Memoryless $H_{\infty}$ Controllers for Discrete-Time Systems with Time Delay*

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Key Words—Time delay; discrete-time control; $H_{\infty}$ disturbance rejection.

Abstract—This paper considers the problem of $H_{\infty}$ stabilization of discrete-time systems with state delays and exogenous bounded-energy $l_2$ disturbances. Specifically, an $H_{\infty}$ state feedback control design problem for discrete-time systems with time delay is addressed. The principal result involves sufficient conditions in terms of a modified Riccati equation for characterizing state feedback controllers that enforce a bound on $H_{\infty}$ performance and guarantee closed-loop stability in the face of system state delay. © 1998 Elsevier Science Ltd. All rights reserved.

Nomenclature

- $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^+$ real numbers, $r \times s$ real matrices, $\mathbb{R}^{+}\times$
- $I_r$ 0, 1, 2, ...
- $I_r^{\bullet}$ 1, 2, 3, ...
- $(\cdot)^T$, $(\cdot)^{-1}$ transpose, inverse, complex conjugate, complex conjugate transpose
- $I_r, 0$, $r \times r$ identity matrix, $r \times r$ zero matrix
- $\sigma_{\text{max}}(X)$ largest singular value of matrix $X$
- $\gamma$ real positive scalar
- $n, m, p, a, d$ positive integers
- $x, u, z$, $n - m - p, d$-dimensional vectors
- $A, B$, $n \times n, n \times m$ matrices
- $A_d$ $n \times n$ matrix
- $K$ $m \times n$ matrix
- $E_1, E_2$, $p \times n, p \times m$ matrices; $E_1^T E_2 = 0$
- $D_K$ $n \times d_K$ matrix
- $w_c$ $d_K \times d_K$ dimensional $l_2$ signals

1. Introduction

Since feedback control system delays arise frequently in practice they can severely degrade closed-loop system performance and in some cases drive the system to instability. In continuous-time systems the presence of system time delay is characterized via mathematical system models governed by delay differential equations (Hale, 1977; Feliachi and Thowsen, 1981). Alternatively, in discrete-time systems the presence of system time delay is characterized via mathematical system models governed by delay difference equations (Verriest and Ivanov, 1995). Since most physical processes evolve naturally in continuous time, it is not surprising that the bulk of stability and control theory for systems with time delay has been developed for continuous-time systems (see, for example, Feliachi and Thowsen, 1981; Mori et al., 1983; Phoojaruenchanchai and Furuta, 1992; Chen and Latchman, 1995; Chen et al., 1995; Haddad et al., 1997; Gu, 1997, and the references therein). Nevertheless, it is the overwhelming trend to implement controllers digitally. For discrete-time systems with time delay characterized via delay difference equations, the system augmentation approach proposed by Åström and Wittenmark (1984) has been extensively used in practice. Specifically, Åström and Wittenmark (1984) propose an approach that converts the delay difference system to a higher-order delay-free system. However, for systems with large known delay amounts this scheme will invariably lead to large-dimensional systems. Furthermore, for systems with unknown delay amounts the augmentation scheme proposed by Åström and Wittenmark (1984) is not applicable. Alternatively, Verriest and Ivanov (1995) develop a Riccati equation framework for analyzing the stability of discrete-time systems with arbitrary unknown delay.

2. $H_{\infty}$ controller design for discrete-time systems with time delay

In this section we introduce the state feedback control problem for discrete-time systems with state delay and an $H_{\infty}$ disturbance attenuation constraint. The goal of this problem is to determine a memoryless state feedback controller which (i) stabilizes the discrete-time system with state delays and (ii) satisfies an $H_{\infty}$ disturbance attenuation constraint. $H_{\infty}$ Constrained state feedback problem. Given the nth-order dynamical system with state delays

\[ x(k + 1) = Ax(k) + A_d x(k - n_d) + Bu(k) + D_K w_c(k), \quad k \in \mathcal{N}, \quad n_d \in \mathcal{N}^+, \quad (2.1) \]

where $u(k) \in \mathbb{R}^n$ and $w_c(k) \in \mathbb{R}^n$ is an $l_2$ sequence each of whose components has norm less than one, determine a state feedback controller

\[ u(k) = K x(k), \quad (2.2) \]

which satisfies the following design criteria:

