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Discrete-time extensions of mixed-μ bounds to monotonic and odd monotonic nonlinearities

WASSIM M. HADDAD†‡ and VIKRAM KAPILA†§

Using the parameter-dependent Lyapunov function framework of Haddad and Bernstein this paper extends and connects some of the classical absolute stability results on discrete-time systems with monotonic and odd monotonic feedback nonlinearities to modern robust stability and performance problems for constant real parameter uncertainty. As in the continuous-time case, it is shown that an immediate application of this framework provides a generalization and reformulation of discrete-time mixed-μ theory in terms of Lyapunov functions and Riccati equations while blending $H_2$ robust performance within the μ framework.

1. Introduction

In a recent series of papers (Haddad and Bernstein 1991, 1993, 1994a, 1994b, 1994c, Haddad et al. 1992, 1994) a refined Lyapunov function technique was developed to overcome some of the current limitations of Lyapunov function theory for the problem of robust stability and performance in the presence of constant real parameter uncertainty. Specifically, a general framework for robust controller analysis and synthesis based on parameter-dependent Lyapunov functions was developed that is both flexible in addressing a large class of uncertainty structures and restrictive in excluding uncertainties that are not physically meaningful. The idea behind this framework is to allow the Lyapunov function to be a function of the uncertainty, thus guaranteeing robust stability and performance via a family of Lyapunov functions. For robust stability, the form of the parametrized Lyapunov function is critical because the presence of the uncertainty within the Lyapunov function does not allow the uncertain parameters to be arbitrarily time-varying, which renders it less conservative for constant real parameter uncertainty than fixed quadratic Lyapunov functions (Haddad and Bernstein 1991).

Connections and extensions to the classical frequency domain stability criteria for time-invariant, monotonic, and odd monotonic nonlinearities that are based on Lur'e-Postnikov and extended Lur'e-Postnikov Lyapunov functions (Popov 1962, Narendra and Neuman 1966, 1967, Thathachar et al. 1967, Narendra and Taylor 1973) are also explored by Haddad and Bernstein (1991) and Haddad et al. (1992, 1994). Specifically, since the linear uncertainty specializations of Lur'e-Postnikov Lyapunov functions are parameter-dependent (Haddad and Bernstein 1991), an immediate application of the parametrized

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Lyapunov function framework is a reinterpretation of the classical absolute stability theory to address the modern robust control problem for real parameter uncertainty. Thus, absolute stability theory concerns the stability of a system for a class of nonlinearities which, as noted by Safonov (1986, 1984), Siljak (1990), and Haddad and Bernstein (1991, 1993, 1994a) can readily be interpreted as an uncertainty model. Of course, the robust real parameter uncertainty problem can be addressed by developing absolute stability criteria for refined classes of nonlinearities (such as time-invariant, locally slope-restricted, and odd monotonic) since, in this case, the nonlinear uncertainty set is a close approximation to the linear time-invariant uncertainty set.

In recent research (Haddad et al. 1992, 1994, How and Hall 1993), the parametrized Lyapunov function framework developed by Haddad and Bernstein (1991) was combined with the frequency-dependent off-axis circle interpretation of the shifted Popov criterion given by Hsu and Meyer (1968) to draw connections between upper bounds to mixed-\(\mu\) (Fan et al. 1991, Young et al. 1991) and absolute stability theory (Narendra and Taylor 1973). As shown by Haddad et al. (1992, 1994) this connection provides new machinery for mixed-\(\mu\) analysis and synthesis in terms of parameter-dependent Lyapunov functions and Riccati equations for full- and reduced-order dynamic compensation. Furthermore, since the stability multipliers in absolute stability theory are analogous to the \(D, N\)-scales in mixed-\(\mu\) theory, the overall framework provides an alternative approach to \(\mu\)-synthesis which potentially circumvents standard \(D, N\)-\(K\) iteration and curve fitting procedures. Similar connections are reported by Safonov and Chiang (1993) for the real/complex \(K_m(=1/\mu)\) synthesis problem.

In this paper we extend the results of Haddad et al. (1992, 1994) to discrete-time systems. Specifically, using the multiplier construction proposed by Narendra and Cho (1968) for monotonic and odd monotonic nonlinearities for SISO discrete-time feedback systems, we extend the results to multivariable feedback systems with time-invariant locally slope-restricted monotonic and odd monotonic nonlinearities using an extended Lur'e-Postnikov Lyapunov function framework. As in the continuous-time case, we show that the stability multipliers for monotonic and odd monotonic nonlinearities provide a parametrization for the \(D, N\)-scales in mixed-\(\mu\) theory and hence establish a direct connection between the results of classical discrete-time absolute stability theory and modern discrete-time mixed-\(\mu\) theory.

2. Mathematical preliminaries

In this section we establish definitions, notation, and several key lemmas. Let \(\mathbb{R}\) and \(\mathbb{C}\) denote the real and complex numbers, let \(\mathcal{N}\) denote \(\{1, 2, 3, \ldots\}\), let \((\cdot)^T\), \((\cdot)^*\), and \((\cdot)^\dagger\) denote transpose, complex conjugate transpose, and Moore-Penrose or generalized inverse, let \(I_n\) or \(I\) denote the \(n \times n\) identity matrix, and let \(0_n\) denote the \(n \times n\) zero matrix. Furthermore, \(M \succ 0 (M > 0)\) denotes the fact that the hermitian matrix \(M\) is non-negative (positive) definite. In this paper a real-rational matrix function is a matrix whose elements are rational functions with real coefficients. Furthermore, a transfer function is a real-rational matrix function each of whose elements is proper, i.e. finite at \(z = \infty\). A strictly proper transfer function is a transfer function that is zero at infinity. Finally, an asymptotically stable transfer function is a transfer function
each of whose poles is in the open unit disc. The space of asymptotically stable transfer functions is denoted by \( \mathcal{H}_\infty \), i.e., the real-rational subset of \( \mathcal{H}_\infty \). Let

\[
G(z) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

denote a state-space realization of a transfer function \( G(z) \), that is, \( G(z) = C(zI - A)^{-1}B + D \). The notation \( G^{\text{min}} \) is used to denote a minimal realization.

A square transfer function \( G(z) \) is called \emph{positive real} (Hitz and Anderson 1969, Siljak 1973) if (1) all poles of \( G(z) \) are in the closed unit disc; and (2) \( G(z) + G^*(z) \) is non-negative definite for \( |z| > 1 \). A square transfer function \( G(z) \) is called \emph{strictly positive real} (Cains 1989) if (1) \( G(z) \) is asymptotically stable and (2) \( G(e^{i\theta}) + G^*(e^{i\theta}) \) is positive definite for all \( \theta \in [0, 2\pi] \). Finally, a square transfer function \( G(z) \) is \emph{strongly positive real} if it is strictly positive real and \( D + D^T > 0 \), where \( D \equiv G(\infty) \). Recall that a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, while a minimal realization of a strictly positive real transfer function is asymptotically stable.

In this paper, \( G(z) \) will denote an \( m \times m \) transfer function with input \( u \in \mathbb{R}^m \), output \( y \in \mathbb{R}^m \), and internal state \( x \in \mathbb{R}^n \). We will omit all matrix dimensions throughout, and assume that all quantities have compatible dimensions. Next, we state the positive real lemma used to characterize positive realness in the state-space setting.

