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Generalized Lyapunov and invariant set theorems for nonlinear dynamical systems ☆

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Abstract

In this paper we develop generalized Lyapunov and invariant set theorems for nonlinear dynamical systems wherein all regularity assumptions on the Lyapunov function and the system dynamics are removed. In particular, local and global stability theorems are given using lower semicontinuous Lyapunov functions. Furthermore, generalized invariant set theorems are derived wherein system trajectories converge to a union of largest invariant sets contained in intersections over finite intervals of the closure of generalized Lyapunov level surfaces. The proposed results provide transparent generalizations to standard Lyapunov and invariant set theorems. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the most basic issues in system theory is stability of dynamical systems. The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov [20]. Lyapunov's results, along with the Barbashin–Krasovskii–LaSalle invariance principle [6,15,16], provide a powerful framework for analyzing the stability of nonlinear dynamical systems. In particular, Lyapunov's direct method can provide local and global stability conclusions of an equilibrium point of a nonlinear dynamical system if a smooth (at least C^1) positive-definite function of the nonlinear system states (Lyapunov function) can be constructed for which its time rate of change due to perturbations in a neighborhood of the system's equilibrium is always negative or zero, with strict negative definiteness ensuring asymptotic stability. Alternatively, using the Barbashin–Krasovskii–LaSalle invariance principle [6,15,16] the positive-definite condition on the Lyapunov function as well as the strict negative-definiteness condition on the Lyapunov derivative can be relaxed while assuring asymptotic stability. In particular, if a smooth function defined on a compact invariant set with respect to the nonlinear dynamical system can be constructed whose derivative along the system's trajectories is negative semidefinite and no system

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trajectories can stay indefinitely at points where the function's derivative identically vanishes, then the system's equilibrium is asymptotically stable.

Most Lyapunov stability and invariant set theorems presented in the literature require that the Lyapunov function candidate for a nonlinear dynamical system be a C^1 function with a negative-definite derivative (see [12–14,17,26,28] and the numerous references therein). This is due to the fact that the majority of the dynamical systems considered are systems possessing continuous dynamics and hence Lyapunov theorems provide stability conditions that do not require knowledge of the system trajectories [12–14,17,26,28]. However, in light of the increasingly complex nature of dynamical systems such as biological systems [19], hybrid systems [27], sampled-data systems [11], discrete-event systems [22], gain scheduled systems [18,21,23], and constrained mechanical systems [5], system discontinuities arise naturally. Even though standard Lyapunov theory is applicable for systems with discontinuous system dynamics and continuous motions, it might be simpler to construct discontinuous "Lyapunov" functions to establish system stability. For example, in gain scheduling control it is not uncommon to use several different controllers designed over several fixed operating points covering the system's operating range and to switch between them over this range. Even though for each operating range one can construct a C^1 Lyapunov function, to show closed-loop system stability over the whole system operating envelope for a given switching control strategy, a *generalized* Lyapunov function involving combinations of the Lyapunov functions for each operating range can be constructed [18,21,23]. However, in this case, it can be shown that the generalized Lyapunov function is nonsmooth and noncontinuous [18,21,23].

In this paper we develop generalized Lyapunov and invariant set theorems for nonlinear dynamical systems wherein all regularity assumptions on the Lyapunov function and the system dynamics are removed. In particular, local and global stability theorems are presented using generalized Lyapunov functions that are lower semicontinuous. Furthermore, generalized invariant set theorems are derived wherein system trajectories converge to a union of largest invariant sets contained in intersections over finite intervals of the closure of generalized Lyapunov level surfaces. In the case where the generalized Lyapunov function is taken to be a C^1 function, our results collapse to the standard Lyapunov stability and invariant set theorems. Finally, we note that nondifferentiable Lyapunov functions have been considered in the literature. Specifically, continuous and lower semicontinuous Lyapunov functions have been considered in [1–4,7,27], with [2–4] focusing on viability theory and differential inclusions. Furthermore, significant extensions of LaSalle's invariance principle for continuous Lyapunov functions are developed in [8,25]. However, the present formulation provides *new* invariant set stability theorem generalizations not considered in [3,8,25] by explicitly characterizing system limit sets in terms of lower semicontinuous Lyapunov functions.

