(L_1 x(k) - \delta_1)^T R_1 (L_1 x(k) - \delta_1) + 2(L_2 x(k) - \delta_2)^T R_2 (L_2 y(k) - \delta_2)] + (L_2 u(k) - \delta_2)^T R_2 (L_2 u(k) - \delta_2)]. \quad (5) \]

Remark 2.1: The cost functional is identical to the LQG criterion with the exception of the shifted set points \( \delta_1 \) and \( \delta_2 \) and matrices \( L_1 \) and \( L_2 \) for selecting linear combinations of components of \( x \) and \( u \).

In general, the development herein incorporates several special features which provide additional flexibility in applications. These include:

1. Constant known disturbance vectors in addition to zero-mean additive plant and measurement noise (i.e., nonzero-mean disturbances \( \gamma_1, \gamma_2 \));
2. Correlated plant and measurement noise \( V_{12} \);
3. State/control performance cross-weighting \( R_{12} \);
4. Arbitrary set points for selected linear combinations of the state and control variables \( L_1 \) and \( L_2 \); and
5. Fixed-order (i.e., full- and reduced-order) compensation.

Because of the last feature, the results obtained in the present paper also generalize the results of Bernstein et al. [7].

The closed-loop system (1)-(4) can be written as

\[ \dot{x}(k+1) = \dot{A}x(k) + \dot{B}u(k) + \dot{w}(k) + \gamma, \quad k = 0, 1, 2, \cdots \quad (6) \]

where \( \dot{x} \) is \( [x^T(k), \dot{x}(k)]^T \) and the closed-loop disturbance \( \dot{w}(k) \) has nonnegative-definite covariance \( \dot{V} \). To analyze (6) define the covariance matrix

\[ \dot{Q}_k = E[(\dot{x}(k) - \hat{m}(k))(\dot{x}(k) - \hat{m}(k))^T] \]

and then

\[ \dot{m}(k+1) = \dot{A}\hat{m}(k) + \dot{B}\dot{u}(k) + \gamma, \quad \hat{m}(k+1) = \dot{A}\hat{m}(k) + \dot{B}u(k) + \gamma. \quad (8) \]

To guarantee that \( J \) is finite and independent of initial conditions, we restrict our attention to the set of admissible stabilizing compensators

\[ \dot{S} = \{(A_c, B_c, C_c); \dot{A} \text{ is asymptotically stable}\}. \]
Hence, for \( (A_c, B_c, C_c) \in \Sigma \), \( \bar{Q} \overset{\Delta}{=} \lim_{k \to -\infty} \bar{Q}(k) \) and 
\( \bar{m} \overset{\Delta}{=} \lim_{k \to -\infty} \bar{m}(k) \) exist and satisfy
\[
\bar{Q} = A\bar{Q}A^T + \bar{V},
\]
\[
\bar{m} = \bar{A}\bar{m} + \bar{B}\tilde{\alpha} + \tilde{\gamma}.
\]

Since the value of \( J \) is independent of the internal realization of the transfer function corresponding to \( (A_c, B_c, C_c) \), the set \( \Sigma' \) is given by

\[\Sigma' = \{(A_c, B_c, C_c) \in \Sigma : (A_c, B_c) \text{ is controllable and } (A_c, C_c) \text{ is observable}\}.
\]

Now \( J(A_c, B_c, C_c, \alpha, \alpha_c) \) is given by
\[
J(A_c, B_c, C_c, \alpha, \alpha_c) = \operatorname{tr}\left\{ (\bar{Q} + \bar{m}\bar{m}^T)\bar{R} - 2m^T L^T R_1 \bar{d}_1 + \bar{d}_1^T R_1 \bar{d}_1 \\
+ 2m^T L^T R_1 L_2 \alpha - 2\bar{m}^T R_1 L_2 \alpha + 2\bar{d}_1^T R_1 L_2 \alpha \\
+ 2m^T C_c^T L_2^T R_2 C_c \alpha \\
- 2m^T C_c^T L_2^T R_2 \bar{d}_2 - 2\alpha^T L_2^T R_2 \bar{d}_2 + \alpha^T L_2^T R_2 C_c \alpha \\
+ \bar{d}_1^T R_2 \bar{d}_2 \right\}.
\]

In this paper we assume that \( S' \) is nonempty.