**Lemma 2.1—Positive real lemma:** The transfer function

\[
G(z) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

is positive real if and only if there exist matrices \( P, L \) and \( W \) with \( P \) positive definite such that

\[
P = A^TPA + L^TL
\] (2.1)

\[
0 = B^TPA - C + W^TL
\] (2.2)

\[
0 = D + D^T - B^TPB - W^TW
\] (2.3)

**Proof:** Sufficiency follows from algebraic manipulation of (2.1)–(2.3) while necessity follows from spectral factorization theory. See Hitz and Anderson (1969) for details. \( \square \)

Next, we show that if \( D + D^T - B^TPB > 0 \) where \( P \) satisfies (2.1)–(2.3), then (2.1)–(2.3) collapse to a single Riccati equation characterizing positive realness. For details see Haddad and Bernstein (1994 a).

**Lemma 2.2:** The transfer function

\[
G(z) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
is strongly positive real if and only if there exist positive-definite matrices \( P \) and \( R \) such that
\[
D + D^T - B^T PB > 0
\]
\[
P = A^T PA + (C - B^T PA)(D + D^T - B^T PB)^{-1}(C - B^T PA) + R
\]

**Proof:** The proof is similar to the proof of Lemma 4.2 for the continuous-time case given by Haddad and Bernstein (1993) and hence is omitted.

3. **Absolute stability for monotonic and odd-monotonic nonlinearities**

As discussed by Narendra and Taylor (1973), the stability multipliers in the absolute stability criteria play a crucial role in distinguishing the class of allowable feedback nonlinearities. Specifically, constructing stability multipliers that effectively place less restrictive conditions on the linear part of the system, results in absolute stability criteria that guarantee stability for a refined class of feedback nonlinearities. In continuous-time theory, absolute stability results extend the positive real Popov stability multiplier for sector-bounded time-invariant nonlinear functions to monotonic and odd monotonic nonlinearities. Specifically, suitable positive real stability multipliers are constructed as driving-point impedances of passive electrical networks involving resistor–inductor (RL) and resistor–capacitor (RC) combinations which exhibit interlacing pole–zero patterns on the negative real axis (Brockett and Willems 1965, Narendra and Neuman 1966, 1967, Narendra and Taylor 1973, Thathachar et al. 1967, Zames and Falb 1968). For discrete-time systems Neuman (1967) and Narendra and Cho (1968) construct discrete-time analogue multipliers for monotonic and odd monotonic nonlinearities. The standard form of the discrete-time stability multiplier for each \( i = 1, \ldots, m \) is
\[
W_i(z) = x_{i0} + \sum_{j=1}^{m_{ii}} \gamma_{ij} \left( 1 - \frac{\gamma_{ij}}{\gamma_{ij} + \beta_{ij} z - \eta_{ij}} \right) + \sum_{j=m_{ii}+1}^{m_{ii+1}} \gamma_{ij} \left( 1 + \frac{\gamma_{ij}}{\gamma_{ij} + \beta_{ij} z - \eta_{ij}} \right)
\]

(3.1)

where the coefficient \( x_{i0} \) is positive, the coefficients \( \gamma_{ij} \) and \( \beta_{ij} \) are non-negative, and the coefficients \( \eta_{ij} \) are positive and satisfy \( \eta_{ij} = \beta_{ij}(\gamma_{ij} + \beta_{ij}) > 0 \). To consider just monotonic nonlinearities, take \( m_{ii+1} = m_{ii} \) in (3.1) which is equivalent to eliminating the last summation. Note that the first sum in (3.1) is a partial fraction expansion of a transfer function with an interlacing pole–zero pattern on the real axis of the \( z \)-plane in the interval \([0, 1]\) with a zero singularity nearest to the point \((1, 0)\) while the last summation in (3.1) involves a partial fraction expansion with an interlacing pole–zero pattern involving a pole singularity nearest to the point \((1, 0)\). Finally, note that \( W_i(z), i = 1, \ldots, m, \) is positive real (Narendra and Cho 1968). As noted by Haddad et al. (1992, 1994) for the continuous-time case, proving stability of the coupled feedback system involving the plant, multipliers, and nonlinearities requires constructing real signals of the form \( W(z) y(z) \), where \( W(z) = \text{diag} W_i(z), i = 1, \ldots, m \). As a result it is necessary to augment the multiplier dynamics to the original system so that the filtered outputs, to be defined later, can be recovered directly from the augmented state vector. The resulting augmented matrix \( A_a \) then contains poles of both the system \( G(z) \) and the multipliers \( W_i(z), i = 1, \ldots, m \).

Since a large body of work on absolute stability theory has been developed
for nonlinearities belonging to the infinite sector $[0, \infty)$, the loop transformation technique discussed by Rekasius and Gibson (1962) and Vidyasagar (1993) can be used to capture nonlinearities in the finite sector. Specifically, as shown in Fig. 1 the double sector nonlinearity belonging in $(M_1, M_2)$, where $M_1, M_2 \in \mathbb{R}^{m \times m}$ are diagonal matrices representing the lower and upper sector bounds for each input-output loop, can be converted to both the one-sided sector condition $[0, M_2 - M_1]$ and the infinite sector condition $[0, \infty)$. In this and the next two sections we shall only consider single sector bounds $[0, M]$ for simplicity of exposition.

Next, we formulate the process involving the loop transformation technique along with the multiplier dynamics augmentation. Let

$$ G(z) \overset{\text{min}}{\sim} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} $$

We begin with the transformation illustrated in Fig. 1, to convert a finite sector restriction on the nonlinearity to an infinite sector restriction. We assume that $M_1 = 0$ and $M_2 = M = \text{diag}(M_{11}, M_{22}, \ldots, M_{nn})$, and consider the set of differentiable monotonic and odd monotonic time-invariant nonlinearities satisfying the finite sector constraint $0 < \sigma \phi_i(\sigma) < M_i \sigma^2$ along with the local slope constraint $0 \leq d\phi_i(\sigma)/d\sigma < M_i$, for all $\sigma \in \mathbb{R}$. From Fig. 1, observe that $\phi(y) = \phi(\tilde{y})$, where $\phi(\tilde{y})$ corresponds to the transformed nonlinearity belonging to the infinite sector, and

$$ \tilde{y} = y - M^{-1}\phi(y) $$

Furthermore, for each nonlinearity, with $y_i \neq 0$

$$ \frac{\tilde{y}_i}{\tilde{y}_i} = \frac{\phi(y_i)/y_i}{1 - M^{-1}\phi(y_i)/y_i} $$

![Figure 1](image)

Figure 1. Loop transformations from finite to infinite sector nonlinearities.
so that if \( \phi(\cdot) \) is sector bounded by \( M_i \), then the equivalent condition for the shifted nonlinearity is \( \tilde{y}_i \tilde{\phi}(\tilde{y}_i) > 0 \). Next, using the chain rule

\[
\frac{d\tilde{\phi}_i(\tilde{y}_i)}{d\tilde{y}_i} = \frac{d\phi_i(y_i)/dy_i}{1 - M_i^{-1} d\phi_i(y_i)/dy_i}
\]

so that if \( \phi(\cdot) \) is differentiable and satisfies the slope restriction \( 0 \leq d\phi_i(\sigma)/d\sigma < M_i \), then \( \tilde{\phi}_i(\cdot) \) is differentiable and satisfies \( 0 \leq d\tilde{\phi}_i(\sigma)/d\sigma \), and is thus monotonic. The same process holds for slope-restricted odd monotonic nonlinearities.

