We describe situations in which this happens in Corollary 4.5 and Corollary 4.6.

**Corollary 4.5:** Suppose that $W_1D_1$ has a zero on $j\tilde{\omega}$. Then $C \in \Sigma(P)$ is not optimal for the weighted sensitivity problem.

**Proof:** Suppose that $C \in \Sigma(P)$ is optimal for (3.3). Then by Theorem 3.7, $|\lambda(C)|$ is constant a.e. on $j\tilde{\omega}$. Since $W_1 D_1$ has a zero at $j\omega_0 \in j\tilde{\omega}$, then $|\lambda(C)| = 0$ a.e. on $j\tilde{\omega}$. But no $K \in \Sigma(P)$ can achieve this.

Thus when a plant $P$ has a $j\tilde{\omega}$ pole with multiplicity $n$, no stabilizing controller will minimize the weighted sensitivity function if $W_1$ does not also have the pole with multiplicity at least $n$. For example consider $P = (s + 1)/s$. For $X = I$ and $D = s/(s + 1)$, $X D^{-1}$ is a coprime factorization of $P$. For any $W_1$ that does not have a pole at $s = 0$, problem (3.3) will not be optimized by a stabilizing controller.

Also, Corollary 4.5 shows that if one wishes to obtain a stabilizing controller as the optimal solution to a weighted sensitivity problem, one should not choose $W_1$ strictly proper or with a $j\tilde{\omega}$ zero.

**Corollary 4.6:** Suppose that $p_2 > 0$ and $W_1 X_1$ has a zero on $j\tilde{\omega}$. Then $C \in \Sigma(P)$ is not optimal for the weighted sensitivity problem.

**Proof:** Suppose that $C \in \Sigma(P)$ is optimal for (3.5). By Theorem 3.8, $|\bar{\sigma}(C)|$ is constant a.e. on $j\tilde{\omega}$. Since $W_1 X_1$ has a zero at $j\omega_0 \in j\tilde{\omega}$, $|\bar{\sigma}(C)| = 0$ a.e. on $j\tilde{\omega}$. But this contradicts the fact that since $p_2 > 0$, by the Maximum Modulus theorem $|\bar{\sigma}(K)|_\omega > 0$ for all $K \in \Sigma(P)$.

Thus when an unstable plant $P$ has a $j\tilde{\omega}$ zero, no stabilizing controller will minimize the weighted complementary sensitivity function if $W_1 P$ also has the zero. Corollary 4.6 also shows that if one wishes to obtain a stabilizing controller as the optimal solution to a weighted complementary sensitivity optimization problem with an unstable plant, one should not use a weighting function with a $j\tilde{\omega}$ zero.

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**A Periodic Fixed-Structure Approach to Multirate Control**

Wassim M. Haddad and Vikram Kapila

**Abstract**—In this note we develop an approach to designing reduced-order multirate controllers. A discrete-time model that accounts for the multirate timing sequence of measurements is presented and is shown to have periodically time-varying dynamics. Using discrete-time stability theory, the optimal projection approach to fixed-order (i.e., full- and reduced-order) dynamic compensation is generalized to obtain reduced-order periodic controllers that account for the multirate architecture. It is shown that the optimal reduced-order controller is characterized by means of a periodically time-varying system of equations consisting of coupled Riccati and Lyapunov equations. In addition, the multirate static output-feedback control problem is considered. For both problems, the design equations are presented in a concise, unified manner to facilitate their accessibility for developing numerical algorithms for practical applications.

**I. INTRODUCTION**

Many applications of feedback control involve continuous-time systems subject to digital (discrete-time) control. Furthermore, in practical applications, the control system actuators and sensors have differing bandwidths. For example, in flexible structure control, it is not unusual to attenuate the low-frequency high-amplitude modes by means of low-bandwidth actuators that are relatively heavy and hence able to exert high force/torque to control the higher frequency modes. Obviously, the high-bandwidth actuators would require sensors that are sampled at high rates, while low-bandwidth actuators require only sensors sampled at low data rates. As a consequence, the use of various sensor data rates leads to a multirate control problem. To properly use such data, a multirate controller must carefully account for the timing sequence of incoming data. The purpose of this note is to develop a general approach to full- and reduced-order steady-state multirate dynamic compensation.

Multirate control problems have been of interest for many years with increased emphasis in recent years [1]-[3], [9]-[11], [15]. A common feature of these papers is the realization that the multirate sampling process leads to periodically time-varying dynamics. Hence, with a suitable reinterpretation, results on multirate control can also be applied to single rate or multirate problems involving systems with periodically time-varying dynamics. The principal challenge of these problems is to arrive at a tractable control design formulation in spite of the extreme complexity of such systems. In order to account for the periodic-time-varying dynamics of multirate systems, a periodically time-varying control law architecture was proposed in [10] and [11] which appears promising in this regard. An alternative approach which has been proposed for the multirate control problem is the use of an expanded state-space formulation [2]. However, this approach results in very high-order systems and is often numerically intractable. Finally, a cost translation and a lifting approach to the multirate LQG problem has been proposed in [15] which does not lead to an increase in the state dimension. Specifically, [15] shows how to translate a multirate sampled-data LQG problem into an equivalent, modified, single rate, shift invariant problem via a lifted isomorphism.