(i) the closed-loop system (2.1), (2.2) is asymptotically stable, and

(ii) the $p_d \times d_K$-closed-loop transfer function from disturbances $w_c(k), k \in \mathcal{N}$, to performance variables $z_c(k) = E_1 x(k) + E_2 u(k)$ given by

\[ H(z) = \bar{E}_d \left[ z I_d - \bar{A} - z^{-n_d} A_d \right]^{-1} D_K, \quad n_d \in \mathcal{N}^+, \quad (2.3) \]

where

\[ \bar{A} = A + BK, \quad \bar{E}_d = E_1 + E_2 K, \]

satisfies the constraint

\[ \| H(z) \|_\infty \leq \gamma, \quad (2.4) \]
where
\[ |H(z)| = \sup_{\|x\|_2} \frac{|x|_2}{|w|_2} = \sup_{w \in N, |w|_2 = 1} \sigma_{\max}[H(w^H)], \tag{2.5} \]
and \( \gamma > 0 \) is a given constant.

3. Sufficient conditions for \( H_\infty \) stabilization of discrete-time systems with time delay

In this section we develop a Riccati equation framework that guarantees that the closed-loop system (2.1), (2.2) consisting of the \( n \)-th order time-delayed system (2.1) and the state feedback controller (2.2) is asymptotically stable and satisfies the disturbance attenuation constraint (2.4). For the statement of this result define \( \mathcal{E}_k, \mathcal{E}_k, \) assume \( \mathcal{E}_k, \mathcal{E}_k > 0 \), and note that for a given controller gain \( K \), the closed-loop system (2.1), (2.2) can be written as
\[ x(k+1) = A \dot{x}(k) + B_k A x(k-n_d) + D_k w(k), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.1} \]

**Theorem 3.1.** Let \( K \) be given and let \( E_k \) be full column rank. Suppose there exists an \( n \times n \) positive-definite matrix \( P \) and a scalar \( \gamma > 0 \) such that
\[ M \sqsubseteq Q - A P A^T > 0, \tag{3.2} \]
\[ N \sqsubseteq \gamma E_k - D_k^T D_k P > 0, \tag{3.3} \]
and
\[ P = \lambda^T P + Q + \lambda^T A P A^T - A^T P A + A^T P A A^T P + \lambda^T D_k P N^T D_k^T P + E_k, \tag{3.4} \]
where \( Q \) is an \( n \times n \) nonnegative-definite matrix. Then the function
\[ V(x) = x^T P x + \sum_{l=-k}^{-1} x^T(l) Q x(l) \]
is a Lyapunov function that guarantees that the closed-loop system (3.1) with \( w(k) = 0 \) is globally asymptotically stable. Furthermore, the closed-loop transfer function \( H(z) \) satisfies the \( H_\infty \) disturbance attenuation constraint
\[ |H(z)| \leq \gamma. \tag{3.6} \]

**Proof.** First note that since \( P \) is positive definite and \( Q \) is nonnegative definite it follows that the Lyapunov function candidate \( V(x) \) given by equation (3.5) is positive definite. The corresponding Lyapunov difference along the trajectories \( x(k) \), \( k \in \mathbb{N} \), of the closed-loop system (3.1) is given by
\[ \Delta V(x(k)) = x^T(k + 1) P x(k + 1) - x^T(k) P x(k) \
+ \sum_{k=-1}^{-k} x^T(l) Q x(l) - x^T(k) P x(k) \]
\[ - \sum_{k=-1}^{-k} x^T(l) Q x(l), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.7} \]
or, equivalently,
\[ \Delta V(x(k)) = x^T(k + 1) P x(k + 1) - x^T(k) P x(k) \]
\[ - x^T(k - n_d) Q x(k - n_d), k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.8} \]

