Structured Matrix Norms for Real and Complex Block-structured Uncertainty*

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Abstract—New upper bounds for robust stability are developed for block-structured real and complex uncertainty involving arbitrary spatial norms. Specifically, norm-bounded, block-structured uncertainty is considered wherein the defining norm is not necessarily the maximum singular value. In the case where the defining norm on the uncertainty characterization is set equal to the spectral norm, the resulting upper bounds generalize upper bounds for mixed-μ by permitting the treatment of non-diagonal real uncertain blocks as well as accounting for internal matrix structure in the uncertainty. The overall framework provides for a considerably simplified proof of the standard mixed-μ bound.

1. Introduction
Chellaboina et al. (1995) have developed new lower and upper bounds for robust stability for block-structured uncertainty involving arbitrary spatial norms. In particular, they considered norm-bounded, block-structured uncertainty wherein the defining norm is not necessarily the maximum singular value. This new uncertainty characterization led to the notion of structured matrix norms for characterizing the allowable size of the nominal system transfer function for robust stability. In the case where the defining norm on the block-structured system uncertainty was set equal to the spectral (maximum singular value) norm, the resulting bounds generalize prior upper bounds for robust stability. In the case where the defining norm on the block-structured system uncertainty is set equal to the spectral norm, the resulting bounds generalize prior upper bounds for mixed-μ (Fan et al., 1991) by permitting the treatment of fully populated real uncertain blocks, which may, in addition, possess internal structure. The ability to address real uncertain blocks is based upon the use of an appropriate class of frequency-dependent scaling functions N that explicitly account for real uncertainty blocks possessing an internal structure. Upper bounds for structured matrix norms are developed wherein the defining norm on the block-structured uncertainty is an arbitrary induced spatial norm. In the case where the defining norm on the block-structured system uncertainty is set equal to the spectral norm, the resulting bounds generalize prior upper bounds for mixed-μ (Fan et al., 1991), the framework presented in this paper provides for a considerably simplified proof of the standard mixed-μ bound.

The following standard notation and definitions are used throughout the paper. Let R and C denote the real and complex numbers and let R^n and C^n denote the real and complex n × n matrices. Let A^T denote the complex-conjugate transpose of A and let det A denote the determinant of a square matrix A. The maximum singular value of an arbitrary matrix A is denoted by σmax(A). The maximum eigenvalue of a Hermitian matrix A is denoted by λmax(A). The Euclidean norm of a vector x ∈ C^n is denoted by ∥x∥_2 = (x^T x)^{1/2} and the Kronecker product of the matrices A and B. Finally, let ∥·∥ and ∥·∥_p denote vector norms on C^n and C^n respectively. Then ∥·∥:C^n→R is defined by ∥A∥ = max_{∥x∥=1} ∥Ax∥ if the matrix norm induced by ∥·∥_1 and ∥·∥_p then ∥·∥ is an equi-induced norm.

2. Necessary and sufficient conditions for robust stability
In this section, we give a generalization of the structured singular value (Fan et al., 1991; Packard and Doyle, 1993) and provide necessary and sufficient conditions for robust stability. First, we consider a nominal square transfer function G(s) ∈ C^m×m in a negative-feedback interconnection with an uncertain complex square matrix Δ ∈ C^m×m. The matrix Δ belongs to the set Δ ∈ C^m×m of block-diagonal uncertain matrices defined by

Δ = \{Δ ∈ C^m×m: Δ = \text{block diag} (\Delta_1, \Delta_2, \ldots, \Delta_r)

l_{i,j} \otimes \Delta_{i,j}, \Delta_i \in C^{m_i×m_i}, i = 1, \ldots, r,

Δ = C^{m×m}, r = r + c \geq 1.

where the dimension m, and the number of repetitions l, of each block are given and r + c ≥ 1. Now let ∥·∥ denote a...
Suppose that $G(s)$ is asymptotically stable. Then the negative induced matrix norms.

where $y > 0$. Henceforth, throughout the paper, the notation $\| \cdot \|$ denotes the matrix norm appearing in the definitions of $v(G(jw))$ and $\Delta_v$. Although $\| \cdot \|$ can be any valid matrix norm (Chellaboina et al., 1995), we restrict ourselves here to induced matrix norms.

**Theorem 2.1.** (Chellaboina et al. (1995).) Let $\gamma > 0$ and suppose that $G(s)$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and $\Delta$ is asymptotically stable for all $A \in A$, if and only if $v(G(jw)) < \gamma$ for all $\omega \in \mathbb{R}$.

The following lemma provides an ordering between different structured matrix norms.

