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Mixed H_2/L_1 fixed-architecture controller design for multi-input/single-output systems[1]

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Abstract

In this paper we develop a mixed-norm H_2/L_1 controller synthesis framework via fixed-order dynamic compensation for multi-input/single-output systems. For flexibility in controller synthesis, we adopt the approach of fixed-structure controller design which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization. Several numerical examples are presented to demonstrate the fixed-structure mixed-norm H_2/L_1 controller synthesis approach. © 1999 The Franklin Institute. Published by Elsevier Science Ltd.

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1. Introduction

One of the principal objectives of control design is to synthesize feedback controllers to reject uncertain exogenous disturbances. In H_∞ theory [1–3] disturbance rejection is achieved for systems with bounded energy (square-integrable) L_2 disturbances while in L_1 theory [4–7] disturbance rejection is achieved for systems with bounded amplitude persistent L_∞ point-wise-in-time disturbances. Despite the significance of H_∞ and L_1 theory in addressing disturbance rejection and robust stability and performance in the presence of norm-bounded uncertainty [8], it is clear that

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a single norm is seldom adequate to capture diverse and often conflicting design objectives. To this end recent research has concentrated on mixed-norm controller synthesis frameworks [9–14] to address the problem of simultaneous disturbance rejection involving exogenous disturbances with disparate characteristics such as white noise, bounded energy, and bounded amplitude.

Even though the mixed-norm H_2/H_∞ control problem has been addressed for general continuous-time and discrete-time multivariable systems controlled by static and dynamic controllers [9–11, 15], the mixed-norm H_2/L_1 problem has been addressed for the limited class of single-input/single-output systems with a single disturbance input and a single performance variable [12, 14]. A notable exception is [13] where a constrained (non-optimal) mixed H_2 bound/L_1 bound control problem is posed using the linear matrix inequality feasibility framework developed in [17]. To address the gap between the actual H_2 performance and the H_2 performance bound given in [13] the authors in [16] developed a multiobjective problem for multi-input/single-output systems involving a convex combination of the actual H_2 norm and the L_1 norm bound proposed in [17]. This approach is reminiscent of scalarization techniques for Pareto optimization [15] and results in a Riccati equation framework for mixed H_2/L_1 static output feedback regulation.

The goal of the present paper is to extend the results in [16] to mixed H_2/L_1 controller synthesis via fixed-order dynamic compensation for multi-input/single-output systems. For flexibility in controller synthesis, we adopt the approach of fixed-structure controller synthesis [18] which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization [19]. To demonstrate the fixed-structure mixed-norm H_2/L_1 control problem, we consider several numerical examples including reduced-order controller design for an Euler–Bernoulli beam involving five flexible modes.

2. Preliminaries

In this section we establish definitions and notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote real and complex numbers, let $(\ )^{\mathrm{T}}$ and $(\ )^*$ denote transpose and complex conjugate transpose, respectively, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|_2$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, "tr" for the trace operator, and $M \geqslant 0$ $(M > 0)$ to denote the fact that the Hermitian matrix M is nonnegative (positive) definite. For a linear time-invariant system with input u and output y, $G(s)$ and $G(t)$ denote real-rational transfer function and impulse response matrix function, respectively.

Let L_∞ denote the space of bounded Lebesgue measurable functions on $[0, \infty)$ and let H_2 denote the Hardy space of real-rational transfer function matrices square-integrable on the imaginary axis with analytic continuation into the right-half plane. The H_2 norm of an asymptotically stable transfer function is defined as

$$\|G\|_2 \triangleq \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \|G(j\omega)\|_{\mathrm{F}}^2 \,\mathrm{d}\omega\right]^{1/2} = \left[\int_0^{\infty} \|G(t)\|_{\mathrm{F}}^2 \,\mathrm{d}t\right]^{1/2}. \tag{1}$$

For a measurable function z: $[0, \infty) \to \mathbb{R}^r$ define the L_∞ function norm with a Euclidean spatial norm by $\|z(\cdot)\|_{\infty,2} \triangleq \text{ess sup}_{t \geqslant 0} \|z(t)\|_2$. For $u(\cdot)$, $y(\cdot) \in L_\infty$ on $[0, \infty)$ with Euclidean spatial norms the L_1 norm of the convolution operator $H: L_\infty \to L_\infty$ of a linear time-invariant system with input u and output y is the equi-induced signal norm

$$\|H\|_1 \triangleq \sup_{u(\cdot) \in L_\infty} \frac{\|y\|_{\infty,2}}{\|u\|_{\infty,2}}. \tag{2}$$

From an input–output point of view the L_1 norm captures the worst-case peak amplification from input disturbance signals to output signals, where the signal size is taken to be the supremum over time of the signal's pointwise-in-time Euclidean norm. Note that the input–output signal norms for inducing the L_1 norm considered in this paper are different from the input–output signal norms considered in [6] where $u(\cdot)$, $y(\cdot) \in L_\infty$ on $[0, \infty)$ with *spectral* spatial norms are used to capture the maximum peak-to-peak system gain.