Now, using Equation (3.1) with \( w(k) = 0, k \in \mathbb{N} \), Equation (3.8) becomes
\[ \Delta V(x(k)) = x^T(k) [A^T P A - P + Q] x(k) \]
\[ + 2 x(k) A^T P A x(k - n_d) \]
\[ + x^T(k - n_d) A^T P A x(k - n_d) \]
\[ - x^T(k - n_d) Q x(k - n_d), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.9} \]
or, equivalently,
\[ \Delta V(x(k)) = x^T(k) [A^T P A - P + Q] x(k) \]
\[ + 2 x(k) A^T P A x(k - n_d) \]
\[ - x^T(k - n_d) M x(k - n_d), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.10} \]

Next, adding and subtracting \( x^T(k) A^T P A M^{-1} A^T P A x(k) \) to and from equation (3.10) and collecting terms yields
\[ \Delta V(x(k)) = x^T(k) [A^T P A - P + Q + A^T P A M^{-1} A^T P A] x(k) \]
\[ - (M^{-1/2} A P A x(k)) - M^{-1/2} x(k - n_d) \]
\[ \times (M^{-1/2} A^T P A x(k) - M^{-1/2} x(k - n_d)), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.11} \]
Finally, using Equation (3.4) we obtain
\[ \Delta V(x(k)) \leq - x^T(k) R x(k) - M^{-1/2} A P A x(k) \]
\[ - M^{-1/2} x(k - n_d) \]
\[ \times (M^{-1/2} A^T P A x(k) - M^{-1/2} x(k - n_d)), \quad k \in \mathbb{N}, n_d \in \mathbb{N}^+. \tag{3.12} \]
where \( R = \mathcal{E}_k + P A M^{-1} A^T P + \lambda^T D_k P N^{-1} D_k^T P A \). Since \( R \) is positive definite it follows that \( \Delta V(x(k)) < 0, \forall x(k) \neq 0, k \in \mathbb{N} \), and hence \( V(x) \) is a Lyapunov function for the closed-loop system (3.1). Next, to prove equation (3.6), replace \( \mathcal{E}_k \) in Equation (3.4) by \( \mathcal{E}_k E_k \) and rewrite equation (3.4) as
\[ \mathcal{E}_k E_k = - \mathcal{T} P A + e^{0} P A e^{-0} - Q - \mathcal{T} P A M^{-1} A^T P A \]
\[ - P A M^{-1} A^T P - \mathcal{T} P D_k N^{-1} D_k^T P A, \tag{3.13} \]
where \( e^{0} \) is arbitrary. Furthermore, define \( z \sqsubseteq e^{0} \) and add and subtract
\[ \mathcal{T} P A, \quad z \sqsubseteq \mathcal{T} P A \]
\[ z = e^{0} P A e^{-0}, \quad z = e^{0} P A e^{-0}, \]
to and from Equation (3.13) so that Equation (3.13) becomes
\[ \mathcal{E}_k E_k = [\mathcal{T} - \mathcal{Z} - e^{0} P A] e^{-0} - e^{0} P A e^{-0}, \quad z = e^{0} P A e^{-0}, \]
\[ - Q - \mathcal{T} P A M^{-1} A^T P A - P A M^{-1} A P \]
\[ - \mathcal{T} P D_k N^{-1} D_k^T P A - \mathcal{T} P A + e^{0} P A e^{-0} + \mathcal{T} P A e^{-0} - z = e^{0} P A e^{-0}, \tag{3.14} \]
Next, using the notation \( L(z) \equiv e^{0} P A e^{-0} \) and rearranging terms in Equation (3.14) yields
\[ \mathcal{E}_k E_k = L(z^*) P L(z) + L(z^*) P A + A^T P L(z) \]
\[ - (\mathcal{T} P D_k N^{-1} D_k^T P A - \mathcal{T} P A e^{-0} - \mathcal{T} P A e^{-0} - z = e^{0} P A e^{-0}, \]
\[ \mathcal{E}_k E_k = L(z^*) P L(z) + L(z^*) P A + A^T P L(z) \]
\[ - (\mathcal{T} P D_k N^{-1} D_k^T P A - \mathcal{T} P A e^{-0} - \mathcal{T} P A e^{-0} - z = e^{0} P A e^{-0}, \tag{3.15} \]
or, equivalently, using \( Q = M + A P A_d \),
\[ \mathcal{E}_k E_k = L(z^*) P L(z) + L(z^*) P A + A^T P L(z) \]
\[ - (\mathcal{T} P D_k N^{-1} D_k^T P A - \mathcal{T} P A e^{-0} - \mathcal{T} P A e^{-0} - z = e^{0} P A e^{-0}, \tag{3.16} \]
where \( w(z) \equiv (z - M^{-1/2} A P - z = e^{0} M^{-1/2} A P) \) and \( S \equiv 2 A P A + \mathcal{T} P A M^{-1} A^T P A \). Now, forming \( D^{-1/2} \) (equation (3.16)) \( L^{-1/2} \) (equation (3.16)) \( D \) yields
\[ H^*(z) H(z) = D^{-1} P D + D^{-1} P A L(z) D + D^{-1} L^{-1} z^* D \]
\[ - D^{-1} L^{-1} z^* \{ W(z) W(z) + S \} L^{-1} z^* D \tag{3.17} \]
Finally, multiplying Equation (3.17) by \( -1 \), adding \( \gamma^2 I_{\mathcal{N}} \) to both sides of Equation (3.17) yields
\[ \gamma^2 I_{\mathcal{N}} - \gamma^2 I_{\mathcal{N}} = U^*(z) U(z) \]
\[ + D^{-1} L^{-1} z^* \{ W(z) W(z) + S \} L^{-1} z^* D \geq 0, \]
where \( U(z) \equiv N^{-1/2} D^{-1} P A L^{-1} (z) D - N^{-1/2} \) and hence \( H^*(z) \)
\[ \times H(z) \leq \gamma^2 I_{\mathcal{N}}. \] This proves equation (3.6). \( \square \)
4. Discrete-time $H_\infty$ controllers for systems with time delay