### III. Necessary Conditions for the Nonzero Set Point Regulation Problem

In this section we obtain necessary conditions for optimality that characterize solutions to the dynamic nonzero set point regulation problem. The following factorization lemma is needed for the statement of the main result.

**Lemma 3.1:** Let \( \bar{Q}, \bar{P} \) be \( n \times n \) nonnegative-definite matrices and suppose \( \operatorname{rank} \bar{Q}\bar{P} = n_c \). Then there exist \( n \times n \) matrices \( G, \Gamma, \) and an \( n \times n \) invertible matrix \( M, \) unique except for a change of basis in \( \mathbb{R}^{n \times n} \), such that
\[
\bar{Q}\bar{P} = G^T M \Gamma M^T G,
\]
\[
\Gamma G^T = I_{n_c}.
\]

Furthermore, the \( n \times n \) matrices
\[
\tau = G^T \Gamma,
\]
\[
\tau_\perp = I_n - \tau
\]
are idempotent and have rank \( n_c \) and \( n - n_c \), respectively.

**Proof:** See [8].

---

**Theorem 3.1:** Suppose \( (A_c, B_c, C_c, \alpha, \alpha_c) \) solves the nonzero set point problem with \( \Sigma \). Then there exist \( n \times n \) nonnegative-definite matrices \( Q, P, \tilde{Q}, \tilde{P} \) such that \( A_c, B_c, C_c, \alpha, \alpha_c \) are given by
\[
A_c = \Gamma [A - BR_1^{-1} P_1 - Q_4 V_2^{-1} C + Q_4 V_2^{-1} D R_1^{-1} P_1] G^T, \tag{15}
\]
\[
B_c = \Gamma Q_4 V_2^{-1}, \tag{16}
\]
\[
C_c = -R_1^{-1} P_1 G^T, \tag{17}
\]
where
\[
\begin{bmatrix}
\alpha \\
\alpha_c
\end{bmatrix} = \Omega_{\perp}^{-1} \left[ \bar{Q} + \bar{B}^T (I - \bar{\Lambda})^{-\top} \bar{S} \right] \bar{S} \tag{18}
\]
and such that \( Q, P, \tilde{Q}, \tilde{P} \) satisfy
\[
Q = A Q A^T + V_1 - Q_4 V_2^{-1} Q_4^T + \tau_1 \tag{19}
\]
\[
P = A^T P A + L_1^T R_1 L_1 - P_2 R_2^{-1} P_0 + \tau_2 \tag{20}
\]
\[
\tilde{Q} = \tau_1 \left[ Q_4 V_2^{-1} Q_4^T + (A - BR_1^{-1} P_1) \bar{Q} (A - BR_1^{-1} P_1)^T \right], \tag{21}
\]
\[
\tilde{P} = \tau_2 \left[ (A - Q_4 V_2^{-1} C)^T \bar{P} (A - Q_4 V_2^{-1} C) + P_2^T R_2^{-1} P_2 \right], \tag{22}
\]
\[
\text{rank} \bar{Q} = \text{rank} \bar{P} = \text{rank} \tilde{Q} = \text{rank} \tilde{P} = n_c. \tag{23}
\]

**Proof:** The proof is similar to the continuous-time proof given in [6] and hence is omitted.

**Remark 3.1:** The results of Bernstein et al. [7] are a special case of Theorem 3.1. To see this let \( \delta_1 = 0, \gamma_1 = 0, \delta_2 = 0, \gamma_2 = 0, L_1 = I_n \) and \( L_2 = I_{n_c} \), which yields the results of Bernstein et al. [7] with the added feature of a direct transmission term \( D \) in the plant dynamics.

**Remark 3.2:** Note that the optimal gains \( \alpha \) and \( \alpha_c \) explicitly depend on the constant disturbances \( \gamma_1 \) and \( \gamma_2 \). As mentioned in Section II we assume that these constant disturbances are known or can be measured accurately.