3. Combined H_2/L_1 fixed-order dynamic compensation

In this section we introduce the mixed H_2/L_1 fixed-order dynamic compensation problem. Without the L_1 performance criterion the problem considered here corresponds to the standard fixed-order H_2 control problem.

3.1. *Combined H_2/L_1 fixed-order dynamic output feedback control problem*

Consider the nth-order stabilizable and detectable system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t) + D_{1\infty} w_\infty(t), \quad t \in [0, \infty), \tag{3}$$

$$y(t) = Cx(t) + D_2 w(t) + D_{2\infty} w_\infty(t), \tag{4}$$

with vector H_2 and scalar L_1 performance variables, respectively,

$$z_2(t) = E_1 x(t) + E_2 u(t), \tag{5}$$

$$z_\infty(t) = E_{1\infty} x(t) + E_{2\infty} u(t), \tag{6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^d$, $w_\infty \in \mathbb{R}^{d_\infty}$, $y \in \mathbb{R}^l$, $z_2 \in \mathbb{R}^p$, $z_\infty \in \mathbb{R}$, and $w(\cdot)$ denotes a unit-intensity white noise signal and $w_\infty(\cdot)$ denotes an L_∞ signal such that $\|w_\infty(\cdot)\|_{\infty,2} \leqslant 1$. We seek an n_cth order $(1 \leqslant n_c \leqslant n)$ dynamic output feedback controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{7}$$

$$u(t) = C_c x_c(t), \tag{8}$$

such that the following design criteria are satisfied:
(i) the closed-loop system (3), (4), (7), and (8) is asymptotically stable; and
(ii) for $\mu \in [0, 1]$ the cost functional

$$J(A_c, B_c, C_c) \triangleq \mu \|\tilde{G}\|_2^2 + (1 - \mu) \|\tilde{H}\|_1^2 \tag{9}$$

is minimized, where $\tilde{G}$ corresponds to the closed-loop impulse response matrix function from disturbances $w(\cdot)$ to H_2 performance variables $z_2(\cdot)$ and $\|\tilde{H}\|_1$ is the L_1 norm of the convolution operator $\tilde{H}$ from L_∞ disturbances $w_\infty(t)$ to L_∞ performance variables $z_\infty(t)$ of the closed-loop system defined by

$$\|\tilde{H}\|_1 \triangleq \sup_{w_\infty(\cdot)\in L_\infty} \frac{\|z_\infty\|_{\infty,2}}{\|w_\infty\|_{\infty,2}}. \tag{10}$$

Note that if criterion (i) is satisfied then $\|\tilde{H}\|_1$ is bounded. Furthermore, note that the problem statement involves both H_2 and L_1 performance variables z_2 and z_∞, respectively, with disturbance inputs w and w_∞.

Remark 3.1. The cost functional (9) involves a convex combination of two scalar costs. By varying $\mu \in [0, 1]$, Eq. (9) can be viewed as a scalar representation of a multiobjective cost (see, e.g., [15] and the references therein). By setting $\mu = 0$ we obtain an L_1 optimal fixed-order dynamic compensation problem. Alternatively, setting $\mu = 1$ recovers the standard H_2 optimal fixed-order dynamic compensation problem. The practical value of this formulation is the case $\mu \in (0, 1)$ in which the optimization problem yields a trade-off between H_2 and L_1 performance.