With the above transformations, the corresponding shifted system is given by

\[
\tilde{G}(z) = G(z) + M^{-1}
\]

(3.5)

Furthermore, each transformed nonlinearity \( \tilde{\phi}_i(\cdot) \) is restricted to lie in the first and third quadrants, so that

\[
\sigma \tilde{\phi}_i(\sigma) > 0, \quad \sigma \in \mathbb{R}
\]

(3.6)

Next, since the transformed nonlinearities are monotonic, they satisfy

\[
0 \leq (\tilde{\phi}(\sigma_1) - \tilde{\phi}(\sigma_2))(\sigma_1 - \sigma_2), \quad \sigma_1, \sigma_2 \in \mathbb{R}
\]

(3.7)

Alternatively, if the transformed nonlinearities considered are odd monotonic, they additionally satisfy (Narendra and Taylor 1973)

\[
0 \leq \sigma_1 \tilde{\phi}_1(\sigma_1) + \sigma_2 \tilde{\phi}_2(\sigma_2) - \sigma_1 \tilde{\phi}_1(\sigma_2) + \sigma_2 \tilde{\phi}_2(\sigma_1), \quad \sigma_1, \sigma_2 \in \mathbb{R}
\]

(3.8)

For a detailed exposition of these transformations the interested reader is referred to Thathachar and Srinath (1967), Narendra and Taylor (1973), and Vidyasagar (1993).

Having discussed the above transformations we now proceed with the multiplier augmentation. As mentioned earlier, proving stability of the coupled feedback system involving the plant, multipliers, and nonlinearities requires the construction of real signals of the form \( W(z)\tilde{y}(z) \). Observing the form of the multiplier in (3.1) it can be seen that the expression \( W_i(z)\tilde{y}_i(z) \) involves signals of the form

\[
\sigma_{ij}(z) = \frac{y_i}{y_{ij} + \beta_i} \frac{z}{z - \eta_{ij}} \tilde{y}_i(z)
\]

(3.9)

Next, taking the inverse \( z \)-transform of (3.9) we obtain

\[
\sigma_{ij}(k) - \eta_{ij}\sigma_{ij}(k - 1) = \frac{y_i}{y_{ij} + \beta_i} \tilde{y}_i(k)
\]

(3.10)

Now, using the notation \( q_{ij}(k) = \sigma_{ij}(k - 1) \), (3.10) becomes

\[
q_{ij}(k + 1) = \eta_{ij}q_{ij}(k) + \frac{y_i}{y_{ij} + \beta_i}[C\tilde{x}(k) - M^{-1}\tilde{\phi}(\tilde{y}(k))]
\]

(3.11)

where \( (\cdot)_i \) denotes the \( i \)-th row of \( (\cdot) \). Writing the multiplier states in a vector \( q^T \triangleq [q_{i1}, \ldots, q_{im_2}] \), the dynamics associated with each multiplier can be written as

\[
q_i(k + 1) = [\hat{C}_i, \hat{A}_i]\begin{bmatrix} x(k) \\ q_i(k) \end{bmatrix} - \hat{M}_i\tilde{\phi}(\tilde{y}(k))
\]

(3.12)
Absolute stability and discrete-time mixed-\( \mu \)

where \( \hat{\lambda}_i \triangleq \text{diag}(\eta_i), j = 1, \ldots, m_{i2} \), and

\[
\hat{C}_i \triangleq \begin{bmatrix}
\gamma_{i1}
\gamma_{i1} + \beta_{i1}
\gamma_{i2}
\gamma_{i2} + \beta_{i2}
\vdots
\gamma_{im_2}
\gamma_{im_2} + \beta_{im_2}
\end{bmatrix}
\quad (C)_i,
\hat{M}_i \triangleq \begin{bmatrix}
\gamma_{i1}
\gamma_{i1} + \beta_{i1}
\gamma_{i2}
\gamma_{i2} + \beta_{i2}
\vdots
\gamma_{im_2}
\gamma_{im_2} + \beta_{im_2}
\end{bmatrix}
\quad (M^{-1})_i
\]

where \((C)_i\) and \((M^{-1})_i\) denote the \(i\)th rows of the respective matrices.

With \(m\) input–output pairs to the system \(G(z)\) we augment the multiplier dynamics to the shifted system \(\hat{G}(z)\) to obtain a state-space representation of \(\hat{G}_a(z)\) given by

\[
x_a(k + 1) = A_a x_a(k) - B_a \hat{\phi}(\bar{y}(k))
\]

\[
\bar{y}(k) = C_a x_a(k) - M^{-1} \hat{\phi}(\bar{y}(k))
\]

where \(x_a \in \mathbb{R}^{n_a}, n_a \triangleq n + \sum_{i=1}^m m_{i2}\) and \(x_a, A_a, B_a\) and \(C_a\) are defined as

\[
x_a \triangleq \begin{bmatrix}
x \\
q_1 \\
q_2 \\
\vdots \\
q_m
\end{bmatrix}, \quad A_a \triangleq \begin{bmatrix}
A & 0 & 0 & \cdots & 0 \\
\hat{C}_1 & \hat{A}_1 & 0 & \cdots & 0 \\
\hat{C}_2 & 0 & \hat{A}_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\hat{C}_m & 0 & 0 & \cdots & \hat{A}_m
\end{bmatrix}, \quad B_a \triangleq \begin{bmatrix}
B \\
\hat{M}_1 \\
\hat{M}_2 \\
\vdots \\
\hat{M}_m
\end{bmatrix}
\]

Next, define \(R_{ij}\) as an output vector for this augmented system, designed to access the \(j\)th element of \(q_i(k)\) so that

\[
a_{ij}(k) = R_{ij} x_a(k)
\]

Then, the only non-zero element of \(R_{ij}\) is the \((n + \sum_{i=1}^{i-1} m_{i2} + j)\)th term, which is 1.

Note that although extra dynamics associated with the multiplier have been added to the system, since \(A_a\) is lower block triangular, it can easily be shown that \(\hat{G}_a(z) = C_a(zI - A_a)^{-1}B_a + M^{-1} = C(zI - A)^{-1}B + M^{-1} = \hat{G}(z)\). Hence, by pole-zero cancellation in each input–output loop, the frequency domain representation of \(\hat{G}(z)\) and \(\hat{G}_a(z)\) are equivalent in terms of their input–output properties. Finally, it is important to note that, unlike the results of Narendra and Neuman (1966, 1967) for the continuous-time case and Neuman (1967) for the discrete-time case, we do not explicitly enforce pole-zero cancellation, between the multiplier and the unaugmented plant, which is often quite restrictive.

In the next sections we provide the main result of this paper for monotonic and odd monotonic nonlinearities using Lyapunov functions, and frequency domain conditions involving \(\hat{G}(z)\) along with the full multiplier \(W(z)\). In order to present these results we define the following vector notation. These definitions are complicated by the fact that \(W_i(z)\) for different input–output loops can
have a different number of expansion terms in (3.1). To overcome this difficulty, extended values of \( \gamma_j, \beta_j \) and \( \eta_j \) are defined as follows. Let \( m_1 = \max_i (m_j) \).

Then, for all \( i = 1, \ldots, m \), and \( j = 1, \ldots, m_1 \), let \( \gamma_j = 0, \beta_j = 0, \eta_j = 0 \) and \( R_j = 0 \), if \( j > m_1 \). Furthermore, define \( M_j \triangleq \text{diag}(\gamma_1, \ldots, \gamma_m), N_j \triangleq \text{diag}(\beta_1, \ldots, \beta_m), S_j \triangleq \text{diag}(\eta_1, \ldots, \eta_m), R_j^T \triangleq [R_{1j}^T, \ldots, R_{mj}^T] \) and \( q_j^T \triangleq [q_{1j}, \ldots, q_{mj}], j = 1, \ldots, m_1 \).

Next, note that using the above notational development and the augmentation scheme proposed in (3.13) and (3.14), simple matrix manipulations yield

\[
(M_j + N_j)R_j B_a = M_j M_j^{-1} 
\]

\[
(M_j + N_j)R_j A_a = N_j R_j + M_j C_a 
\]

Recall that similar features for a single feedback nonlinearity were achieved by Narendra and Neuman (1966) by enforcing pole–zero cancellation between the unaugmented plant transfer function and the multiplier dynamics.