Next, for clarity, we specialize Theorem 3.1 to the full-order LQG case. As discussed in Bernstein et al. [7], in the full-order LQG case \( n_c = n \) and the Lyapunov equations (21) and (22) for \( \tilde{Q} \) and \( \tilde{P} \) are superfluous. In this case \( G = \Gamma = \tau = I_n \) without loss of generality. To develop further connections with standard LQG theory, assume
\[
L_1 = I_n, \quad L_2 = I_{n_c}, \quad R_{12} = 0, \quad V_{12} = 0 \tag{24}
\]
and define
\[
\bar{\mathcal{R}}_1 \overset{\Delta}{=} \begin{bmatrix} R_1 & 0 \\ C_c R_2 C_c & 0 \end{bmatrix}, \quad \bar{\mathcal{R}}_2 \overset{\Delta}{=} \begin{bmatrix} R_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{R}}_12 \overset{\Delta}{=} \begin{bmatrix} 0 & R_2 C_c \\ 0 & 0 \end{bmatrix},
\]
\[
\bar{\mathcal{S}} \overset{\Delta}{=} \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{R}}_2 \overset{\Delta}{=} \begin{bmatrix} R_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{S}} \overset{\Delta}{=} \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{S}} \overset{\Delta}{=} \begin{bmatrix} 0 & R_2 C_c \\ 0 & 0 \end{bmatrix}.
\]

In this case \( S' \) becomes
\[
S' = \{(A_c, B_c, C_c) \in \Sigma : \bar{Q} > 0\}
\]
where
\[
\begin{bmatrix}
\alpha \\
\alpha_c
\end{bmatrix} = \Omega_{\perp}^{-1} \left[ \bar{Q} + \bar{B}^T (I - \bar{\Lambda})^{-\top} \bar{S} \right] \bar{S} \tag{18}
\]
and such that \( Q, P, \tilde{Q}, \tilde{P} \) satisfy
\[
Q = A Q A^T + V_1 - Q_4 V_2^{-1} Q_4^T + \tau_1 \tag{19}
\]
\[
P = A^T P A + L_1^T R_1 L_1 - P_2 R_2^{-1} P_0 + \tau_2 \tag{20}
\]
\[
\tilde{Q} = \tau_1 \left[ Q_4 V_2^{-1} Q_4^T + (A - BR_1^{-1} P_1) \bar{Q} (A - BR_1^{-1} P_1)^T \right], \tag{21}
\]
\[
\tilde{P} = \tau_2 \left[ (A - Q_4 V_2^{-1} C)^T \bar{P} (A - Q_4 V_2^{-1} C) + P_2^T R_2^{-1} P_2 \right], \tag{22}
\]
\[
\text{rank} \bar{Q} = \text{rank} \bar{P} = \text{rank} \tilde{Q} = \text{rank} \tilde{P} = n_c. \tag{23}
\]
and define the set
\[ \text{controllers point regulation problem. Specifically, we seek full state feedback for the full-state feedback nonzero set point regulation problem. For convenience in stating this result, recall the definitions of } P_a \text{, such that } A_c, B_c, C_c, \alpha, \alpha_c \text{ are given by}
\]
\[ A_c = A - B(R_c + B^T PB)^{-1} B^T PA - AQ C^T (V_2 + C Q C^T)^{-1} C + A Q C^T (V_2 + C Q C^T)^{-1} D(R_c + B^T PB)^{-1} B^T PA, \]
\[ B_c = A Q C^T (V_2 + C Q C^T)^{-1}, \]
\[ C_c = -(R_c + B^T PB)^{-1} B^T PA, \]
\[ \begin{bmatrix} \alpha \\ \alpha_c \end{bmatrix} = \Omega^{-1} (N^2 + B^T (I - \hat{A})^T S \hat{A}) \delta \\
-(\hat{R}_{12} + B^T (I - \hat{A})^T R)(I - \hat{A})^{-1} z \]
and such that \( Q, P \) satisfy
\[ Q = A Q A^T + V_1 - A Q C^T (V_2 + C Q C^T)^{-1} C Q A^T, \]
\[ P = A^T P A + R_1 - A^T P B (R_c + B^T PB)^{-1} B^T P A. \]

**Remark 3.3:** Note that by setting \( b_1 = \gamma_1 = 0, b_2 = 0, \gamma_2 = 0, \) and \( D = 0, \) Corollary 3.1 yields the standard discrete-time LQG result.