2. Mathematical preliminaries

In this section we establish definitions, notation, and a key result used later in the paper. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors. Furthermore, let $\partial\mathscr{S}$, $\overset{\circ}{\mathscr{S}}$, and $\overline{\mathscr{S}}$ denote the boundary, the interior, and the closure of the set $\mathscr{S} \subset \mathbb{R}^n$, respectively. Let $\|\cdot\|$ denote the Euclidean vector norm, let $\mathscr{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, denote the open ball centered at α with radius ε, let $V'(x)$ denote the gradient of V at x, and let $D^+V(x)$ denote the lower Dini derivative of V at x [24]. Finally, let C^0 denote the set of continuous functions and C^r denote the set of functions with r-continuous derivatives.

In this paper we consider the general nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geqslant 0, \tag{1}$$

where $x(t) \in \mathscr{D} \subseteq \mathbb{R}^n$ is the system state vector, $\mathscr{D}$ is an open set, $0 \in \mathscr{D}$, $f : \mathscr{D} \to \mathbb{R}^n$, and $f(0) = 0$. A function $x : \mathscr{I} \to \mathscr{D}$ is said to be a solution to (1) on the interval $\mathscr{I} \subseteq \mathbb{R}$ if $x(t)$ satisfies (1) for all $t \in \mathscr{I}$. Note that we do not assume any regularity conditions on the function $f(\cdot)$. However, we do assume that $f(\cdot)$ is such that the solution $x(t)$, $t \geqslant 0$, to (1) is well defined on the time interval $\mathscr{I} = [0, \infty)$. That is, we assume that for every $y \in \mathscr{D}$ there exists a unique solution $x(\cdot)$ of (1) defined on $[0, \infty)$ satisfying $x(0) = y$. Furthermore, we assume that all the solutions $x(t)$, $t \geqslant 0$, to (1) are continuous functions of the initial conditions $x_0 \in \mathscr{D}$. The following definition introduces three types of stability and attractivity corresponding to the zero solution $x(t) \equiv 0$ of (1).

Definition 2.1. The zero solution $x(t) \equiv 0$ to (1) is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \geqslant 0$. The zero solution $x(t) \equiv 0$ to (1) is *attractive* if there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\to\infty} x(t) = 0$. The zero solution $x(t) \equiv 0$ to (1) is *asymptotically stable if* it is Lyapunov stable and attractive. The zero solution $x(t) \equiv 0$ to (1) is *globally asymptotically stable* if it is Lyapunov stable and for all $x(0) \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t) = 0$.

Definition 2.2. A set $\mathscr{M} \subset \mathscr{D} \subseteq \mathbb{R}^n$ is an *invariant set* for the nonlinear dynamical system (1) if $x(0) \in \mathscr{M}$ implies that $x(t) \in \mathscr{M}$ for all $t \geqslant 0$.

Definition 2.3. $p \in \bar{\mathscr{D}} \subseteq \mathbb{R}^n$ is a *positive limit point* of the solution $x(t)$, $t \geqslant 0$, of (1) if there exists a sequence $\{t_n\}_{n=0}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$, $t \geqslant 0$, is the *omega limit set* $\omega(x_0)$ of $x(t)$, $t \geqslant 0$.

The following result on omega limit sets is fundamental and forms the basis for all later developments.

Lemma 2.1 (Khalil [14]). *Suppose the solution $x(t)$, $t \geqslant 0$, to (1) corresponding to an initial condition $x(0) = x_0$ is bounded. Then the omega limit set $\omega(x_0)$ of $x(t)$, $t \geqslant 0$, is a nonempty, compact, connected invariant set. Furthermore, $x(t) \to \omega(x_0)$ as $t \to \infty$.*

Remark 2.1. It is important to note that Lemma 2.1 holds for time-invariant nonlinear dynamical systems (1) possessing unique solutions forward in time with the solutions being continuous functions of the initial conditions. More generally, letting $s(\cdot, x_0)$ denote the solution of a dynamical system with initial condition $x(0) = x_0$, Lemma 2.1 holds if $s(t+\tau, x_0) = s(t, s(\tau, x_0))$, $t, \tau \geqslant 0$, and $s(\cdot, x_0)$ is a continuous function of $x_0 \in \mathscr{D}$.