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However, this approach results in an equivalent system involving more inputs and outputs than the original system. The interested reader is referred to [7], [8], [10], and [14] for further discussions on multirate and periodic control.

For generality in our development, we consider both full- and reduced-order dynamic compensators as well as static output-feedback controllers. In the discrete-time case, this problem was considered in [4], while single rate sampled-data aspects were addressed in [5]. The approach of the present note is the fixed-structure Riccati equation technique developed in [4]. Essentially, this approach addresses controller complexity by explicitly imposing implementation constraints on the controller structure, and optimizing over that class of controllers. Specifically, in addressing the problem of reduced-order dynamic compensation, it is shown in [4] that optimal reduced-order steady-state dynamic compensators can be characterized by means of an algebraic system of Riccati/Lyapunov equations coupled by a projection matrix which arises as a direct consequence of optimality and which represents a breakdown of the separation between the operations of state estimation and state estimate feedback; that is, the certainty equivalence principle is no longer valid. The proof is based on expressing the closed-loop quadratic cost functional as a function of the design parameters, i.e., the compensator gains, and the utilization of Lagrange multipliers for optimization over the parameter space. Thus, this approach provides a constrained optimal control methodology in which we do not seek to optimize a performance measure per se, but rather a performance measure within a class of a priori fixed-structure controllers.

In the present note, analog-to-digital conversions are employed within a multirate setting to obtain periodically time-varying dynamics. The compensator is thus assigned a corresponding discrete-time periodic structure to account for the multirate sampled-data measurements. It is of the problem, the necessary conditions for optimality now involve to the order of the compensator. Similar extensions to reduced-order dynamic compensators as well as static output-feedback compensators characterized by systems of four quadratic cost functionals as functions of the design parameters, i.e., optimizing over that class of controllers. Specifically, in addressing the problem of reduced-order dynamic compensation, it is shown in [4] that optimal reduced-order steady-state dynamic compensators can be characterized by means of an algebraic system of Riccati/Lyapunov equations. The compensator is thus assigned a corresponding discrete-time periodic structure to account for the multirate sampled-data measurements.

\[ \dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \quad t \in [0, \infty). \] (2.1)

\[ y(t_k) = C(t_k)x(t_k) + w_2(t_k), \quad k = 1, 2, \ldots. \] (2.2)

Then design a static output-feedback multirate sampled-data control law

\[ u(t_k) = D_1(t_k)y(t_k) \] (2.3)

which, with D/A zero-order-hold controls \( u(t) = u(t_k), t \in [t_k, t_{k+1}), \) minimizes the quadratic performance criterion

\[ J_r(D_1) = \lim_{t \to \infty} E \left[ \int_0^t [x^T(s)R_1x(s) + 2x^T(s)R_2u(s) + u^T(s)R_3u(s)]ds \right]. \] (2.4)

\textbf{Fixed Structure Multirate Dynamic Output-Feedback Control Problem:} Given the \( n \)-th-order continuous-time system (2.1) with multirate sampled-data measurements (2.2), design an \( n \)-th-order (1 \( \leq n \leq n_r) \) multirate sampled-data dynamic compensator

\[ x_{n_r}(t_{k+1}) = A_{n_r}(t_k)x_{n_r}(t_k) + B_{n_r}(t_k)y(t_k) \] (2.5)

\[ u(t_k) = C_{n_r}(t_k)x_{n_r}(t_k) + D_{n_r}(t_k)y(t_k) \] (2.6)

which, with D/A zero-order-hold controls \( u(t) = u(t_k), t \in [t_k, t_{k+1}), \) minimizes the quadratic performance criterion (2.4) with \( J_r(D_{n_r}) \) denoted by \( J_r(A_{n_r}, B_{n_r}, C_{n_r}, D_{n_r}). \)

The key feature of both problems is the time-varying nature of the output equation (2.2) which represents sensor measurements available at different rates. Fig. 1 provides a typical multirate timing diagram for a three-sensor model. For generality, we do not assume that the sample intervals \( h_k \triangleq t_{k+1} - t_k \) are uniform (note the sample times for sensor #3 in Fig. 1). However, we do assume that the overall timing sequence of intervals \( [t_k, t_{k+1}], k = 1, 2, \ldots \) is periodic over \([0, \infty), \) where \( N \) represents the periodic interval. Note that \( h_{k+N} = h_k, k = 1, 2, \ldots. \) Since different sensor measurements are available at different times \( t_k, \) the dimension \( l_k \) of the measurements \( y(t_k) \) may also vary periodically. Finally, in subsequent analysis, the static output-feedback law (2.3) and dynamic compensator (2.5) and (2.6) are assigned periodic gains corresponding to the periodic timing sequence of the multirate measurements.