4. Combined H_2/L_1 fixed-order dynamic compensation: decentralized static output feedback formulation

In this section we use the fixed-structure control framework of [19] to transform the combined H_2/L_1 fixed-order strictly proper, centralized dynamic compensation problem to a decentralized static output feedback setting. Specifically, note that for every dynamic controller (7), (8) the closed-loop system (3), (4), (7), and (8) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix} w(t) + \begin{bmatrix} D_{1\infty} \\ B_cD_{2\infty} \end{bmatrix} w_\infty(t). \tag{11}$$

Furthermore, by treating A_c, B_c, and C_c as decentralized static output feedback gains we obtain

$$\dot{\tilde{x}}(t) = \mathscr{A}\tilde{x}(t) + \sum_{i=1}^{3} \mathscr{B}_i\hat{u}_i(t) + \mathscr{D}_1 w(t) + \mathscr{D}_{1\infty} w_\infty(t), \tag{12}$$

$$\hat{y}_i(t) = \mathscr{C}_i\tilde{x}(t) + \mathscr{D}_{2_i} w(t) + \mathscr{D}_{2\infty_i} w_\infty(t), \quad i = 1, 2, 3 \tag{13}$$

$$z_2(t) = \mathscr{E}_1\tilde{x}(t) + \sum_{i=1}^{3} \mathscr{E}_{2_i}\hat{u}_i(t), \tag{14}$$

$$z_{2\infty}(t) = \mathscr{E}_{1\infty}\tilde{x}(t) + \sum_{i=1}^{3} \mathscr{E}_{2\infty_i}\hat{u}_i(t), \tag{15}$$

$$\hat{u}_1(t) = A_c\hat{y}_1(t), \quad \hat{u}_2(t) = B_c\hat{y}_2(t), \quad \hat{u}_3(t) = C_c\hat{y}_3(t), \tag{16}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \mathscr{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{B}_1 \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathscr{B}_2 \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathscr{B}_3 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$\mathscr{C}_1 = [0 \;\; I], \quad \mathscr{C}_2 \triangleq [C \;\; 0], \quad \mathscr{C}_3 \triangleq [0 \;\; I],$$

$$\mathscr{D}_1 \triangleq \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \quad \mathscr{D}_{2_1} \triangleq 0, \quad \mathscr{D}_{2_2} \triangleq D_2, \quad \mathscr{D}_{2_3} \triangleq 0,$$

$$\mathscr{D}_{1\infty} \triangleq \begin{bmatrix} D_{1\infty} \\ 0 \end{bmatrix}, \quad \mathscr{D}_{2\infty_1} \triangleq 0, \quad \mathscr{D}_{2\infty_2} \triangleq D_{2\infty}, \quad \mathscr{D}_{2\infty_3} \triangleq 0,$$

$$\mathscr{E}_1 \triangleq [\mathscr{E}_1 \;\; 0], \quad \mathscr{E}_{2_1} \triangleq 0, \quad \mathscr{E}_{2_2} \triangleq 0, \quad \mathscr{E}_{2_3} \triangleq E_2,$$

$$\mathscr{E}_{1\infty} \triangleq [E_{1\infty} \;\; 0], \quad \mathscr{E}_{2\infty_1} \triangleq 0, \quad \mathscr{E}_{2\infty_2} \triangleq 0 \quad \mathscr{E}_{2\infty_3} \triangleq E_{2\infty}.$$

Next, defining

$$\hat{u}(t) \triangleq \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \\ \hat{u}_3(t) \end{bmatrix}, \quad \hat{y}(t) \triangleq \begin{bmatrix} \hat{y}_1(t) \\ \hat{y}_2(t) \\ \hat{y}_3(t) \end{bmatrix},$$

Equations (12–(15) can be rewritten as

$$\dot{\tilde{x}}(t) = \mathscr{A}\tilde{x}(t) + \mathscr{B}\hat{u}(t) + \mathscr{D}_1 w(t) + \mathscr{D}_{1\infty} w_\infty(t), \tag{17}$$

$$\hat{y}(t) = \mathscr{C}\tilde{x}(t) + \mathscr{D}_2 w(t) + \mathscr{D}_{2\infty} w_\infty(t), \tag{18}$$

$$z_2(t) = \mathscr{E}_1 \tilde{x}(t) + \mathscr{E}_2 \hat{u}(t), \tag{19}$$

$$z_\infty(t) = \mathscr{E}_{1\infty} \tilde{x}(t) + \mathscr{E}_{2\infty} \hat{u}(t), \tag{20}$$

where

$$\mathscr{B} \triangleq [\mathscr{B}_1 \; \mathscr{B}_2 \; \mathscr{B}_3], \quad \mathscr{E}_2 \triangleq [\mathscr{E}_{2_1} \; \mathscr{E}_{2_2} \; \mathscr{E}_{2_3}], \quad \mathscr{E}_{2\infty} \triangleq [\mathscr{E}_{2\infty_1} \; \mathscr{E}_{2\infty_2} \; \mathscr{E}_{2\infty_3}],$$

$$\mathscr{C} \triangleq \begin{bmatrix} \mathscr{C}_1 \\ \mathscr{C}_2 \\ \mathscr{C}_3 \end{bmatrix}, \quad \mathscr{D}_2 \triangleq \begin{bmatrix} \mathscr{D}_{2_1} \\ \mathscr{D}_{2_2} \\ \mathscr{D}_{2_3} \end{bmatrix}, \quad \mathscr{D}_{2\infty} \triangleq \begin{bmatrix} \mathscr{D}_{2\infty_1} \\ \mathscr{D}_{2\infty_2} \\ \mathscr{D}_{2\infty_3} \end{bmatrix}.$$