4. Absolute stability criteria for monotonic nonlinearities

To state the main result of this section the following definitions are needed. Let \( M \in \mathbb{R}^{m \times m} \) be a positive definite diagonal matrix and define the set \( \Phi \) of allowable sector-bounded locally slope-restricted monotonic differentiable nonlinearities \( \phi \) by

\[
\Phi \triangleq \{ \phi: \mathbb{R}^m \rightarrow \mathbb{R}^m: \phi^T(y)[M^{-1}\phi(y) - y] < 0, y \in \mathbb{R}^m, y \neq 0, \]

\[
\phi(y) = [\hat{\phi}_1(y_1), \ldots, \hat{\phi}_m(y_m)]^T, 0 \leq \frac{d\phi(y_i)}{dy_i} < M_{ii}, \text{ and} \\
[\hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_2)](\alpha_1 - \alpha_2) + \hat{\phi}_i(\sigma_1)M^{-1}\hat{\phi}_i(\sigma_2) \geq 0, \\
i = 1, \ldots, m, \sigma_1, \sigma_2 \in \mathbb{R} \}
\]

(4.1)

In the special case when \( m = 1 \) the sector condition characterizing \( \Phi \) is equivalent to the more familiar condition \( 0 < \phi(y)y < My^2 \) along with the local slope bounded \( 0 \leq d\phi/dy < M, \ y \in \mathbb{R} \). Note that it follows from the discussion in §3 that the transformed nonlinearities \( \hat{\phi}(\cdot) \) belonging to the infinite sector are defined by the set \( \hat{\Phi} \) given by

\[
\hat{\Phi} \triangleq \{ \hat{\phi}: \mathbb{R}^m \rightarrow \mathbb{R}^m: \hat{\phi}^T(\bar{y})\bar{y} > 0, \ \bar{y} \in \mathbb{R}^m, \]

\[
\hat{\phi}(\bar{y}) = [\hat{\phi}_1(\bar{y}_1), \ldots, \hat{\phi}_m(\bar{y}_m)]^T, \ \frac{d\hat{\phi}_i(\bar{y}_i)}{d\bar{y}_i} \geq 0, \text{ and} \\
[\hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_2)](\alpha_1 - \alpha_2) \geq 0, i = 1, \ldots, m, \sigma_1, \sigma_2 \in \mathbb{R} \}
\]

(4.2)

For convenience in stating our next result, we define \( H_0 \triangleq \text{diag}(\alpha_{j0}, \ldots, \alpha_{m0}) \).

**Theorem 4.1:** Let

\[
G(z) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}
\]

let \( H_0 \in \mathbb{R}^{m \times m} \) be diagonal and positive definite, and let \( N_j, M_j, S_j \in \mathbb{R}^{m \times m}, j = 1, \ldots, m_1 \), be diagonal and non-negative definite. Then
\[
\mathcal{Q}(z) \triangleq \left[ H_0 + \sum_{j=1}^{m_1} M_j (I - z(M_j + N_j)^* M_j (zI - S_j)^*) \right] [G(z) + M^{-1}] 
\] (4.3)

is positive real if and only if there exist matrices \( P, L \) and \( W \) with \( P \) positive definite satisfying
\[
P = A_a^T PA_a + L^T L 
\] (4.4)
\[
0 = B_a^T PA_a - H_0 C_a - \sum_{j=1}^{m_1} M_j (C_a - R_j A_a) + W^T L 
\] (4.5)
\[
0 = 2H_0 M^{-1} - B_a^T PB_a - \sum_{j=1}^{m_1} M_j (R_j B_a - M^{-1}) 
+ (R_j B_a - M^{-1})^T M_j - W^T W 
\] (4.6)

In this case
\[
V(x_a) = x_a^T P x_a + 2 \sum_{j=1}^{m_1} \sum_{i=1}^{m} \int_{q_{ij}(k)}^{q_{ij}(k+1)} \beta_{ij} \hat{\Phi}_i(s) \, ds 
\] (4.7)

where \( q_{ij}(k) = R_j x_a(k) \), is a Lyapunov function that guarantees the negative feedback interconnection of \( G(z) \) and \( \phi \) is globally asymptotically stable for all \( \phi \in \Phi \).

**Proof:** First note that \( \mathcal{Q}(z) \) has a minimal realization given by
\[
\mathcal{Q}(z) \min \begin{bmatrix} A_a & B_a \\ H_0 C_a - \sum_{j=1}^{m_1} M_j (C_a - R_j A_a) & H_0 M^{-1} - \sum_{j=1}^{m_1} M_j (R_j B_a - M^{-1}) \end{bmatrix} 
\]

Now it follows from Lemma 2.1 that \( \mathcal{Q}(z) \) is positive real if and only if there exists \( P, L \), and \( W \) with \( P \) positive definite satisfying (4.4)–(4.6).

Next, for \( \phi \in \Phi \) consider the Lyapunov function candidate (4.7). Since \( P \) is positive definite, \( \tilde{\phi} \in \Phi \), and \( \beta_{ij} \) are non-negative scalars, it follows that \( V(x_a) \) given by (4.7) is positive definite. The corresponding Lyapunov difference is given by
\[
\Delta V(x_a) = x_a^T (A_a^T PA_a - P)x_a - \tilde{\phi}^T B_a^T PA_a x_a - x_a^T A_a^T PB_a \tilde{\phi} 
+ \tilde{\phi}^T B_a^T PB_a \tilde{\phi} + 2 \sum_{j=1}^{m_1} \sum_{i=1}^{m} \int_{q_{ij}(k)}^{q_{ij}(k+1)} \beta_{ij} \hat{\Phi}_i(s) \, ds 
\] (4.8)

Now using the fact that \( \tilde{\phi} (\cdot) \) is a monotonic function, the integral term in (4.8) can be bounded from above by
\[
2 \int_{q_{ij}(k)}^{q_{ij}(k+1)} \beta_{ij} \hat{\Phi}_i(s) \, ds \leq 2 \left[ \hat{\Phi}_i(q_{ij}(k+1)) \beta_{ij} [q_{ij}(k+1) - q_{ij}(k)] \right] 
\]

so that
\[
\Delta V(x_a) \leq x_a^T (A_a^T PA_a - P)x_a - \tilde{\phi}^T B_a^T PA_a x_a - x_a^T A_a^T PB_a \tilde{\phi} + \tilde{\phi}^T B_a^T PB_a \tilde{\phi} 
+ 2 \sum_{j=1}^{m_1} \sum_{i=1}^{m} \int_{q_{ij}(k)}^{q_{ij}(k+1)} N_j \hat{\Phi}_i(s) \, ds \left[ q_{ij}(k+1) - q_{ij}(k) \right] 
\] (4.9)
Furthermore, note that since \( \bar{q}_j(k+1) = R_j x_a(k+1) \) and \( \bar{q}_j(k+1) - \bar{q}_j(k) = R_j(A_a - I)x_a - R_j B_a \tilde{\phi} \), Eq. (4.9) becomes

\[
\Delta V(x_a) \leq x_a^T \left[ A_a^T P A_a - P \right] x_a - \tilde{\phi}^T B_a^T P A_a x_a - x_a^T A_a^T P B_a \tilde{\phi} + \tilde{\phi}^T B_a^T P B_a \tilde{\phi} \\
+ 2 \sum_{j=1}^{m_1} \tilde{\phi}^T (\bar{q}_j(k+1)) N_j R_j(A_a - I)x_a - R_j B_a \tilde{\phi}
\]  

(4.10)