In the above problem formulation, \( w_1(t) \) denotes a continuous-time stationary white noise process with nonnegative-definite intensity \( V_1(t) \in \mathbb{R}^{n \times n}, \) while \( w_2(t_k) \) denotes a variable-dimensional discrete-time white noise process with positive-definite covariance \( V_2(t_k) \in \mathbb{R}^{n\times n\times n}. \) We assume \( w_2(t_k) \) is cyclostationary, that is, \( V_2(t_k) \) is periodic. In what follows, we shall simplify the notation considerably by replacing the continuous-time sample instant \( t_k \) by the discrete-time index \( k. \) With this minor abuse of notation, we replace \( x(t_k) \) by \( x_k, x(t_k) = x_k, y(t_k) = y_k, u(t_k) = u_k, y_2(t_k) = y_2(k), A_{n_r}(t_k) = A_k, \) and similarly for \( B_{n_r}. \)

The key feature of both problems is the time-varying nature of the output equation (2.2) which represents sensor measurements available at different rates. Fig. 1 provides a typical multirate timing diagram for a three-sensor model. For generality, we do not assume that the sample intervals \( h_k \triangleq t_{k+1} - t_k \) are uniform (note the sample times for sensor #3 in Fig. 1). However, we do assume that the overall timing sequence of intervals \( [t_k, t_{k+1}], k = 1, 2, \ldots \) is periodic over \([0, \infty), \) where \( N \) represents the periodic interval. Note that \( h_{k+N} = h_k, k = 1, 2, \ldots. \) Since different sensor measurements are available at different times \( t_k, \) the dimension \( l_k \) of the measurements \( y(t_k) \) may also vary periodically. Finally, in subsequent analysis, the static output-feedback law (2.3) and dynamic compensator (2.5) and (2.6) are assigned periodic gains corresponding to the periodic timing sequence of the multirate measurements.

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clarify whether the argument is periodicity assumption on the compensator implies that (2.4) have the equivalent discrete-time representation

\[
x(k+1) = A(k)x(k) + B(k)u(k) + w_k(k).
\]

Also, by assumption, \( C(k+1) = C(k), \) for \( k = 1, 2, \ldots \).

Next, we model the propagation of the plant over one time step. For notational convenience, define \( H(k) \equiv \int_0^h e^{A(k)\tau} d\tau \).

**Theorem 2.1:** For the fixed-order multirate sampled-data control problem, the plant dynamics (2.4) and quadratic performance criterion (2.9) have equivalent discrete-time representation

\[
J = \delta_\infty + \lim_{h \to \infty} \frac{1}{h} \sum_{k=1}^h \left\{ J^T(k) + 2r^T(k)R_2(k)u(k) + u^T(k)R_2(k)u(k) \right\}
\]

where

\[
A(k) \triangleq e^{A(k)h}, \quad B(k) \triangleq H(k)B,
\]

\[
u_i(k) = \int_0^h e^{A(k)h} w_i(k + s) ds,
\]

\[
\delta_\infty \triangleq \lim_{h \to \infty} \frac{1}{h} \text{tr} \left\{ \sum_{k=1}^h \int_0^h e^{A(k)h} R_i(k) e^{A(k)h} ds \right\},
\]

\[
R_1(k) \triangleq \frac{1}{h} \int_0^h e^{A(k)h} R_1(k) e^{A(k)h} ds,
\]

\[
R_2(k) \triangleq R_2 + \frac{1}{h} \int_0^h \left[ D(k)^T H(s) R_2(k) H(s) D(k) \right] ds + \frac{1}{h} H^T(k)R_1(k)H(k) R_2(k) H(s) B \]

\[
+ R_2^T(k) H(s) B + B^T R_1(k) H(s) R_2(k) ds
\]

and \( u_i(k) \) is a zero-mean discrete-time white noise process with

\[
\mathcal{E}\{u_i(k)u_i^T(k)\} = \mathcal{Y}_i(k) = \int_0^h e^{A(k)h} V_i(k) e^{A(k)h} ds.
\]

The proof of this theorem is a straightforward calculation involving integrals of white noise signals, and hence is omitted. Note that by the sampling periodicity assumption, \( A(k + N) = A(k), k = 1, 2, \ldots \).

The above formulation assumes that a discrete-time multirate measurement model is available. One can assume, alternatively, that analog measurements corrupted by continuous-time noise signals are available instead, that is, \( y(t) = Cx(t) + w_2(t) \). In this case, one can develop an equivalent discrete-time model that employs an averaging-type \( A/D \) device [5].

**Remark 2.1:** The equivalent discrete-time quadratic performance criterion (2.9) involves a constant offset \( \delta_\infty \), which is a function of sampling rates, and effectively imposes a lower bound on sampled-data performance due to the discretization process.

**Remark 2.1:** The equivalent discrete-time quadratic performance criterion (2.9) involves a constant offset \( \delta_\infty \), which is a function of sampling rates, and effectively imposes a lower bound on sampled-data performance due to the discretization process.