**Remark 3.4:** It is easy to see that in the full-order case \( n_c = n \) a solution to the nonzero set point problem exists as long as \( \dot{\Omega} \) is positive definite. In the reduced-order case, however, the situation is more complex.

**IV. FULL-STATE FEEDBACK NONZERO SET POINT REGULATION**

In this section we consider the full-state feedback nonzero set point regulation problem. Specifically, we seek full state feedback controllers
\[ u(k) = K x(k) + \alpha \]
such that \( A + BK \) is asymptotically stable and (5) is minimized with \( J(A_c, B_c, C_c, \alpha, \alpha_c) \) replaced by \( J(K, \alpha). \) Next we present necessary conditions for optimality that characterize solutions to the full-state feedback nonzero set point regulation problem. For convenience in stating this result, recall the definitions of \( R_{2a} \) and \( P_a, \) define
\[ \ddot{A} = I_n - A - BK, \]
\[ \begin{bmatrix} \Psi_f \\ \Lambda_f \end{bmatrix} = L_{2a}^T R_{2a} L_{2a} \tilde{A}^{-1} + B^T \tilde{A}^{-T} \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} K \tilde{A}^{-1}
+ B^T \tilde{A}^{-T} \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} K \tilde{A}^{-1} + K^T L_{2a}^T R_{2a} L_{2} K \tilde{A}^{-1}, \]
\[ \begin{bmatrix} \phi_f \\ \psi_f \end{bmatrix} = L_{2a}^T R_{2a} + B^T \tilde{A}^{-T} L_{1}^T R_{1} L_{1} + B^T \tilde{A}^{-T} L_{2}^T R_{12} L_{2}, \]
and define the set
\[ \mathcal{S}_f = \{ K: \ A + BK \text{ is asymptotically stable and } \Omega_f > 0 \} \]
where
\[ \Omega_f \triangleq B^T \tilde{A}^{-T} \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} (I_n + K \tilde{A}^{-1} B)
+ (I_n + K \tilde{A}^{-1} B)^T \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} (I_n + K \tilde{A}^{-1} B) \]
\[ + (I_n + K \tilde{A}^{-1} B)^T \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} \tilde{A}^{-1} B \]
\[ + (I_n + K \tilde{A}^{-1} B)^T \begin{bmatrix} L_{1}^T R_{1} L_{1} \\ L_{2}^T R_{12} L_{2} \end{bmatrix} \tilde{A}^{-1} B. \]

**Theorem 4.1:** Suppose \( (K, \alpha) \) solves the full-state nonzero set point problem with \( K \in \mathcal{S}_f. \) Then there exists an \( n \times n \) nonnegative-definite matrix \( P \) such that \( K \) and \( \alpha \) are given by
\[ K = -R_{2a}^{-1} P_a, \]
\[ \alpha = \Omega_f^{-1} [\phi_f + \gamma_1 b_1 - \phi_f b_2] \]
and such that \( P \) satisfies
\[ P = A^T P A + L_{1}^T R_{1} L_{1} - P_a R_{2a} P_a. \]

**Proof:** The proof is similar to the proof given in [1].

Once again, note that the closed-loop controller can be designed independently of the open-loop control. As will be seen in the next section, this feature is quite useful since it implies that the feedback gain \( K \) can be designed without regard to the set point. Hence a change in the desired set point during on-line operation only requires updating \( \alpha. \) As in the dynamic compensation case, we require that the constant disturbance be known.