Next, adding and subtracting

\[
2\tilde{\phi}^T H_0 \bar{y}, \sum_{j=1}^{m_1} \tilde{\phi}^T M_j(C_a - R_j A_a)x_a, \text{ and } 2 \sum_{j=1}^{m_1} \tilde{\phi}^T (\bar{q}_j(k+1)) M_j R_j B_a \tilde{\phi}
\]

to and from (4.10) and grouping terms yields

\[
\Delta V(x_a) \leq x_a^T \left[ A_a^T P A_a - P \right] x_a \\
- \tilde{\phi}^T \left[ B_a^T P A_a - H_0 C_a - \sum_{j=1}^{m_1} M_j(C_a - R_j A_a) \right] x_a \\
- x_a^T \left[ B_a^T P A_a - H_0 C_a - \sum_{j=1}^{m_1} M_j(C_a - R_j A_a) \right]^T \tilde{\phi} \\
- \tilde{\phi}^T \left[ 2H_0 M^{-1} - B_a^T P B_a \right] \tilde{\phi} \\
+ 2 \sum_{j=1}^{m_1} \tilde{\phi}^T (\bar{q}_j(k+1)) N_j R_j A_a x_a - \tilde{\phi}^T (\bar{q}_j(k+1)) N_j R_j x_a \\
- \tilde{\phi}^T (\bar{q}_j(k+1)) (M_j + N_j) R_j B_a \tilde{\phi} - \tilde{\phi}^T M_j(C_a - R_j A_a) x_a \\
+ \tilde{\phi}^T (\bar{q}_j(k+1)) M_j R_j B_a \tilde{\phi} - 2\tilde{\phi}^T H_0 \bar{y}
\]  

(4.11)

Now, adding and subtracting

\[
2 \sum_{j=1}^{m_1} \left[ \tilde{\phi}(\bar{q}_j(k+1)) - \bar{\phi}(\bar{y}(k)) \right]^T M_j(\bar{q}_j(k+1) - \bar{y}(k))
\]

to and from (4.11) and using (3.16) gives

\[
\Delta V(x_a) \leq x_a^T \left[ A_a^T P A_a - P \right] x_a \\
- \tilde{\phi}^T \left[ B_a^T P A_a - H_0 C_a - \sum_{j=1}^{m_1} M_j(C_a - R_j A_a) \right] x_a \\
- x_a^T \left[ B_a^T P A_a - H_0 C_a - \sum_{j=1}^{m_1} M_j(C_a - R_j A_a) \right]^T \tilde{\phi} \\
- \tilde{\phi}^T \left[ 2H_0 M^{-1} - B_a^T P B_a \right] \tilde{\phi} \\
+ 2 \sum_{j=1}^{m_1} \tilde{\phi}^T (\bar{q}_j(k+1)) N_j R_j A_a x_a - \tilde{\phi}^T (\bar{q}_j(k+1)) N_j R_j x_a \\
- \tilde{\phi}^T (\bar{q}_j(k+1)) R_j B_a \tilde{\phi} - \tilde{\phi}^T M_j(C_a - R_j A_a) x_a \\
+ \tilde{\phi}^T (\bar{q}_j(k+1)) M_j R_j B_a \tilde{\phi} - 2\tilde{\phi}^T H_0 \bar{y}
\]
\[ - \tilde{\phi}^T(\tilde{\varphi}_j(k + 1))M_jM^{-1}\tilde{\phi} - \tilde{\phi}^T M_j(C_a - R_jA_a)x_a \\
+ \tilde{\phi}^T(\tilde{\varphi}_j(k + 1))M_jR_jB_a\tilde{\phi} \\
+ [\tilde{\phi}(\tilde{\varphi}_j(k + 1)) - \tilde{\phi}(\tilde{\gamma}(k))]^T M_j[\tilde{\varphi}_j(k + 1) - \tilde{\gamma}(k)] - 2\tilde{\phi}^T H_0\tilde{\gamma} \\
- 2\sum_{j=1}^{m_1} [\tilde{\phi}(\tilde{\varphi}_j(k + 1)) - \tilde{\phi}(\tilde{\gamma}(k))]^T M_j[\tilde{\varphi}_j(k + 1) - \tilde{\gamma}(k)] \tag{4.12} \]

or, equivalently, using \( \tilde{\varphi}_j(k + 1) - \tilde{\gamma}(k) = (R_jA_a - C_a)x_a - (R_jB_a - M^{-1})\tilde{\phi} \) and (3.17) yields

\[ \Delta V(x_a) \leq x_a^T[A_a^TPA_a - P]x_a \\
- \tilde{\phi}^T \left[ B_a^TPA_a - H_0C_a - \sum_{j=1}^{m_1} M_j(C_a - R_jA_a) \right]x_a \\
- \tilde{\phi}^T \left[ B_a^TPA_a - H_0C_a - \sum_{j=1}^{m_1} M_j(C_a - R_jA_a) \right] \tilde{\phi} \\
- \tilde{\phi}^T \left[ 2H_0M^{-1} - B_a^TPB_a - \sum_{j=1}^{m_1} M_j(R_jB_a - M^{-1}) \right] \tilde{\phi} \\
+ (R_jB_a - M^{-1})^T M_j \tilde{\phi} - 2\tilde{\phi}^T H_0\tilde{\gamma} \\
- 2\sum_{j=1}^{m_1} [\tilde{\phi}(\tilde{\varphi}_j(k + 1)) - \tilde{\phi}(\tilde{\gamma}(k))]^T M_j[\tilde{\varphi}_j(k + 1) - \tilde{\gamma}(k)] \tag{4.13} \]

Now using (4.4)-(4.6) yields

\[ \Delta V(x_a) \leq -[Lx_a - W\tilde{\phi}]^T[Lx_a - W\tilde{\phi}] - 2\tilde{\phi}^T H_0\tilde{\gamma} \\
- 2\sum_{j=1}^{m_1} [\tilde{\phi}(\tilde{\varphi}_j(k + 1)) - \tilde{\phi}(\tilde{\gamma}(k))]^T M_j[\tilde{\varphi}_j(k + 1) - \tilde{\gamma}(k)] \tag{4.14} \]

Since \( \tilde{\phi} \in \tilde{\Phi} \) it follows that \( \Delta V(x_a) \leq 0 \), which proves stability in the sense of Lyapunov.

To show global asymptotic stability we need to show that \( \Delta V(x_a) = 0 \) implies that \( x = 0 \). Note that \( \Delta V(x_a) = 0 \) implies that \( \tilde{\gamma}(k) = 0, \ k \in \mathcal{N} \), which further implies that \( \tilde{\phi}(\tilde{\gamma}) = 0 \). Next, using \( y = \tilde{\gamma} + M^{-1}\tilde{\phi}(\tilde{\gamma}), \ \tilde{\gamma} = 0 \) and \( \tilde{\phi}(\tilde{\gamma}) = 0 \), implies that \( y(k) = 0 \) and thus \( \phi(y(k)) = 0, \ k \in \mathcal{N} \). Hence, in this case \( Cx(k) = 0 \) implies \( CA^Ix(k) = 0, \ l \geq 0, \ k \in \mathcal{N} \). Since \( (A, C) \) is observable, it follows that \( x(k) = 0, \ k \in \mathcal{N} \). Thus, the only solution that remains in the set where \( \Delta V(x_a) = 0 \) is the \( x(k) = 0, \ k \in \mathcal{N} \), solution and hence global asymptotic stability is established as a direct consequence of the invariant set theorem. \( \square \)

**Remark 4.1:** If \( \varphi(z) \) is strongly positive real, then it follows from Lemma 2.2 that the term \( W^Tw \) in (4.6) is positive definite, and hence \( F > 0 \) satisfies (4.4)-(4.6) if and only if there exists \( R > 0 \) such that \( P \) satisfies the Riccati equation.
\[ P = A^T_a PA_a + (\hat{C}_a - B^T_a PA_a)^T \hat{R}^{-1} (\hat{C}_a - B^T_a PA_a) + R \] (4.15)

where
\[ \hat{C}_a \triangleq H_0 C_a + \sum_{j=1}^{m_1} M_j (C_a - R_j A_a) \] (4.16)
\[ \hat{R} \triangleq 2 H_0 M^{-1} - \sum_{j=1}^{m_1} [M_j (R_j B_a - M^{-1}) + (R_j B_a - M^{-1})^T M_j] - B^T_a P B_a \] (4.17)

**Remark 4.2:** Note that with the definitions of \( N_j, \ M_j, \ S_j \) and \( H_0 \), (4.3) can be written as
\[ \mathcal{G}(z) = W(z)[G(z) + M^{-1}] \] (4.18)
where \( W(z) = \text{diag} W_i(z), \ i = 1, \ldots, m \), and \( W_i(z) \) is given by (3.1) with \( m_1 = m_{i2} \).