As will be shown by Lemma 3.1, due to the periodicity of \( h_i \), \( \delta_\infty \) is a constant.
attention to output-feedback controllers having the property that the closed-loop transition matrix over one period \( \Phi_p(\alpha) \) is stable for \( \alpha = 1, \cdots, N \). Note that since \( \tilde{A}(\cdot) \) is periodic, the eigenvalues of \( \Phi_p(\alpha) \) are actually independent of \( \alpha \). Hence, it suffices to require that \( \Phi_p(1) = \tilde{A}(N) \tilde{A}(N-1) \cdots \tilde{A}(1) \) is stable.

**Lemma 3.1** [12]: Suppose \( \Phi_p(1) \) is stable. Then for given \( D(\cdot) \), the covariance Lyapunov equation (3.2) reaches a steady-state periodic trajectory as \( k \to \infty \), that is,

\[
\lim_{k \to \infty} [Q(k), Q(k+1), \ldots, Q(k+N-1)] = [Q(\alpha), Q(\alpha+1), \ldots, Q(\alpha+N-1)].
\]

(3.4)

In this case, the covariance \( Q(k) \) satisfies

\[
Q(\alpha + 1) = \tilde{A}(\alpha)Q(\alpha)\tilde{A}^T(\alpha) + \tilde{V}(\alpha), \quad \alpha = 1, \cdots, N
\]

(3.5)

where \( Q(N+1) = Q(1) \). Furthermore, the quadratic performance criterion (3.3) is given by

\[
\mathcal{J}_s(D(\cdot)) = \delta + \frac{1}{N} \text{tr} \sum_{\alpha=1}^{N} [Q(\alpha)R(\alpha) + D^T(\alpha)R^0(\alpha)D(\alpha)V(\alpha)]
\]

(3.6)

where

\[
\delta = \frac{1}{N} \text{tr} \sum_{\alpha=1}^{N} \int_{0}^{\frac{1}{N}} \int_{0}^{T} e^{\lambda T} V_1 e^{T \lambda T} R_1 d\lambda d\tau.
\]

For the statement of the main result of this section, define the set

\[
\mathcal{S} \triangleq \{ D(\cdot); \Phi_p(\alpha) \text{ is stable for } \alpha = 1, \cdots, N \}.
\]

(3.7)

In addition to ensuring that the covariance Lyapunov equation (3.2) reaches a steady-state periodic trajectory as \( k \to \infty \), the set \( \mathcal{S} \) constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the fixed-order multirate sampled-data static output-feedback control problem. The asymptotic stability of the transition matrix \( \Phi_p(\alpha) \) serves as a normality condition which further implies that the dual \( P(\alpha) \) of \( Q(\alpha) \) is nonnegative-definite.

For notational convenience in stating the multirate sampled-data static output-feedback result, define the notation

\[
R_{2o}(\alpha) \triangleq B^T(\alpha)P(\alpha+1)B(\alpha) + \frac{1}{N} R_2(\alpha),
\]

\[
V_{2o}(\alpha) \triangleq C(\alpha)Q(\alpha)C^T(\alpha) + V_2(\alpha),
\]

\[
P_{2o}(\alpha) \triangleq B^T(\alpha)P(\alpha+1)A(\alpha) + \frac{1}{N} B_2(\alpha),
\]

\[
Q_{2o}(\alpha) \triangleq A(\alpha)Q(\alpha)C^T(\alpha),
\]

for arbitrary \( Q(\alpha) \) and \( P(\alpha) \in \mathbb{R}^{n \times n} \) and \( \alpha = 1, \cdots, N \).

**Theorem 3.1**: Suppose \( D(\cdot) \in \mathcal{S} \), solves the multirate sampled-data static output-feedback control problem. Then there exist \( n \times n \) nonnegative-definite matrices \( Q(\alpha) \) and \( P(\alpha) \) such that, for \( \alpha = 1, \cdots, N \), \( D(\alpha) \) is given by

\[
D(\alpha) = R_{2o}(\alpha)P_{2o}(\alpha)Q_{2o}(\alpha)V_{2o}^{-1}(\alpha)
\]

(3.8)

and such that \( Q(\alpha) \) and \( P(\alpha) \) satisfy

\[
Q(\alpha + 1) = A(\alpha)Q(\alpha)A^T(\alpha) + V_1(\alpha) - Q_{2o}(\alpha)V_{2o}^{-1}(\alpha)Q_{2o}^T(\alpha)
\]

\[
+ [Q_{2o}(\alpha) + B(\alpha)D(\alpha)V_{2o}(\alpha)]
\]

\[
\cdot V_{2o}^{-1}(\alpha)[Q_{2o}(\alpha) + B(\alpha)D(\alpha)V_{2o}(\alpha)]^T.
\]

(3.9)

Furthermore, the minimal cost is given by

\[
\mathcal{J}_s(D(\cdot)) = \delta + \frac{1}{N} \sum_{\alpha=1}^{N} \sum_{n} Q(\alpha)[R_1(\alpha) - 2R_2(\alpha)R_{2o}(\alpha)]
\]

\[
\cdot P_{2o}(\alpha)Q(\alpha)C^T(\alpha)V_{2o}^{-1}(\alpha)C(\alpha)
\]

\[
+ P_{2o}(\alpha)Q(\alpha)C^T(\alpha)V_{2o}^{-1}(\alpha)C(\alpha).
\]

(3.10)

**Proof**: The necessary conditions for optimality follow from standard Lagrange multiplier arguments. See [14] for a similar proof.