Remark 2.2. If $f(\cdot)$ is Lipschitz continuous on $\mathscr{D}$ then there exists a unique solution to (1) and hence the required semi-group property $s(t+\tau, x_0) = s(t, s(\tau, x_0))$, $t, \tau \geqslant 0$, and the continuity of $s(t, \cdot)$ on $\mathscr{D}$, $t \geqslant 0$, hold. Alternatively, uniqueness of solutions in forward time along with the continuity of $f(\cdot)$ ensure that the solutions to (1) satisfy the semi-group property and are continuous functions of the initial conditions $x_0 \in \mathscr{D}$ even when $f(\cdot)$ is not Lipschitz continuous on $\mathscr{D}$ (see [9, Theorem 4.3, p. 59]). More generally, $f(\cdot)$ need not be continuous. In particular, if $f(\cdot)$ is discontinuous but bounded and $x(\cdot)$ is the unique solution to (1) in the sense of Filippov [10], then the required semi-group property along with the continuous dependence of solutions on initial conditions hold [10].

Finally, the following definitions are used in the paper.

Definition 2.4. A function $V : \mathscr{D} \to \mathbb{R}$ is *lower semicontinuous* on $\mathscr{D}$ if for every sequence $\{x_n\}_{n=0}^{\infty} \subset \mathscr{D}$ such that $\lim_{n\to\infty} x_n = x$, $V(x) \leqslant \liminf_{n\to\infty} V(x_n)$.

Definition 2.5. Let $\mathscr{Q} \subseteq \mathscr{D}$ and let $V : \mathscr{Q} \to \mathbb{R}$. For $\alpha \in \mathbb{R}$, the set $V^{-1}(\alpha) \triangleq \{x \in \mathscr{Q}: V(x) = \alpha\}$ is called the *α-level set*. For $\alpha, \beta \in \mathbb{R}$, $\alpha \leqslant \beta$, the set $V^{-1}([\alpha, \beta]) \triangleq \{x \in \mathscr{Q}: \alpha \leqslant V(x) \leqslant \beta\}$ is called the *$[\alpha, \beta]$-sublevel set*.

Definition 2.6. A function $V : \mathscr{D} \to \mathbb{R}$ is *positive definite* on $\mathscr{D}$ if $V(0) = 0$ and $V(x) > 0$, $x \in \mathscr{D}$, $x \neq 0$. A function $V : \mathscr{D} \to \mathbb{R}$ is *radially unbounded* if $V(x) \to \infty$ as $\|x\| \to \infty$.

3. Generalized stability theorems

In this section, we present several generalized stability theorems where we relax all regularity assumptions on the Lyapunov function while guaranteeing local and global stability of a nonlinear dynamical system. The following result gives sufficient conditions for Lyapunov stability of a nonlinear dynamical system.

Theorem 3.1. *Consider the nonlinear dynamical system (1) and let $x(t)$, $t \geqslant 0$, denote the solution to (1). Assume that there exists a lower semicontinuous, positive-definite function $V : \mathscr{D} \to \mathbb{R}$ such that $V(\cdot)$ is continuous at the origin and $V(x(t)) \leqslant V(x(\tau))$, for all $0 \leqslant \tau \leqslant t$. Then the zero solution $x(t) \equiv 0$ to (1) is Lyapunov stable.*

Proof. Let $\varepsilon > 0$ be such that $\mathscr{B}_\varepsilon(0) \subset \mathscr{D}$. Since $\partial\mathscr{B}_\varepsilon(0)$ is compact and $V(x)$, $x \in \mathscr{D}$, is lower semicontinuous it follows from Proposition 8.10 of Royden [24, p. 195] that there exists $\alpha = \min_{x\in\partial\mathscr{B}_\varepsilon(0)} V(x)$. Note $\alpha > 0$ since $0 \notin \partial\mathscr{B}_\varepsilon(0)$ and $V(x) > 0$, $x \in \mathscr{D}$, $x \neq 0$. Next, since $V(0) = 0$ and $V(\cdot)$ is continuous at the origin it follows that there exists $\delta \in (0, \varepsilon)$ such that $V(x) < \alpha$, $x \in \mathscr{B}_\delta(0)$. Now, it follows that for all $x(0) \in \mathscr{B}_\delta(0)$, $V(x(t)) \leqslant V(x(0)) < \alpha$, $t \geqslant 0$, which, since $V(x)\geqslant\alpha$, $x \in \partial\mathscr{B}_\varepsilon(0)$, implies that $x(t) \notin \partial\mathscr{B}_\varepsilon(0)$, $t\geqslant 0$. Hence, for all $\varepsilon > 0$ such that $\mathscr{B}_\varepsilon(0) \subset \mathscr{D}$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t\geqslant 0$, which proves Lyapunov stability. □