5. **Absolute stability criteria for odd monotonic nonlinearities**

In this section we extend the results of §4 to odd monotonic nonlinearities. To state the main result of this section, we define the set \( \Phi \) of allowable sector-bounded locally slope-restricted odd monotonic differentiable nonlinearities \( \phi \) by

\[ \Phi \triangleq \left\{ \phi: \mathbb{R}^m \to \mathbb{R}^m: \phi^T(y)[M^{-1}\phi(y) - y] < 0, \ y \in \mathbb{R}^m, \ y \neq 0, \right. \]

\[ \phi(y) = [\hat{\phi}_1(y_1), \ldots, \hat{\phi}_m(y_m)]^T, \ 0 \leq \frac{d\hat{\phi}_i(y_i)}{dy_i} < M_{ii}, \]

\[ [\hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_2)][\sigma_1 - \sigma_2] + \hat{\phi}_i(\sigma_1)M^{-1}\hat{\phi}_i(\sigma_2) \geq 0, \text{ and} \]

\[ [\sigma_1 \hat{\phi}_i(\sigma_2) - \sigma_2 \hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_1)M^{-1}\hat{\phi}_i(\sigma_2)] \leq \sigma_1 \hat{\phi}_i(\sigma_1) + \sigma_2 \hat{\phi}_i(\sigma_2), \]

\[ i = 1, \ldots, m, \ \sigma_1, \ \sigma_2 \in \mathbb{R} \] (5.1)

Once again, it follows from the discussion in §3 that the transformed nonlinearities \( \hat{\phi}^T(\cdot) \) belonging to the infinite sector are defined by the set \( \hat{\Phi} \) given by

\[ \hat{\Phi} \triangleq \left\{ \hat{\phi}: \mathbb{R}^m \to \mathbb{R}^m: \hat{\phi}^T(\tilde{y}) \tilde{y} > 0, \ \tilde{y} \in \mathbb{R}^m, \right. \]

\[ \hat{\phi}(\tilde{y}) = [\hat{\phi}_1(\tilde{y}_1), \ldots, \hat{\phi}_m(\tilde{y}_m)]^T, \ \frac{d\hat{\phi}_i(\tilde{y}_i)}{d\tilde{y}_i} \geq 0, \]

\[ [\hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_2)][\sigma_1 - \sigma_2] \geq 0, \text{ and} \]

\[ \sigma_1 \hat{\phi}_i(\sigma_2) + \sigma_2 \hat{\phi}_i(\sigma_1) - \hat{\phi}_i(\sigma_1)M^{-1}\hat{\phi}_i(\sigma_2) \geq 0, \]

\[ i = 1, \ldots, m, \ \sigma_1, \ \sigma_2 \in \mathbb{R} \] (5.2)

In stating our next result recall \( m_1 = \max_i (m_{i1}) \) and define \( m_2 \triangleq \max_i (m_{i2}) \).
Theorem 5.1: Let

\[ G(z) = \min \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \]

let \( H_0 \in \mathbb{R}^{m \times m} \) be diagonal and positive definite, and let \( N_j, M_j, S_j \in \mathbb{R}^{m \times m}, j = 1, \ldots, m_2, \) be diagonal and non-negative definite. Then

\[ \mathcal{G}(z) \triangleq H_0 + \sum_{j=1}^{m_1} M_j \{ I - z(M_j + N_j)^\# M_j(zI - S_j)^\# \} \]

\[ + \sum_{j=m_1+1}^{m_2} M_j \{ I + z(M_j + N_j)^\# M_j(zI - S_j)^\# \} \]

\[ (G(z) + M^{-1}) \] (5.3)

is positive real if and only if there exist matrices \( P, L \) and \( W \) with \( P \) positive definite satisfying

\[ P = A_x^T P A_x + L^T L \] (5.4)

\[ 0 = B_x^T P A_x - H_0 C_a - \sum_{j=1}^{m_1} M_j (C_a - R_j A_a) - \sum_{j=m_1+1}^{m_2} M_j (C_a + R_j A_a) + W^T L \] (5.5)

\[ 0 = 2H_0 M^{-1} - B_x^T P B_x - \sum_{j=1}^{m_1} [M_j (R_j B_a - M^{-1}) + (R_j B_a - M^{-1})^T M_j] \]

\[ + \sum_{j=m_1+1}^{m_2} [M_j (R_j B_a + M^{-1}) + (R_j B_a + M^{-1})^T M_j] - W^T W \] (5.6)

In this case

\[ V(x_a) = x_a^T P x_a + 2 \sum_{j=1}^{m} \sum_{i=1}^{m} q_i(k) \beta_{ij} \phi_i(s) ds \] (5.7)

where \( q_i(k) = R_i x_a(k) \), is a Lyapunov function that guarantees the negative feedback interconnection of \( G(z) \) and \( \phi \) is globally asymptotically stable for all \( \phi \in \Phi \).

Proof: The proof is similar to the proof of Theorem 4.1 with the additional terms arising due to the odd monotone constraint on the nonlinear feedback elements.

In the next section, we make explicit connections between the absolute stability frequency domain conditions (developed in this and the previous section) and discrete-time mixed-\( \mu \) theory.

6. Connections to mixed-\( \mu \) bounds

In this section we make explicit connections between recent results on mixed-\( \mu \) and discrete-time absolute stability with positive real multipliers. To make these connections with mixed-\( \mu \) theory it is useful to consider nonlinearities with both upper and lower sector bounds. To do this, let diagonal \( M_1, \)


$M_2 \in \mathbb{S}^{m \times m}$ be such that $M_2 - M_1$ is positive definite. Using the loop transformation techniques discussed in §3 and replacing $\phi(y)$ by $\phi(y)$, $G(z)$ by $G(z)$, and $M$ by $M_2 - M_1$, where

$$
\phi(y) \hat{=} \phi(y) - M_1 y, \quad y \in \mathbb{R}^m \quad (6.1)
$$

$$
G(z) \hat{=} (I + G(z)M_1)^{-1}G(z) \sim \begin{bmatrix}
A - BM_1C \\
B \\
C \\
0
\end{bmatrix} \quad (6.2)
$$

it follows from Theorem 4.1 and Theorem 5.1 with Remark 4.2 that, in this case, the frequency domain conditions (4.3) and (5.3) can be written as

$$
\mathcal{G}_h(z) \hat{=} W(z)[(I + G(z)M_1)^{-1}G(z) + (M_2 - M_1)^{-1}] \quad (6.3)
$$

or, equivalently

$$
\mathcal{G}_h(z) = W(z)[(M_2 - M_1)^{-1}(I + M_2 G(z))(I + M_1 G(z))^{-1}] \quad (6.4)
$$

where it is assumed that $G(z)$ is asymptotically stable so that $I + G(z)M_1$ is invertible for all $z = e^{j\omega}$, $\omega \in [-\pi, \pi]$. In order to make explicit connections to real-$\mu$ we consider symmetric sector bounds so that $M_1 = -M_2 < 0$. In this case it follows from Theorems 4.1 and 5.1 that absolute stability of the linear system $G(z)$ with feedback nonlinearities $\phi \in \Phi$ given by (4.1) for monotonic nonlinearities and (5.1) for odd monotonic nonlinearities is guaranteed if $\mathcal{G}_h(z)$ is positive real or, equivalently,

$$
0 \leq \mathcal{G}_h(z) + \mathcal{G}_h^*(z), \quad z = e^{j\omega}, \quad \omega \in [-\pi, \pi] \quad (6.5)
$$