**Remark 3.2**: In the full-state feedback case, we take \( C(\alpha) = 1 \). \( V_2(\alpha) = 0 \), and \( R_2(\alpha) = 0 \) for \( \alpha = 1, \cdots, N \). In this case, (3.8) becomes \( D(\alpha) = -R_{2o}(\alpha)B^T(\alpha)P(\alpha+1)A(\alpha) \), and (3.10) specializes to

\[
P(\alpha) = A^T(\alpha)P(\alpha+1)A(\alpha) + \frac{1}{N} R_1(\alpha)
\]

\[
- A^T(\alpha)P(\alpha+1)B(\alpha)R_{2o}(\alpha)B^T(\alpha)P(\alpha+1)A(\alpha)
\]

(3.12)

while (3.9) is superfluous and can be omitted. Finally, we note that if we assume a single-rate architecture, the plant dynamics are constant and (3.12) collapses to the standard discrete-time regulator Riccati equation.

### IV. THE FIXED-STRUCTURE MULTIRATE SAMPLED-DATA DYNAMIC OUTPUT-FEEDBACK CONTROL PROBLEM

In this section, we consider the fixed-order multirate sampled-data dynamic compensation problem. As in Section III, we first form the closed-loop system for (2.7), (2.8), (2.5), and (2.6) to obtain

\[
\dot{j}(k+1) = A(k)\dot{j}(k) + a(k)
\]

(4.1)

where

\[
J(k) = \begin{bmatrix} x(k) \\ x_2(k) \end{bmatrix},
\]

\[
\Lambda(k) = \begin{bmatrix} A(k) + B(k)D(k)C(k) & B(k)C_2(k) \\ B(k)C_1(k) & A_2(k) \end{bmatrix}
\]

\[
\Lambda(k) + \Lambda(k+X) = \Lambda(k), \quad k = 1, 2, \cdots.
\]

The closed-loop disturbance

\[
\sigma(k) = \begin{bmatrix} a_1(k) + B(k)D(k)w_2(k) \\ B(k)w_2(k) \end{bmatrix}, \quad k = 1, 2, \cdots
\]

has nonnegative-definite covariance shown by \( \sigma \) at the bottom of the next page, where once again we assume that the continuous-time plant noise and the discrete-time measurement noise are uncorrelated, i.e., \( V_2(\alpha) = 0 \). As for the static output-feedback case, the cost functional (2.9) can now be expressed in terms of the closed-loop second-moment matrix. Specifically, Proposition 3.1 and Lemma 3.1 hold for the dynamic-output feedback problem with \( x(k), \dot{x}(k), \dot{y}(k), \) and \( R(k) \) replaced by \( \dot{j}(k), \dot{\Lambda}(k), \dot{y}(k), \) and \( R(k) \), respectively, where \( R(k) \) is shown in (y) at the bottom of the next page.

For the next result, define the closed-loop transition matrix and the compensator transition matrix over one period by \( \Phi_p(\alpha) \triangleq \tilde{A}(\alpha) + \tilde{V}(\alpha) \)

(4.2)
for arbitrary $P(\alpha) \in \mathbb{R}^{n \times n}$ and $\alpha = 1, \ldots, N$.

**Theorem 4.1:** Suppose $[A(\cdot), B(\cdot), C(\cdot), D(\cdot)] \in \mathcal{S}$ solves the fixed-order multirate sample-data dynamic output-feedback control problem. Then there exist $n \times n$ nonnegative-definite matrices $Q(\alpha)$, $P(\alpha)$, and $\hat{P}(\alpha)$ such that, for $\alpha = 1, \ldots, N$, $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$ are given by

\[
A(\alpha) = \Gamma(\alpha + 1)[A(\alpha) - B(\alpha)R_{2\alpha}(\alpha)P_{\alpha}(\alpha)] - Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha).
\]

\[
B(\alpha) = \Gamma(\alpha + 1)[Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha) + D(\alpha)C(\alpha)G^T(\alpha)].
\]

\[
C(\alpha) = -[R_{2\alpha}(\alpha)P_{\alpha}(\alpha) + D(\alpha)C(\alpha)G^T(\alpha)].
\]

\[
D(\alpha) = -[P_{\alpha}(\alpha) + D(\alpha)C(\alpha)]T_{2\alpha}(\alpha + 1).
\]

and such that $Q(\alpha)$, $P(\alpha)$, $\hat{Q}(\alpha)$, and $\hat{P}(\alpha)$ satisfy

\[
Q(\alpha + 1) = A(\alpha)Q(\alpha)A^T(\alpha) + V(\alpha) - Q_{2\alpha}(\alpha)V_{2\alpha}^{-1}(\alpha)Q_{2\alpha}(\alpha) + r(\alpha)[(A(\alpha) - B(\alpha)R_{2\alpha}(\alpha)P_{\alpha}(\alpha)]^T
\]

\[
+ [Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha) + D(\alpha)C(\alpha)G^T(\alpha)]^T R_{2\alpha}(\alpha) + [P_{\alpha}(\alpha) + D(\alpha)C(\alpha)]T_{2\alpha}(\alpha + 1).
\]

For details see [6].