A lower semicontinuous, positive-definite function $V(\cdot)$, with $V(\cdot)$ being continuous at the origin, is called a *generalized Lyapunov function candidate* for the nonlinear dynamical system (1). If, additionally, $V(\cdot)$ satisfies $V(x(t))\leqslant V(x(\tau))$, for all $t\geqslant\tau\geqslant 0$, $V(\cdot)$ is called a *generalized Lyapunov function* for the nonlinear dynamical system (1). Note that in the case where the function $V(\cdot)$ is C^1 on $\mathscr{D}$ in Theorem 3.1, it follows that $V(x(t))\leqslant V(x(\tau))$, for all $t\geqslant\tau\geqslant 0$, is equivalent to $\dot V(x) \triangleq V'(x)f(x)\leqslant 0$, $x\in\mathscr{D}$. In this case, Theorem 3.1 specializes to the standard Lyapunov stability theorem [12,14,17].

Next, we generalize the Barbashin–Krasovskii–LaSalle invariant set theorems [6,14–17] to the case in which the function $V(\cdot)$ is lower semicontinuous. For the remainder of the results of this paper define the notation $\mathscr{R}_\gamma \triangleq \bigcap_{c>\gamma}\overline{V^{-1}([\gamma,c])}$, for arbitrary $V:\mathscr{D}\subseteq\mathbb{R}^n\to\mathbb{R}$ and $\gamma\in\mathbb{R}$, and let $\mathscr{M}_\gamma$ denote the largest invariant set (with respect to (1)) contained in $\mathscr{R}_\gamma$.

Theorem 3.2. *Consider the nonlinear dynamical system* (1), *let* $x(t)$, $t\geqslant 0$, *denote the solution to* (1), *and let* $\mathscr{D}_\mathrm{c}\subset\mathscr{D}$ *be a compact invariant set with respect to* (1). *Assume that there exists a lower semicontinuous function* $V:\mathscr{D}_\mathrm{c}\to\mathbb{R}$ *such that* $V(x(t))\leqslant V(x(\tau))$, *for all* $0\leqslant\tau\leqslant t$ *and* $x_0\in\mathscr{D}_\mathrm{c}$. *If* $x_0\in\mathscr{D}_\mathrm{c}$, *then* $x(t)\to\mathscr{M}\triangleq\bigcup_{\gamma\in\mathbb{R}}\mathscr{M}_\gamma$ *as* $t\to\infty$.

Proof. Let $x(t)$, $t\geqslant 0$, be the solution to (1) with $x_0\in\mathscr{D}_\mathrm{c}$. Since $V(\cdot)$ is lower semicontinuous on the compact set $\mathscr{D}_\mathrm{c}$, there exists $\beta\in\mathbb{R}$ such that $V(x)\geqslant\beta$, $x\in\mathscr{D}_\mathrm{c}$. Hence, since $V(x(t))$, $t\geqslant 0$, is nonincreasing, $\gamma_{x_0}\triangleq\lim_{t\to\infty}V(x(t))$, $x_0\in\mathscr{D}_\mathrm{c}$, exists. Now, for all $p\in\omega(x_0)$ there exists an increasing unbounded sequence $\{t_n\}_{n=0}^\infty$, with $t_0=0$, such that $x(t_n)\to p$ as $n\to\infty$. Next, since $V(x(t_n))$, $n\geqslant 0$, is nonincreasing it follows that for all $N\geqslant 0$, $\gamma_{x_0}\leqslant V(x(t_n))\leqslant V(x(t_N))$, $n\geqslant N$, or, equivalently, since $\mathscr{D}_\mathrm{c}$ is invariant, $x(t_n)\in V^{-1}([\gamma_{x_0},V(x(t_N))])$, $n\geqslant N$. Now, since $\lim_{n\to\infty}x(t_n)=p$ it follows that $p\in\overline{V^{-1}([\gamma_{x_0},V(x(t_n))])}$, $n\geqslant 0$. Furthermore, since $\lim_{n\to\infty}V(x(t_n))=\gamma_{x_0}$ it follows that for every $c>\gamma_{x_0}$, there exists $n\geqslant 0$ such that $\gamma_{x_0}\leqslant V(x(t_n))\leqslant c$ which implies that for every $c>\gamma_{x_0}$, $p\in\overline{V^{-1}([\gamma_{x_0},c])}$. Hence, $p\in\mathscr{R}_{\gamma_{x_0}}$ which implies that $\omega(x_0)\subseteq\mathscr{R}_{\gamma_{x_0}}$. Now, since $\mathscr{D}_\mathrm{c}$ is compact and invariant it follows that the solution $x(t)$, $t\geqslant 0$, to (1) is bounded for all $x_0\in\mathscr{D}_\mathrm{c}$ and hence it follows from Lemma 2.1 that $\omega(x_0)$ is a nonempty compact invariant set which further implies that $\omega(x_0)$ is a subset of the largest invariant set contained in $\mathscr{R}_{\gamma_{x_0}}$, that is, $\omega(x_0)\subseteq\mathscr{M}_{\gamma_{x_0}}$. Hence, for all $x_0\in\mathscr{D}_\mathrm{c}$, $\omega(x_0)\subseteq\mathscr{M}$. Finally, since $x(t)\to\omega(x_0)$ as $t\to\infty$ it follows that $x(t)\to\mathscr{M}$ as $t\to\infty$. □