In this section we present the main theorem characterizing state feedback controllers for $H_\infty$ stabilization of equations (2.1) and (2.2). For the statement of this result define the notation

$$ R_{1,2} \triangleq E_{12}^T E_{12}, R_{2,2} \triangleq E_{22}^T E_{22} $$

and

$$ P_2 \triangleq B^T P A + B^T P A M^{-1} A_2^T P A + B^T D_2 N^{-1} D_1^T P A, $$

$$ R_{2,2} \triangleq B^T P A + B^T P A M^{-1} A_2^T P A + B^T D_2 N^{-1} D_1^T P A, $$

for arbitrary $P \in \mathbb{R}^{n \times n}$.

**Theorem 4.1.** Let $R_{1,2} \in \mathbb{R}^{n \times n}$ and $R_{2,2} \in \mathbb{R}^{n \times n}$ be positive-definite matrices and let $Q$ be an $n \times n$ nonnegative-definite matrix. Suppose there exists an $n \times n$ positive-definite matrix $P$ and a scalar $\gamma > 0$ satisfying

$$ Q - A_1^T P A_2 > 0, \tag{4.1} $$

$$ \gamma^2 I_{2n} - D_1^T P D_2 > 0, \tag{4.2} $$

and

$$ P = A^T P A + A^T P A M^{-1} A_2^T P A + P A M^{-1} A_2^T P $$

$$ + A^T P D_2 N^{-1} D_1^T P A + R_{1,2} - P_2^T R_{2,2} P, \tag{4.3} $$

and let $K$ be given by

$$ K = - R_{2,2}^{-1} P. \tag{4.4} $$

Then the closed-loop system (3.1) with $w(k) = 0$ is globally asymptotically stable for all $\mu \in \mathbb{R}^+$ and the closed-loop transfer function $H(z)$ satisfies the $H_\infty$ disturbance attenuation constraint (3.6).