**Lemma 4.1:** Let $\hat{Q}$, $\hat{P}$ be $n \times n$ nonnegative-definite matrices and assume rank $\hat{Q}P = n$. Then there exist $n \times n$ matrices $\Gamma$, $\Gamma$, and $\Delta$ unique except for a change of basis in $\mathbb{R}^{n \times n}$, such that $\hat{Q}P = \Gamma G^T(\Gamma)\Gamma$. Furthermore, the $n \times n$ matrices $\Gamma$, $\Gamma$, and $\Delta$ are idempotent and have rank $n$ and $n - n$, respectively.

The following result gives necessary conditions that characterize solutions to the fixed-order multirate sample-data control problem. For convenience in stating this result, recall the definitions of $R_{2\alpha}(\cdot)$, $V_{2\alpha}(\cdot)$, $P_{\alpha}(\cdot)$, and $Q_{\alpha}(\cdot)$ and define the additional notation

\[
\mathcal{M}(\alpha) = \begin{bmatrix} I_n & -R_{2\alpha}(\alpha)P_{\alpha}(\alpha) \\
D(\alpha)C(\alpha) \end{bmatrix}, \quad \mathcal{M}(\alpha) = \begin{bmatrix} I_n & -R_{2\alpha}(\alpha)P_{\alpha}(\alpha) \\
D(\alpha)C(\alpha) \end{bmatrix}.
\]

\[
\hat{P}(\alpha) = \Gamma^T(\alpha)[(A(\alpha) - Q_{2\alpha}(\alpha)V_{2\alpha}^{-1}(\alpha)C(\alpha)]^T R_{2\alpha}(\alpha) + [P_{\alpha}(\alpha) + R_{2\alpha}(\alpha)C(\alpha)]T_{2\alpha}(\alpha + 1).
\]

\[
\hat{Q}(\alpha + 1) = \hat{Q}(\alpha + 1)[(A(\alpha) - B(\alpha)R_{2\alpha}(\alpha)P_{\alpha}(\alpha)]^T
\]

\[
+ [Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha) + D(\alpha)C(\alpha)]^T R_{2\alpha}(\alpha) + [P_{\alpha}(\alpha) + R_{2\alpha}(\alpha)C(\alpha)]T_{2\alpha}(\alpha + 1).
\]

Theorem 4.1: Suppose $[A(\cdot), B(\cdot), C(\cdot), D(\cdot)] \in \mathcal{S}$ solves the fixed-order multirate sample-data dynamic output-feedback control problem. Then there exist $n \times n$ nonnegative-definite matrices $Q(\alpha)$, $P(\alpha)$, $\hat{Q}(\alpha)$, and $\hat{P}(\alpha)$ such that, for $\alpha = 1, \ldots, N$, $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, and $D(\alpha)$ are given by

\[
A(\alpha) = \Gamma(\alpha + 1)[A(\alpha) - B(\alpha)R_{2\alpha}(\alpha)P_{\alpha}(\alpha)] - Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha).
\]

\[
B(\alpha) = \Gamma(\alpha + 1)[Q_{2\alpha}(\alpha)C(\alpha)G^T(\alpha) + D(\alpha)C(\alpha)G^T(\alpha)].
\]

\[
C(\alpha) = -[R_{2\alpha}(\alpha)P_{\alpha}(\alpha) + D(\alpha)C(\alpha)G^T(\alpha)].
\]

\[
D(\alpha) = -[P_{\alpha}(\alpha) + D(\alpha)C(\alpha)]T_{2\alpha}(\alpha + 1).
\]
Furthermore, the minimal cost is given by
\[
J_{\alpha}(A(\cdot), B(\cdot), C(\cdot), D(\cdot)) = \delta + \frac{1}{N} \text{tr} \left[ \sum_{\alpha=1}^{N} \{(M(\alpha)Q(\alpha)M^T(\alpha) + \bar{M}(\alpha)Q(\alpha)\bar{M}^T(\alpha))B(\alpha)\} \right].
\]  

(4.14)

**Proof:** The proof is identical to the proof of Theorem 3.3 of [14] concerning reduced-order control of discrete-time linear periodic systems with suitable reinterpretations for capturing the equivalent multirate sampled-data model. □

Theorem 4.1 provides necessary conditions for the fixed-order multirate sampled-data control problem. These necessary conditions consist of a system of two modified periodic difference Lyapunov equations and two modified periodic difference Riccati equations coupled by projection matrices \( \tau(\alpha) \), \( \alpha = 1, \ldots, N \). As expected, these equations are periodically time-varying over the period \( 1 \leq \alpha \leq N \) in accordance with the multirate nature of the measurements. As discussed in [4], the fixed-order constraint on the compensator gives rise to the projection \( \tau \) which characterizes the optimal reduced-order compensator gains. In the multirate case, however, it is interesting to note that the time-varying nature of the problem gives rise to multiple projections corresponding to each of the intermediate points of the periodicity interval, and whose rank along the periodic interval is equal to the order of the compensator.