Remark 3.1. If, in Theorem 3.2, $0\in\mathring{\mathscr{D}}_\mathrm{c}$ and $\mathscr{M}$ contains no solution other than the trivial solution $x(t)\equiv 0$, then the zero solution $x(t)\equiv 0$ to (1) is attractive and $\mathscr{D}_\mathrm{c}$ is a subset of the domain of attraction.

Remark 3.2. Note that if $V:\mathscr{D}_\mathrm{c}\to\mathbb{R}$ is a lower semicontinuous function such that all the conditions of Theorem 3.2 are satisfied, then for every $x_0\in\mathscr{D}_\mathrm{c}$ there exists $\gamma_{x_0}\leqslant V(x_0)$ such that $\omega(x_0)\subseteq\mathscr{M}_{\gamma_{x_0}}\subseteq\mathscr{M}$.

It is important to note that even though the stability conditions appearing in Theorem 3.2 are system trajectory dependent, in [18] a hierarchical switching nonlinear control strategy is developed using Theorem 3.2 *without* requiring knowledge of the system trajectories. Furthermore, note that as in standard Lyapunov and invariant set theorems involving C^1 functions, Theorem 3.2 allows one to characterize the invariant set $\mathscr{M}$ *without* knowledge of the system trajectories $x(t)$, $t\geqslant 0$. Similar remarks hold for the rest of the theorems in this section. To illustrate the utility of Theorem 3.2 consider the simple scalar nonlinear dynamical system given by

$$\dot x(t)=-x(t)(x(t)-1)(x(t)+2),\quad x(0)=x_0,\ t\geqslant 0, \tag{2}$$

with generalized Lyapunov function candidate $V(x)$ given by

$$V(x)=\begin{cases}(x+2)^2, & x<0,\\ (x-1)^2, & x\geqslant 0.\end{cases}$$

Now, note that

$$\begin{aligned}\dot V(x) &\triangleq D^+V(x)[-x(x-1)(x+2)]\\ &=\begin{cases}-2x(x-1)(x+2)^2, & x<0,\\ -2x(x-1)^2(x+2), & x\geqslant 0,\end{cases}\leqslant 0,\quad x\in\mathbb{R},\end{aligned}$$

which implies that $V(x(t))$, $t\geqslant 0$, is nonincreasing along the system trajectories. Next, note that $\mathscr{R}_\gamma = V^{-1}(\gamma)$, $\gamma \in \mathbb{R}\setminus\{4\}$, and $\mathscr{R}_4 = V^{-1}(4)\cup\{0\}$. Since the only invariant sets for the dynamical system (2) are the equilibrium points $x_{e1}=-2$, $x_{e2}=0$, $x_{e3}=1$, it follows that $\mathscr{M}_\gamma=\emptyset$, $\gamma\notin\{0,\ 1,\ 4\}$, $\mathscr{M}_0=\{-2,\ 1\}$, $\mathscr{M}_1=\{0\}$, and $\mathscr{M}_4=\{0\}$ which implies that $\mathscr{M}=\{-2,\ 0,\ 1\}$. Hence, it follows from Theorem 3.2 that for every $x_0\in\mathbb{R}$ the solution to (2) approaches the invariant set $\mathscr{M}=\{-2,\ 0,\ 1\}$ as $t\to\infty$ which can be easily verified. As shown by the above example, Theorem 3.2 allows for a systematic way of constructing system Lyapunov functions by piecing together a collection of functions.