**Lemma 2.1** (Chellaboina et al. (1995).) Let $\omega \in \mathbb{R}$ and let $\| \cdot \|$ and $\| \cdot \|$ denote induced matrix norms on $\mathbb{C}^{m \times m}$. Assume that there exists $\Delta \in \Delta$ such that $\det |I + G(jw)\Delta| = 0$ and let $k_1, k_2 > 0$ satisfy

$$k_1 \| \Delta \| = \| \Delta \| \leq k_2 \| \Delta \|$$

for all $\Delta \in \Delta$ such that $\det |I + G(jw)\Delta| = 0$. Furthermore, let $v'(G(jw))$ and $v''(G(jw))$ denote the structured matrix norms with defining norms $\| \cdot \|$ and $\| \cdot \|$ respectively. Then

$$k_1 v'(G(jo)) \leq v'(G(jo)) \leq k_2 v'(G(jo))$$

**Remark 2.1.** Lemma 2.1 can be used to construct upper bounds for structured matrix norms in terms of alternative structured matrix norms.

It is important to note that one cannot obtain the results of Theorem 2.1 from the standard small-$\mu$ theorem by using the equivalence of matrix norms. Specifically, using the necessary and sufficient conditions of the standard small-$\mu$ theorem for robust stability for block-structured uncertainty characterized by spectral norm bounds, we can arrive at sufficient but not necessary conditions for robust stability. For further details, see Chellaboina et al. (1995).

### 3. Upper bounds for the structured matrix norm

In this section, we provide upper bounds for the structured matrix norm. As in standard mixed-$\mu$ theory, in order to account for the structure of the elements of $\Delta$ involving real and complex multiple-block structured uncertainty, we introduce frequency-dependent scaling matrix functions $\varphi$ and $N$ defined by

$$\varphi \triangleq \{D : \mathbb{R} \rightarrow \mathbb{C}^{m \times m} : D(jw) = D^*(jw)\},$$

$$\det D(jw) \neq 0, D(jw) = D(jw), \Delta \in \Delta, \omega \in \mathbb{R}\}$$

and $N \triangleq \{N : \mathbb{R} \rightarrow \mathbb{C}^{m \times m} : N(jw) = N^*(jw)\}$,

$$\det N(jw) = D^*(jw), \Delta \in \Delta, \omega \in \mathbb{R}\}$$

Although the condition $D(jw) = D^*(jw)$ is required for stability, in general, this condition is not satisfied by all elements of $\Delta$. Therefore, we define frequency-dependent scaling matrix functions $\varphi$ and $N$.

Next, let $\| \cdot \|$ be a matrix norm on $\mathbb{C}^{m \times m}$ induced by the vector norms $\| \cdot \|$ and $\| \cdot \|$ on $\mathbb{C}^m$. For example, let $\| \cdot \|$ denote the matrix norm on $\mathbb{C}^{m \times m}$ induced by vector norms $\| \cdot \|$ and $\| \cdot \|$ on $\mathbb{C}^m$, and define $\varphi \subset \mathbb{C}^m$ and $\varphi(G(jw))$ by

$$\varphi \triangleq \{x \in \mathbb{C}^m : x^*G(jw)N(jw) = N(jw)G(jw)x = 0, x \neq 0, N(jw) \in N, \omega \in \mathbb{R}\}$$

**Theorem 3.1.** Let $\omega \in \mathbb{R}$. Then $v(G(jw)) \leq \eta(G(jw))$.

**Proof.** Suppose that $\eta(G(jw)) < \gamma$, and let $y > 0$ be such that $\eta(G(jw)) < \gamma < v(G(jw))$. Then it follows from Lemma 3.1 that $\det |I + G(jw)\Delta| = 0$ for all $\omega \in \mathbb{R}$.

Next, we present the main result of this paper, which provides an upper bound for the structured matrix norm.

**Theorem 3.2.** Let $\omega \in \mathbb{R}$. Then $v(G(jw)) \leq \eta(G(jw))$.

**Proof.** Suppose that $\eta(G(jw)) < \gamma$, and let $y > 0$ be such that $\eta(G(jw)) < \gamma < v(G(jw))$. Then it follows from Lemma 3.1 that $\det |I + G(jw)\Delta| = 0$ for all $\omega \in \mathbb{R}$.