**Remark 4.3:** As in the linear time-invariant case [4] to obtain the full-order multirate LQG controller, set \( n_x = n \). In this case, the projections \( \tau(\alpha) \), and \( \Gamma(\alpha) \) and \( G(\alpha) \), for \( \alpha = 1, \ldots, N \), become the identity. Consequently, (4.11) and (4.12) play no role and hence can be omitted. In order to draw connections with existing full-order multirate results, set \( \bar{D}(\alpha) = 0 \) and \( \bar{R}(\alpha) = 0, \alpha = 1, \ldots, N \), so that

\[
A(\alpha) = \bar{A} + B(\alpha)R_{\bar{E}}^{-1}(\alpha)B^T(\alpha)P(\alpha + 1)A(\alpha) - A(\alpha)Q(\alpha)C^T(\alpha)\bar{V}_{\bar{E}}^{-1}(\alpha)C(\alpha). \tag{4.15}
\]

\[
B(\alpha) = \bar{A} + B(\alpha)R_{\bar{E}}^{-1}(\alpha)B^T(\alpha)P(\alpha + 1)A(\alpha). \tag{4.16}
\]

\[
C(\alpha) = -R_{\bar{E}}^{-1}(\alpha)B^T(\alpha)P(\alpha + 1)A(\alpha). \tag{4.17}
\]

where \( Q(\alpha) \) and \( P(\alpha) \) satisfy

\[
Q(\alpha + 1) = \bar{A}Q(\alpha)\bar{A}^T(\alpha) + \bar{V}_{\bar{E}}(\alpha) - A(\alpha)Q(\alpha)C^T(\alpha)\bar{V}_{\bar{E}}^{-1}(\alpha)C(\alpha)A^T(\alpha). \tag{4.18}
\]

\[
P(\alpha) = \bar{A}^T(\alpha)P(\alpha + 1)A(\alpha) + \frac{1}{N} \bar{R}(\alpha) \tag{4.19}
\]

Thus, the full-order multirate sampled-data controller is characterized by two decoupled periodic difference Riccati equations (observer and regulator Riccati equations) over the period \( \alpha = 1, \ldots, N \). This corresponds to the results obtained in [10]. Next, assuming a single rate architecture yields time-invariant plant dynamics, while (4.18) and (4.19) specialize to the discrete-time observer and regulator Riccati equations. Alternatively, retaining the reduced-order constraint and assuming single rate sampling, Theorem 4.1 yields the sampled-data optimal projection equations for reduced-order dynamic compensation given in [5].

V. NUMERICAL EVALUATION OF INTEGRALS

**IN Volving MATRIX EXPONENTIALS**

To evaluate the integrals involving matrix exponentials appearing in Theorem 2.1, we utilize the approach of [16]. The idea is to eliminate the need for integration by computing the matrix exponential of appropriate block matrices.

**Proposition 5.1:** For \( \alpha = 1, \ldots, N \), consider the following partitioned matrix exponentials

\[
\begin{bmatrix}
E_1 & E_2 & E_3 & E_4 \\
0 & E_5 & E_6 & E_7 \\
0 & 0 & E_8 & E_9 \\
0_{m \times n} & 0_{m \times n} & 0_{n \times m} & I_m
\end{bmatrix}
\]

\[
\begin{bmatrix}
-E^T & I_n & 0 & 0_{m \times m} \\
0 & -A^T & R_1 & 0_{n \times m} \\
0 & 0 & A & B \\
0_{m \times n} & 0_{m \times n} & 0_{n \times n} & 0_{m \times m}
\end{bmatrix}
\]

\[
\begin{bmatrix}
E_{10} & E_{11} & E_{12} & E_{13} \\
0 & E_{14} & E_{15} & E_{16} \\
0 & 0 & E_{17} & E_{18} \\
0_{m \times n} & 0_{m \times n} & 0_{n \times m} & I_m
\end{bmatrix}
\]

\[
\begin{bmatrix}
-E^T & I_n & 0 & 0_{m \times m} \\
0 & -A^T & R_1 & 0_{n \times m} \\
0 & 0 & A & B \\
0_{m \times n} & 0_{m \times n} & 0_{n \times n} & 0_{m \times m}
\end{bmatrix}
\]

\[
\begin{bmatrix}
E_{19} & E_{20} & E_{21} \\
0 & E_{22} & E_{23} \\
0 & 0 & E_{24}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_1 & A_2 & A_3 \\
0 & A_1 & 0 \\
0 & 0 & A_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_1 & R_2 & R_3 \\
0 & R_1 & 0 \\
0 & 0 & R_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
R_1 & R_2 & R_3 \\
0 & R_1 & 0 \\
0 & 0 & R_1
\end{bmatrix}
\]