The following corollary to Theorem 3.2 presents sufficient conditions that guarantee local asymptotic stability of the nonlinear dynamical system (1).

Corollary 3.1. *Consider the nonlinear dynamical system* (1), *let* $x(t)$, $t\geqslant 0$, *denote the solution to* (1), *and let* $\mathscr{D}_c\subset\mathscr{D}$ *with* $0\in\mathring{\mathscr{D}}_c$ *be a compact invariant set with respect to* (1). *Assume that there exists a lower semicontinuous, positive-definite function* $V:\mathscr{D}_c\to\mathbb{R}$ *such that* $V(\cdot)$ *is continuous at the origin and* $V(x(t))\leqslant V(x(\tau))$, *for all* $0\leqslant\tau\leqslant t$ *and* $x_0\in\mathscr{D}_c$. *Furthermore, assume* $\mathscr{M}\triangleq\bigcup_{\gamma\geqslant 0}\mathscr{M}_\gamma$ *contains no solution other than the trivial solution* $x(t)\equiv 0$. *Then the zero solution* $x(t)\equiv 0$ *to* (1) *is asymptotically stable and* $\mathscr{D}_c$ *is a subset of the domain of attraction of* (1).

Proof. The result is a direct consequence of Theorems 3.1 and 3.2. □

Next, we specialize Theorem 3.2 to the Barbashin–Krasovskii–LaSalle invariant set theorem wherein $V(\cdot)$ is a C^1 function.

Corollary 3.2. *Consider the nonlinear dynamical system* (1), *assume* $\mathscr{D}_c\subset\mathscr{D}$ *is a compact invariant set with respect to* (1), *and assume that there exists a* C^1 *function* $V:\mathscr{D}_c\to\mathbb{R}$ *such that* $V'(x)f(x)\leqslant 0$, $x\in\mathscr{D}_c$. *Let* $\mathscr{R}\triangleq\{x\in\mathscr{D}_c\colon V'(x)f(x)=0\}$ *and let* $\mathscr{M}$ *be the largest invariant set contained in* $\mathscr{R}$. *If* $x_0\in\mathscr{D}_c$, *then* $x(t)\to\mathscr{M}$ *as* $t\to\infty$.

Proof. The result follows from Theorem 3.2. Specifically, since $V'(x)f(x)\leqslant 0$, $x\in\mathscr{D}_c$, it follows that

$$V(x(t))-V(x(\tau))$$

$$=\int_\tau^t V'(x(s))f(x(s))\,\mathrm{d}s\leqslant 0,\quad t\geqslant\tau,$$

and hence $V(x(t))\leqslant V(x(\tau))$, $t\geqslant\tau$. Now, since $V(\cdot)$ is C^1 it follows that $\mathscr{R}_\gamma=V^{-1}(\gamma)$, $\gamma\in\mathbb{R}$. In this case, it follows from Theorem 3.2 and Remark 3.2 that for every $x_0\in\mathscr{D}_c$ there exists $\gamma_{x_0}\in\mathbb{R}$ such that $\omega(x_0)\subseteq\mathscr{M}_{\gamma_{x_0}}$, where $\mathscr{M}_{\gamma_{x_0}}$ is the largest invariant set contained in $\mathscr{R}_{\gamma_{x_0}}=V^{-1}(\gamma_{x_0})$ which implies that $V(x)=\gamma_{x_0}$, $x\in\omega(x_0)$. Hence, since $\mathscr{M}_{\gamma_{x_0}}$ is an invariant set it follows that for all $x(0)\in\mathscr{M}_{\gamma_{x_0}}$, $x(t)\in\mathscr{M}_{\gamma_{x_0}}$, $t\geqslant 0$, and thus $\dot V(x(0))\triangleq \mathrm{d}V(x(t))/\mathrm{d}t|_{t=0}=V'(x(0))f(x(0))=0$, which implies the $\mathscr{M}_{\gamma_{x_0}}$ is contained in $\mathscr{M}$ which is the largest invariant set contained in $\mathscr{R}$. Hence, since $x(t)\to\omega(x_0)\subseteq\mathscr{M}$ as $t\to\infty$, it follows that $x(t)\to\mathscr{M}$ as $t\to\infty$. □

Next, we sharpen the results of Theorem 3.2 by providing a refined construction of the invariant set $\mathscr{M}$. In particular, we show that the system trajectories converge to a union of largest invariant sets contained in intersections over the largest limit value of $V(\cdot)$ at the origin of the closure of generalized Lyapunov surfaces. First, however, the following key lemma is needed.