**V. ILLUSTRATIVE NUMERICAL EXAMPLE**

In this section we apply Theorem 4.1 to an illustrative example. Our example is adopted from [2] and involves a position servo system. Specifically, the system involves a moving object in a plane with a rotating antenna at the origin of the plane, driven by an electric motor,
that tracks the object at all times. The dynamics of the antenna are described by

$$\dot{h}(t) + B \dot{\theta}(t) + K \theta(t) = T(t) + w(t)$$  \hspace{1cm} (35)$$

where $I$ is the mass moment of inertia, $B$ is the viscous damping, $K$ is the rotational stiffness, $T(t)$ is the control torque applied by the motor, and $w(t)$ is the disturbance torque caused by wind. The state space description of the system is given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ - \frac{K}{I} & - \frac{B}{I} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} w(t)$$  \hspace{1cm} (36)$$

where $x(t) = [\theta(t) \dot{\theta}(t)]^T$, $k_1$ is a voltage proportionality constant such that $T(t) = k_1 u(t)$ and $w(t)$ is the input voltage to the motor. The following numerical values are used

$$\frac{K}{I} = 6 \frac{1}{\text{rad sec}^2}, \quad \frac{B}{I} = 4.6 \frac{1}{\text{rad sec}},$$

$$k_1 = 0.187 \frac{1}{\text{V sec}^2}, \quad I = 10 \text{ kg m}^2.$$  

Next, note that the continuous-time system (36) has the form

$$\dot{x}(t) = \hat{A} x(t) + \hat{B} u(t) + \hat{D} w(t).$$

Furthermore, we assume a continuous-time cost of the form

$$J = \lim_{T \to \infty} \mathbb{E}[(L_1 x(t) - \delta_1)^T \hat{R}_1 (L_1 x(t) - \delta_1) + 2 x(t) - \delta_1^T \hat{R}_2 (2 x(t) - \delta_1) + (L_2 u(t) - \delta_2)^T \hat{R}_2 (L_2 u(t) - \delta_2)]$$

with

$$\hat{R}_1 = \begin{bmatrix} 100 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \hat{R}_{12} = 0, \quad \hat{R}_2 = 0.2,$$

$$\delta_1 = [0.1 \ 0]^T, \quad \delta_2 = 3.1.$$  

To convert to the discrete-time problem with discretization interval $h$, let (see [2], [9])

$$A = e^{h \hat{A}}, \quad B = \int_0^h e^{h \hat{A}} \hat{B} \, dt, \quad V_1 = \int_0^h e^{h \hat{A}} \hat{D} \hat{D}^T e^{h \hat{A}}^T \, dt,$$

$$R_1 = \hat{R}_1, \quad R_{12} = 0, \quad R_2 = \hat{R}_2.$$  

Assuming $L_1 = I_2$, $L_2 = 1$, $R_{12} = 0$, $V_{12} = 0$, and $\gamma = 0$, Theorem 4.1 was used to obtain the nonzero set point controllers for $h = 0.05$ sec. The results are compared with those obtained by simply employing the shifting technique discussed in [2]. The results are summarized as follows. Fig. 1 shows the responses of the angular positions and angular velocities of the closed-loop system to a step in the set point for both the optimal and shifting schemes. Fig. 2 shows the input voltage to the electric motor versus time. Note that the optimal scheme achieves both the tracking and regulation requirements with significantly less control effort over the shifting technique.

**REFERENCES**


**Nonlinear Active Vibration Damping**

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Abstract—A multivariable nonlinear controller is investigated for vibration suppression of flexible structures. The closed-loop gain matrix consists of a varying term added to a constant term. The varying term is proportional to the measured signal, providing rapid cancellation of large deflections and maintaining low noise sensitivity. $L_2$-stability is proved, and experimental results show improvement of performance compared to a corresponding linear controller with constant gain.

**I. INTRODUCTION**

In active vibration damping of large flexible space structures, it is important to maintain stability and performance in the presence of parameter variations. Robust stability can be obtained using a dissipative compensator [1], which is based on collocation of sensors and actuators. In practical applications, tradeoffs must often be made between high performance and low noise sensitivity, and the measurement noise hence defines an upper bound on the controller gain. To overcome this limitation, we propose to use a nonlinear controller. Sandor and Williamson [2] presented a procedure for designing stable nonlinear control laws for linear plants. The design was based on an inverse optimal control problem. Here we propose a somewhat simpler approach to improve the collocated integral force feedback scheme reported in [3]. A gain proportional to the force measurement is added to the constant gain, resulting in improved performance maintaining a low noise sensitivity. The stability properties are investigated using the theory of passivity, and