of orders \( (3n + m) \times (3n + m) \), \((3n + m) \times (3n + m) \), respectively. Then, for \( \alpha = 1, \ldots, N \),

\[
A(\alpha) = E_{24}, \quad B(\alpha) = E_{21}, \quad V_1(\alpha) = E_{12}^T E_{24},
\]

\[
R_1(\alpha) = \frac{1}{h_n} E_{17} E_{15}, \quad R_2(\alpha) = \frac{1}{h_n} E_{17} E_{16},
\]

\[
R_3(\alpha) = R_n + \frac{1}{h_n} [B^T E_{17} E_{13} + E_{17} B^T - B^T E_{17} E_{4}],
\]

\[
\delta = \frac{1}{h_n} \sum_{\alpha=1}^{N} \text{tr} R_n E_{17} E_{21}.
\]

The proof of the above proposition involves straightforward manipulations of matrix exponentials.

VI. ILLUSTRATIVE NUMERICAL EXAMPLE

For illustrative purposes, we consider a numerical example involving a rigid body with a flexible appendage. This example is reminiscent of a single-axis spacecraft involving unstable dynamics and sensor fusion of slow, accurate spacecraft attitude sensors (such as horizon sensors or start trackers) with fast, less accurate rate gyroscopes. The motivation for slow/fast sensor configuration is that rate information can be used to improve the attitude control between attitude measurements. Hence define

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & -0.03
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0.1 & 0 & 0.1 \\
0 & 1 & 0 & 1
\end{bmatrix}.
\]
\[ Y_1 = DD^T, \quad V_1 = I, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
\[ R_1 = E^T E, \quad R_2 = 1. \]

Note that the dynamic model involves one rigid body mode along with one flexible mode at a frequency of 1 rad/s with 0.5% damping. The matrix \( C \) captures the fact that the rigid body angular position and tip velocity of the flexible appendage are measured. Also, note that the rigid body position measurement is corrupted by the flexible mode (i.e., observation spillover). To reflect a plausible mission, we assume that the rigid body angular position is measured by an attitude sensor sampling at 1 Hz, while the tip appendage velocity is measured by a rate gyro sensor sampling at 5 Hz. The matrix \( R_1 \) expresses the desire to regulate the rigid body and tip appendage positions, and the matrix \( V_1 \) was chosen to capture the type of noise correlation that arises when the dynamics are transformed into a modal basis.

Using the homotopy algorithm based on a prediction and a Newton correction scheme for periodic difference Riccati equations reported in [13], the following designs were obtained. For \( n_r = 4 \) discrete-time single rate and multirate controllers were obtained from (4.15)-(4.19) using Theorem 2.1 for continuous-time to discrete-time conversions. These designs were compared using the performance criterion (4.14). The results are summarized as follows:

<table>
<thead>
<tr>
<th>Measurement Scheme</th>
<th>Optimal Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two 1 Hz sensors</td>
<td>65,6922</td>
</tr>
<tr>
<td>Two 5 Hz sensors</td>
<td>53,9930</td>
</tr>
<tr>
<td>Multirate scheme (1 Hz and 5 Hz sensors)</td>
<td>54,6061</td>
</tr>
</tbody>
</table>

Note that the improvement in the cost of the two 5 Hz sensor scheme over the multirate scheme is minimal, which clearly demonstrates that the multirate scheme provides sensor complexity reduction over the two 5 Hz sensor scheme.

REFERENCES


Pursuing a Maneuvering Target Which Uses a Random Process for Its Control

V. E. Beneš, K. L. Helmes, and R. W. Rishel

Abstract—Since a pursuer pursuing a maneuvering target does not know what the evader's control law appears as a random process to the pursuer, however, he has opinions about what the evader will do. From these, he can assign a prior probability distribution to the evader's maneuvers. For a linear pursuit evasion problem in which the evader's control law is modeled as a random process, in which the pursuer has partial noisy linear measurements of its own and the evader's relative position, and a quadratic optimality criterion is used, past results of the authors imply that the optimal control is a linear function of the "predicted miss." Determining the predicted miss involves estimating the evader's terminal position from past system measurements. Nonlinear filtering techniques are used to give expressions for computing the conditional expectation of the evader's terminal position even in the presence of the random unknown maneuvers of the evader.

I. INTRODUCTION

Recently, Helmes and Rishel [5], [6], and Beneš [1] have shown, for a linear system,

\[ dx = (Ax + Bu) dt + dz, \]

whose driving noises \( z \) may not be Gaussian, and for an optimality criteria which is a quadratic function of the terminal state plus a quadratic running cost in the control, that the optimal control is a linear function of a quantity called the "predicted miss."

The purpose of this note is to apply these results, and results from optimal nonlinear filtering and extrapolation, to describe the optimal control for a linear pursuit-evasion problem in which the evader's control is modeled as an unknown random process.

The main problem in computing the predicted miss is to compute the conditional expectation of the evader's terminal position given...