Lemma 3.1. *Let* $\mathscr{Q}\subseteq\mathbb{R}^n$, *let* $V:\mathscr{Q}\to\mathbb{R}$, *and let* $\gamma_0\triangleq\limsup_{x\to 0}V(x)$. *If* $0\in\mathscr{R}_\gamma$ *for some* $\gamma\in\mathbb{R}$, *then* $\gamma\leqslant\gamma_0$.

Proof. If $0\in\mathscr{R}_\gamma$ for $\gamma\in\mathbb{R}$, then there exists a sequence $\{x_n\}_{n=0}^\infty\subset\mathscr{R}_\gamma$ such that $\lim_{n\to\infty}x_n=0$. Now, since $\gamma_0=\limsup_{x\to 0}V(x)$, it follows that $\limsup_{n\to\infty}V(x_n)\leqslant\gamma_0$. Next, note that $x_n\in\overline{V^{-1}([\gamma,c])}$, $c>\gamma$, $n=0,1,\ldots$, which implies that $V(x_n)\geqslant\gamma$, $n=0,1,\ldots$. Thus, using the fact that $\limsup_{n\to\infty}V(x_n)\leqslant\gamma_0$ it follows that $\gamma\leqslant\gamma_0$. □

Remark 3.3. Note that if in Lemma 3.1 $V(\cdot)$ is continuous at the origin then $\gamma_0=V(0)$.

Theorem 3.3. *Consider the nonlinear dynamical system* (1), *let* $x(t)$, $t\geqslant 0$, *denote the solution to* (1), *and let* $\mathscr{D}_c\subset\mathscr{D}$ *with* $0\in\mathring{\mathscr{D}}_c$ *be a compact invariant set with respect to* (1). *Assume that there exists a lower semicontinuous, positive-definite function* $V:\mathscr{D}_c\to\mathbb{R}$ *such that* $V(x(t))\leqslant V(x(\tau))$, *for all* $0\leqslant\tau\leqslant t$ *and* $x_0\in\mathscr{D}_c$. *Furthermore, assume that for all* $x_0\in\mathscr{D}_c$, $x_0\neq 0$, *there exists an increasing unbounded sequence* $\{t_n\}_{n=0}^\infty$, *with* $t_0=0$, *such that* $V(x(t_{n+1}))<V(x(t_n))$, $n=0,1,\ldots$. *If* $x_0\in\mathscr{D}_c$, *then*

$x(t) \to \hat{\mathscr{M}} \triangleq \bigcup_{\gamma\in\mathscr{G}} \mathscr{M}_\gamma$ as $t \to \infty$, where $\mathscr{G} \triangleq \{\gamma \in [0,\gamma_0]\colon 0 \in \mathscr{R}_\gamma\}$ and $\gamma_0 \triangleq \limsup_{x\to 0} V(x)$. If, in addition, $V(\cdot)$ is continuous at the origin then the zero solution $x(t) \equiv 0$ to (1) is locally asymptotically stable and $\mathscr{D}_\mathrm{c}$ is a subset of the domain of attraction. Finally, if $\mathscr{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded and such that $V(x(t)) \leqslant V(x(\tau))$ for all $0 \leqslant \tau \leqslant t$ and $x_0 \in \mathbb{R}^n$, then the above results are global.