Expanding (6.5) yields

$$
0 \leq W(z)(2M_2)^{-1}(I + M_2 G)(I - M_2 G)^{-1} + (I - M_2 G)^{-1}* (I + M_2 G)^*(2M_2)^{-1}W^*(z) \quad (6.6)
$$

Next, defining $W(z) \hat{=} W_{Re}(z) + jW_{Im}(z), \quad z = e^{j\omega}, \quad \omega \in [-\pi, \pi]$ and forming $M_2(I - M_2 G)^*(6.6) (I - M_2 G)M_2$ we obtain

$$
0 \leq M_2(I - M_2 G)^*(W_{Re} + jW_{Im})(2M_2)^{-1}(I + M_2 G)M_2 \\
+ M_2(I - M_2 G)^*(2M_2)^{-1}(W_{Re} - jW_{Im})(I - M_2 G)M_2 \\
= \frac{1}{2}[M_2(W_{Re} + jW_{Im}) + M_2(W_{Re} + jW_{Im})GM_2 - M_2G*M_2(W_{Re} + jW_{Im}) \\
- M_2G*M_2(W_{Re} + jW_{Im})GM_2 + (W_{Re} - jW_{Im})M_2 \\
- (W_{Re} - jW_{Im})M_2GM_2 \\
+ M_2G*(W_{Re} - jW_{Im})M_2 - M_2G*(W_{Re} - jW_{Im})M_2GM_2] \quad (6.7)
$$

Dividing through by $M_2^2$ and collecting terms yields

$$
G^*W_{Re}M_2G - j(W_{Im}G - G^*W_{Im}) - W_{Re}M_2^{-1} \leq 0 \quad (6.8)
$$

To make connections with mixed-$\mu$ theory we consider the uncertain system with multiple-block-structured uncertainty $\Delta \in \Delta$ where

$$
\Delta \hat{=} \{\Delta: \sigma_{\max}(\Delta) < \gamma^{-1}\} \quad (6.9)
$$

and $\Delta = \text{block-diag}(\Delta_1, \ldots, \Delta_r)$, with $\Delta_i, i = 1, \ldots, r$, corresponding to scalar
real block uncertainties, scalar complex block uncertainties, and matrix complex block uncertainties.

Next recall that for uncertainty \( \Delta \in \Delta \), the structured singular value \( \mu(G(e^{i\omega})) \) is defined by Doyle (1982) (see Dahleh and Khannamash 1993 for discrete-time systems)

\[
\mu(G(e^{i\omega})) \triangleq \left( \min_{\Delta \in \Delta} \{ \sigma_{\max}(\Delta) : \det(I - G(e^{i\omega})\Delta) = 0 \} \right)^{-1}
\] (6.10)

while \( \mu(G(e^{i\omega})) = 0 \) if there exists no \( \Delta \in \Delta \) such that \( \det(I - G(e^{i\omega})\Delta) = 0 \). Since the mixed-\( \mu \) problem is NP complete, recent work (in continuous-time) has led to approximations using upper bounds (Fan et al. 1991, Lee and Tits 1991 and Young et al. 1991). Furthermore, since the bound on \( \mu \) is defined for a constant fixed matrix it holds for every fixed \( z = e^{i\omega}, \omega \in [-\pi, \pi] \). Hence, the mixed-\( \mu \) bounds developed by Fan et al. (1991) and Young et al. (1991) for continuous-time systems also hold for discrete-time systems. Specifically, it follows from Fan et al. (1991) that for \( \omega \in [-\pi, \pi] \), \( G(e^{i\omega}) \in \mathcal{M}^{m \times n} \), and multiple-block-structured uncertainty \( \Delta \subset \Delta \)

\[
\mu(G(e^{i\omega})) \leq \mu_{\text{cir}}
\] (6.11)

where

\[
\mu_{\text{cir}} \triangleq \inf_{D \in \mathbb{D}_\Delta, N \in \mathbb{N}_\Delta, \gamma \in \mathbb{R}} \left[ \min \{ \gamma; (G^*DG + j\gamma(NG - G^*N) - \gamma^2 D) \leq 0 \} \right]
\] (6.12)

where \( \mathbb{D}_\Delta \) and \( \mathbb{N}_\Delta \) denote sets of positive-definite and hermitian scaling matrices, respectively, which are compatible with the uncertainty structure \( \Delta \).

Now, noting that \( W(z) \) is positive real so that \( W_{\text{Re}} > 0 \) and \( W_{\text{Im}} = W_{\text{Im}}^* \) and comparing the frequency domain inequality (6.8) to the inequality satisfied by the \( D, N \)-scales of real-\( \mu \) theory given by (6.12) it follows that with \( D = W_{\text{Re}}M_2, N = -\gamma^{-1}W_{\text{Im}} \) and \( \gamma I = M_2^{-1} \) that \( \mu(G(e^{i\omega})) \leq M_2^{-1} \). Hence, as in the continuous-time case (Haddad et al. 1992, 1994, How and Hall 1993) direct connections between classical absolute stability theory and mixed-\( \mu \) theory have been established for discrete-time systems.

As noted by Lee and Tits (1991), Haddad et al. (1992, 1994), and How and Hall (1993) in the scalar case the inequality in (6.8) or, equivalently, in (6.12) has a graphical interpretation involving frequency-dependent off-axis circles. Specifically, since \( W_{\text{Re}} > 0 \) and \( W_{\text{Im}} \) is real with \( G(z) = x + iy \), the inequality in (6.8) is equivalent to

\[
x^2 + \left( y + \frac{W_{\text{Im}}(e^{i\omega})}{W_{\text{Re}}(e^{i\omega})M_2} \right)^2 \leq \frac{1}{M_2^2} + \left( \frac{W_{\text{Im}}(e^{i\omega})}{W_{\text{Re}}(e^{i\omega})M_2} \right)^2
\] (6.13)

This inequality corresponds to a circle in the Nyquist plane with a frequency dependent centre located at \( -W_{\text{Im}}(e^{i\omega})/W_{\text{Re}}(e^{i\omega})M_2 \) and constant real axis intercepts at \( \pm M_2^{-1} \). Furthermore, condition (6.13) requires that, at each frequency \( \omega \), the transfer function \( G(e^{i\omega}) \) lies inside the circle centred at \( -W_{\text{Im}}(e^{i\omega})/W_{\text{Re}}(e^{i\omega})M_2 \) with radius

\[
\left[ M_2^{-2} + \left( \frac{W_{\text{Im}}(e^{i\omega})}{W_{\text{Re}}(e^{i\omega})M_2} \right)^2 \right]^{1/2}
\]

Finally, note that for a given choice of \( W_{\text{Re}} \) and \( W_{\text{Im}} \), optimizing over \( M_2 \)
corresponds to minimizing the length of the real axis segment contained within the off-axis circles and hence maximizes the sector (robustness) boundary $M_2$ or, equivalently, provides the tightest real-$\mu$ upper bound since $\mu(G(e^{j\omega})) \leq M_2^{-1}$.