Theorem 3.1 provides an upper bound for the structured matrix norm. However, since $\eta(G(jw))$ given by (7) defines a non-convex optimization problem, it is possible that the resulting maximum may correspond to a local maximum, which would not necessarily provide an upper bound for the structured matrix norm. Hence we present the following theorem, which provides an upper bound for the structured matrix norm in terms of a quasi-convex optimization problem involving a linear matrix inequality (LMI). For notational convenience, we write $D$ and $N$ for $D(j\omega)$ and $N(j\omega)$ respectively. Furthermore, we define

$$\eta(G(jw)) \triangleq \max \left\{ 0, \inf_{N(jw)} \lambda_{\max}(G(jw)N(jw)) \right\}$$

and let $c', c'' > 0$ be such that

$$c' \|x\| \leq \|x\| \leq c'' \|x\|$$

for all $x \in \mathbb{R}$ if $X \neq \emptyset$ and arbitrary otherwise.

**Theorem 3.3.** Let $\omega \in \mathbb{R}$. Then $v(G(jw)) \leq \eta(G(jw))$.
and hence

\[ c^2 \left[ \frac{\| G(j\omega) x \|_2}{\| x \|_2} \right] \leq c^2 \left[ \frac{\| G(j\omega) x \|}{\| x \|} \right] \]

\[ = c^2 \lambda^\star = \left\{ G^*(j\omega)G(j\omega) + j(NG(j\omega) - G^*(j\omega)N)x \right\} \]

\[ = c^2 \lambda_{\text{max}} = \left\{ G^*(j\omega)G(j\omega) + j(NG(j\omega) - G^*(j\omega)N) \right\} \]

for all \( x \in \mathbb{R} \) and \( N(\cdot) \in \mathcal{N} \). Since the above inequality holds for all \( N(\cdot) \in \mathcal{N} \), it follows that \( \eta(G(j\omega)) \leq (c^2/c^\star)(G(j\omega)) \).

**Corollary 3.1.** Let \( \omega \in \mathbb{R} \). Then

\[ v(G(j\omega)) \leq \frac{1}{c^\star} \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \{ D(j\omega)G(j\omega)D^{-1}(j\omega) \} \leq \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \{ (c^2/c^\star)(G(j\omega)) \} = (11) \]

**Proof:** The result is a direct consequence of Theorem 3.2 by noting that \( v(G(j\omega)) = v(D(j\omega)G(j\omega)D^{-1}(j\omega)) \) for all \( D(\cdot) \in \mathcal{D} \).

**Remark 3.1.** In order to obtain the tightest upper bound for the structured matrix norm \( v(G(j\omega)) \), it follows from Theorem 3.3 and Corollary 3.1 that the largest \( c^\star \) need to be computed such that (9) is satisfied for all \( x \in \mathcal{X} \). However, since in general it is difficult to characterize \( \mathcal{X} \), we can compute the largest \( c^\star \) and smallest \( c^\star \) such that (9) holds for all \( x \in \mathcal{X} \). See, for example, Stone (1962) for further details.

Note that \( \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \{ (DG(j\omega))D^{-1}(j\omega) \} \) in (11) can equivalently be characterized by

\[ \inf\{ a > 0 : \text{there exist } D(\cdot) \in \mathcal{D} \text{ and } N(\cdot) \in \mathcal{N} \text{ such that} \]

\[ G^*(j\omega)DG(j\omega) + j(NG(j\omega) - G^*(j\omega)N) - a^2 D < 0 \],

which can be computed using standard LMI techniques. Furthermore, in the case where \( \| \cdot \| = \sigma_{\text{max}}(\cdot) \) and \( c^\star = c^\star = 1 \), (11) specializes to

\[ v(G(j\omega)) \leq \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \left\{ \frac{\zeta(DG(j\omega))D^{-1}}{\| \cdot \|} \right\} \]

\[ = \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \left\{ \alpha > 0 : G^*(j\omega)DG(j\omega) + j(NG(j\omega) - G^*(j\omega)N) - \alpha^2 D < 0 \right\} \leq \inf_{D(\cdot) \in \mathcal{D}, N(\cdot) \in \mathcal{N}} \sigma_{\text{max}}(DG(j\omega)) \]

yielding the standard mixed-\( \mu \) and complex-\( \mu \) bounds with the additional refinement for capturing internal matrix structure.

4. **Conclusions**

This paper has developed new robust stability bounds for systems with block-structured real and complex uncertainty involving arbitrary spatial norms. In particular, we have considered a norm-bounded, block-structured uncertainty characterization wherein the defining norm is not the maximum singular value. In the case where the defining norm is set equal to the maximum singular value, the resulting stability bounds generalize upper bounds to mixed-\( \mu \) by permitting the treatment of nondiagonal real uncertain blocks as well as accounting for internal matrix structure in the uncertainty.

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**References**