Proof. It follows from Theorem 3.2, Remark 3.2, and the fact that $V(\cdot)$ is positive definite that, for every $x_0 \in \mathscr{D}_\mathrm{c}$, there exists $\gamma_{x_0} \geqslant 0$ such that $\omega(x_0) \subseteq \mathscr{M}_{\gamma_{x_0}} \subseteq \mathscr{R}_{\gamma_{x_0}}$. Furthermore, since all solutions $x(t)$, $t \geqslant 0$, to (1) are bounded it follows from Lemma 2.1 that $\omega(x_0)$ is a nonempty, compact, invariant set. Now, ad absurdum, suppose $0 \notin \omega(x_0)$. Since $V(\cdot)$ is lower semicontinuous it follows from Proposition 8.10 of Royden [24, p. 195] that $\alpha \triangleq \min_{x\in\omega(x_0)} V(x)$ exists. Furthermore, there exists $\hat{x} \in \omega(x_0)$ such that $V(\hat{x}) = \alpha$. Now, with $x(0) = \hat{x} \neq 0$ it follows that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$, which implies that there exists $t > 0$ such that $V(x(t)) < \alpha$ which further implies that $x(t) \notin \omega(x_0)$ contradicting the fact that $\omega(x_0)$ is an invariant set. Hence, $0 \in \omega(x_0) \subseteq \mathscr{R}_{\gamma_{x_0}}$ which, using Lemma 3.1, implies that $\gamma_{x_0} \leqslant \gamma_0$ for all $x_0 \in \mathscr{D}_\mathrm{c}$, which further implies that $\omega(x_0) \subseteq \hat{\mathscr{M}}$. Now, since $x(t) \to \omega(x_0) \subseteq \hat{\mathscr{M}}$ as $t \to \infty$ it follows that $x(t) \to \hat{\mathscr{M}}$ as $t \to \infty$.

If $V(\cdot)$ is continuous at the origin then Lyapunov stability follows from Theorem 3.1. Furthermore, in this case, $\gamma_0 = V(0) = 0$ which implies that $\hat{\mathscr{M}} \equiv \{0\}$. Hence, $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathscr{D}_\mathrm{c}$ establishing local asymptotic stability with a subset of the domain of attraction given by $\mathscr{D}_\mathrm{c}$.

Finally, to show global attraction and global asymptotic stability let $\mathscr{D} = \mathbb{R}^n$ and note that since $V(x) \to \infty$ as $\|x\| \to \infty$ it follows that for every $\beta > 0$ there exists $r > 0$ such that $V(x) > \beta$ for all $x \notin \mathscr{B}_r(0)$, or, equivalently, $V^{-1}([0,\beta]) \subseteq \bar{\mathscr{B}}_r(0)$ which implies that $V^{-1}([0,\beta])$ is bounded for all $\beta > 0$. Hence, for all $x_0 \in \mathbb{R}^n$, $V^{-1}([0,\beta_{x_0}])$ is bounded where $\beta_{x_0} \triangleq V(x_0)$. Furthermore, since $V(\cdot)$ is a lower semicontinuous, positive-definite function it follows that $V^{-1}([0,\beta_{x_0}])$ is closed and since $V(x(t))$, $t \geqslant 0$, is nonincreasing it follows that $V^{-1}([0,\beta_{x_0}])$ is an invariant set. Hence, for every $x_0 \in \mathbb{R}^n$, $V^{-1}([0,\beta_{x_0}])$ is a compact invariant set. Now, with $\mathscr{D}_\mathrm{c} = V^{-1}([0,\beta_{x_0}])$, global attraction and global asymptotic stability are a direct consequence of the first part of the theorem. □

Remark 3.4. In some applications the construction of the function $V(x)$ in Theorem 3.3 will itself guarantee the existence of a compact invariant set $\mathscr{D}_\mathrm{c}$. In particular, if $V(x)$ is a generalized Lyapunov function for (1) then all the solutions $x(t)$, $t \geqslant 0$, to (1) that are bounded approach $\mathscr{M}$ as $t \to \infty$ since $\mathscr{D}_\mathrm{c}$ can be taken as the union of all bounded solutions of (1).

Remark 3.5. If in Theorem 3.3 the function $V(\cdot)$ is C^1 on $\mathscr{D}_\mathrm{c}$ and $\mathbb{R}^n$, respectively, and $V'(x)f(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then every increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, is such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$. In this case, Theorem 3.3 specializes to the standard Lyapunov stability theorems for local and global asymptotic stability.

4. Conclusion

Generalized Lyapunov and invariant set stability theorems for nonlinear dynamical systems were developed. In particular, local and global stability theorems were presented using generalized lower semicontinuous Lyapunov functions providing a transparent generalization of standard Lyapunov and invariant set theorems.

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