**Remark 6.1:** For odd monotonic nonlinearities, Narendra and Cho (1968) present additional positive real multipliers of the form

$$W_i(z) = (3.1) + \sum_{j=m_{n+1}}^{m_3} c_{ij} \frac{z + \mu_{ij}}{z + v_{ij}} + \sum_{j=m_{n+1}}^{m_4} d_{ij} \frac{z + \kappa_{ij}}{z + \lambda_{ij}} \quad (6.14)$$

where $c_{ij} \geq 0$, $d_{ij} \geq 0$, $1 \geq v_{ij} > \mu_{ij} \geq 0$ and $1 \geq \kappa_{ij} > \lambda_{ij} \geq 0$. This case can be handled exactly as the case considered in this paper. However, the benefit of these additional terms provides additional degrees of freedom in constructing more general parametrizations of the $D$, $N$-scales in mixed-$\mu$ theory. In this case, the extra degrees of freedom allow for more rapid phase variation in the frequency domain test which potentially may give less conservative results.

7. Specialization to linear uncertainty and robust stability and performance via parameter-dependent Lyapunov functions

As discussed in the Introduction, in order to address the constant real parameter uncertainty problem it is crucial to restrict the allowable time variation of the uncertainty. In this section we use the parameter-dependent Lyapunov function framework for robust stability and performance, as developed by Haddad and Bernstein (1991, 1994b, 1994c), to draw connections to the absolute stability criteria for monotonic and odd monotonic nonlinearities developed in the previous sections.

The results of §4 and 5 can be specialized to the case of linear uncertainty guaranteeing robust stability and performance. Results of this type were obtained by Haddad and Bernstein (1991, 1994b, 1994c) for a Popov-type robustness criterion involving uncertainties $\phi(y) = Fy$.

The general framework involves the system

$$x_a(k + 1) = (A_a + \Delta A_a)x_a(k) + Dw(k) \quad (7.1)$$

$$z(k) = Ex_a(k) \quad (7.2)$$

where $x_a(k) \in \mathbb{R}^{n_a}$, $A_a \in \mathbb{R}^{n_a \times n_a}$ denotes the nominal dynamic matrix, $\Delta A_a$ denotes a perturbation of $A_a$ belonging to a given set $\mathcal{U}$, $Dw(k)$ is a white noise signal with intensity $V \triangleq DD^T$ and $z(k) \in \mathbb{R}^{d}$ is a vector of outputs. Note that in order to account for the extra dynamics introduced by the frequency domain multiplier, the resulting state-space model is of increased dimension. Furthermore, note that since $A_a$ is lower block triangular, it follows that if $A_a + \Delta A_a$ is asymptotically stable, then $A + \Delta A$ is asymptotically stable for all perturbations $\Delta A$. The system given by (7.1) and (7.2) could, for example, represent a control system in the closed-loop configuration. For this system the worst-case $H_2$ performance is given by

$$J(\mathcal{U}) \triangleq \sup_{\Delta A_a \in \mathcal{U}} \lim_{k \to \infty} \sup_{z(k)} \mathbb{E}[z^T(k)z(k)] \quad (7.3)$$

By standard results, $J(\mathcal{U})$ is given by

$$J(\mathcal{U}) = \sup_{\Delta A_a \in \mathcal{U}} \text{tr} D^T P_{A_a} D \quad (7.4)$$
where $P_{\Delta A_a}$ satisfies
\[ P_{\Delta A_a} = (A_a + \Delta A_a)^T P_{\Delta A_a} (A_a + \Delta A_a) + E^T E \] (7.5)

Now define $\mathcal{U}$ by
\[ \mathcal{U} \triangleq \{ \Delta A_a \in \mathbb{R}^{n_a \times n_a}; \Delta A_a = -B_a F (I + MF)^{-1} C_a, F \in \mathcal{F} \} \] (7.6)

where $\mathcal{F}$ is defined by
\[ \mathcal{F} \triangleq \{ F \in \mathbb{R}^{m \times m}; \; F \succeq 0 \} \] (7.7)

and where $B_a \in \mathbb{R}^{n_a \times m}$ and $C_a \in \mathbb{R}^{m \times n_a}$ are fixed matrices denoting the structure of the uncertainty, $M \in \mathbb{R}^{m \times m}$ is a given diagonal positive-definite matrix and $F \in \mathbb{R}^{m \times m}$ is a diagonal uncertain matrix.

Note that it follows from Proposition 2 and Lemma 6 of Haddad et al. (1992, 1994) that an equivalent representation for the uncertainty set $\mathcal{U}$ is
\[ \mathcal{U} \triangleq \{ \Delta A_a \in \mathbb{R}^{n_a \times n_a}; \Delta A_a = -B_a \hat{F} C_a, \hat{F} \in \mathcal{F} \} \] (7.8)

where $\mathcal{F}$ is defined by
\[ \mathcal{F} \triangleq \{ F \in \mathbb{R}^{m \times m}; \; 0 < \hat{F} < M \} \] (7.9)

It now follows from Theorem 5.1 by setting $\tilde{\phi}(\tilde{y}) = F \tilde{y} = F (I + M^{-1} F)^{-1} C_a x_a$ that if $\mathcal{S}(z)$ is positive real then $A_a + \Delta A_a$ is asymptotically stable for all $\Delta A_a \in \mathcal{U}$. In this case, the Lyapunov function that establishes robust stability is of the form
\[ V(x_a) = x_a^T P x_a + 2 \sum_{j=1}^{m_2} x_a^T R_j^T (I - \hat{F} M^{-1})^{-1} \hat{F} N_j R_j x_a \] (7.10)

In the terminology of Haddad and Bernstein (1991, 1994 b, 1994 c) this is a parameter-dependent Lyapunov function. Finally, it can be shown that the worst-case $H_2$ performance is bounded by
\[ J(\mathcal{U}) \leq \text{tr} PV + \sup_{\hat{F} \in \mathcal{F}} \sum_{j=1}^{m_2} R_j^T (I - \hat{F} M^{-1})^{-1} \hat{F} N_j R_j V \] (7.11)

\[ = \text{tr} PV \] (7.12)

since the second term in (7.11) vanishes due to the fact that $R_j V = 0, j = 1, \ldots, m_2$.

Now, using the synthesis framework developed by Haddad and Bernstein (1991, 1994 c) the above results can be used to synthesize robust feedback controllers for full- and reduced-order dynamic compensation. As in Haddad et al. (1992, 1994), to reduce conservatism further, one can view the matrices $H_0$, $M_j$ and $N_j$, $j = 1, \ldots, m_2$, as free parameters and optimize the worst case $H_2$ performance bound (7.12) with respect to $H_0$, $M_j$, and $N_j$. Since the structure of the dynamic stability multiplier is a priori fixed, within the controller synthesis framework, this approach eliminates the need for iterating between controller design and optimal multiplier evaluation and curve fitting procedures.

8. Conclusion

In this paper we make explicit connections between discrete-time classical absolute stability theory and modern discrete-time mixed-$\mu$ analysis. Specifically,
multivariable generalizations of discrete-time systems with uncertain monotonic
and odd monotonic locally slope-restricted feedback nonlinearities are used to
provide a tight approximation for the robust stability and performance problem
with constant real parameter uncertainty. The overall framework extends the
discrete-time parameter-dependent Lyapunov function framework developed by
Haddad and Bernstein (1994 b) for robust stability and performance analysis for
real parameter uncertainty, while simultaneously blending $H_2/\mu$ theory.

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