**Proof.** With $K$ given by equation (4.4), it follows that equation (4.3) is equivalent to

$$ P = (A + BK)^T P (A + BK) + Q $$

$$ + (A + BK)^T P A M^{-1} A_2^T P (A + BK) $$

$$ + P A M^{-1} A_2^T P + (A + BK)^T P D_2 N^{-1} D_1^T P (A + BK) $$

$$ + R_{1,2} + K^T R_{2,2} K. \tag{4.5} $$

Now, it follows from Theorem 3.1 that the closed-loop system (3.1) with $w(k) = 0$ is globally asymptotically stable and the $H_\infty$ disturbance rejection constraint (2.4) is satisfied. □

**Remark 4.1.** Theorem 4.1 presents sufficient conditions for designing full-state feedback controllers for discrete-time systems with state delays while guaranteeing an $H_\infty$ disturbance attenuation constraint. Using the fixed-structure controller synthesis framework developed by Davis et al. (1991) these results can be readily extended to fixed-order (i.e. full- and reduced-order) dynamic compensation.

**Remark 4.2.** It is interesting to note that the problem considered in this paper can alternatively be handled via the scaled small gain theorem. See for example Chen and Latchman (1995) and Chen et al. (1995). Specifically, the delay-independent $H_\infty$ stabilization problem can be converted into a $\mu$-synthesis problem where an additional bounded real block can be used to capture the $H_\infty$ performance.

5. Illustrative numerical example

In this section we present an illustrative numerical example to demonstrate the proposed delay stabilization approach. The design equation (4.3) was solved using a homotopy continuation algorithm. Specifically, we parameterize equation (4.3) as

$$ P_{i+1} = A^T P_{i+1} A + Q + \gamma A^T P_{i+1} M^{-1} A_2^T P_{i+1} A + P_{i+1} A M^{-1} A_2^T P_{i+1} A $$

$$ + A^T P_{i+1} D_2 N^{-1} D_1^T P_{i+1} A + R_{1,2} - P_2^T R_{2,2} P_{i+1} A, \tag{5.1} $$

where

$$ M \triangleq Q - A_1^T P_{i+1} A, N \triangleq \gamma I_{2n} - D_1^T P_{i+1} D_2, $$

$$ P_{i+1}(z) \triangleq B^T P_{i+1} A + \alpha B^T P_{i+1} A M^{-1} A_2^T P_{i+1} A $$

$$ + B^T P_{i+1} D_2 N^{-1} D_1^T P_{i+1} A, $$

$$ R_{2,2}(z) \triangleq B^T P_{i+1} A + \alpha B^T P_{i+1} A M^{-1} A_2^T P_{i+1} A $$

$$ + B^T P_{i+1} D_2 N^{-1} D_1^T P_{i+1} A, $$

and $\alpha \in [0, 1]$. The algorithm is initialized with $\alpha = 0$ and solves for $P_{i+1}$ with small increments in $\alpha$ until $\alpha = 1$. Next, with $\alpha = 1$, we iteratively solve for $P_{i+2}$.

**Example 5.1.** Consider the dynamical system (2.1) with problem data

$$ A = \begin{bmatrix} 0 & 1 \\ -0.14 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_2 = B, $$

and design data

$$ R_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad R_{2,2} = 1, \quad Q = 0.48 I_2, \quad \gamma = 1.95. $$

Using the homotopy continuation algorithm outlined above a positive-definite solution to equation (4.3) is given by

$$ P = \begin{bmatrix} 1.8296 & -0.1665 \\ -0.1665 & 3.7887 \end{bmatrix} \tag{5.2} $$

and

$$ K = [0.1399 - 0.8552] \tag{5.3} $$

Table 1 provides the $H_\infty$ norm of the closed-loop transfer function $H(z)$ for several values of $\mu_d$. Next, we use an extended version of Theorem 3.1 of Verriest and Ivanov (1995) to design a full-state feedback controller for systems with time delay but without the $H_\infty$ disturbance rejection guarantees. This controller is given by

$$ K = [0.1052 - 0.6489]. \tag{5.4} $$

In this case, for the values of $\mu_d$ listed in Table 1, $\|H(z)\|_\infty = 2.0065$.

6. Conclusion

In this paper we extended the analysis results of Verriest and Ivanov (1995) to address the problem of $H_\infty$ stabilization of discrete-time systems with time delay. Specifically, for systems with state delay we presented constructive sufficient conditions in terms of a solution of a modified Riccati equation for characterizing state feedback $H_\infty$ controllers.

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**References**