Furthermore, by rewriting the decentralized controls (16) in the compact form

$$\hat{u}(t) = \mathscr{K}\hat{y}(t), \tag{21}$$

where

$$\mathscr{K} \triangleq \begin{bmatrix} A_c & 0 & 0 \\ 0 & B_c & 0 \\ 0 & 0 & C_c \end{bmatrix},$$

the closed-loop system is given by

$$\dot{\tilde{x}}(t) = \tilde{\mathscr{A}}\tilde{x}(t) + \tilde{D}w(t) + \tilde{D}_\infty w_\infty(t), \tag{22}$$

$$z_2(t) = \tilde{E}\tilde{x}(t), \tag{23}$$

$$z_\infty(t) = \tilde{E}_\infty \tilde{x}(t), \tag{24}$$

where

$$\tilde{A} \triangleq \mathscr{A} + \mathscr{B}\mathscr{K}\mathscr{C}, \quad \tilde{D} \triangleq \mathscr{D}_1 + \mathscr{B}\mathscr{K}\mathscr{D}_2, \quad \tilde{D}_\infty \triangleq \mathscr{D}_{1\infty} + \mathscr{B}\mathscr{K}\mathscr{D}_{2\infty},$$

$$\tilde{E} \triangleq \mathscr{E}_1 + \mathscr{E}_2\mathscr{K}\mathscr{C}, \quad \tilde{E}_\infty \triangleq \mathscr{E}_{1\infty} + \mathscr{E}_{2\infty}\mathscr{K}\mathscr{C}.$$

Note that the closed-loop multi-input/multi-output transfer function from disturbances w to H_2 performance variables is characterized by the triple $(\tilde{A}, \tilde{D}, \tilde{E})$ and the closed-loop multi-input/single-output transfer function from disturbances w_∞ to L_1 performance variables is characterized by the triple $(\tilde{A}, \tilde{D}_\infty, \tilde{E}_\infty)$.

It is useful to note that if $\tilde{A}$ is asymptotically stable for a given feedback gain $K \in \mathbb{R}^{(2n_c+m)\times(2n_c+l)}$, then $\|\tilde{G}\|_2^2$ is given by

$$\|\tilde{G}\|_2^2 = \int_0^\infty \|\tilde{E}\mathrm{e}^{\tilde{A}t}\tilde{D}\|_\mathrm{F}^2\,\mathrm{d}t = \operatorname{tr}\tilde{Q}\tilde{R}, \tag{25}$$

where $\tilde{R} \triangleq \tilde{E}^\mathrm{T}\tilde{E}$ and $\tilde{Q}$ is the unique, $\tilde{n}\times\tilde{n}$ nonnegative-definite solution to the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^\mathrm{T} + \tilde{V}, \tag{26}$$

where $\tilde{n} \triangleq n + n_c$ and $\tilde{V} \triangleq \tilde{D}\tilde{D}^\mathrm{T}$.

Since minimizing the L_1 norm directly results in irrational (infinite dimensional) controllers [4, 6, 8] we minimize an upper bound on the L_1 norm to avoid this complexity. Next, we present a key lemma that provides an upper bound on the L_1 performance in terms of a solution to a modified Lyapunov equation.

Lemma 4.1. (Haddad and Kapila [16]). *Let $\alpha > 0$ and $\mathscr{K} \in \mathbb{R}^{(2n_c+m)\times(2n_c+l)}$ be given and assume there exists a positive-definite matrix $\hat{Q} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ satisfying*

$$0 = \tilde{A}\hat{Q} + \hat{Q}\tilde{A}^\mathrm{T} + \alpha\hat{Q} + \tilde{V}_\infty, \tag{27}$$

where $\tilde{V}_\infty \triangleq \tilde{D}_\infty\tilde{D}_\infty^\mathrm{T}$. Then $\tilde{A}$ is Hurwitz. Furthermore, the L_1 norm of the convolution operator $\tilde{H}$ of the closed-loop system from disturbances w_∞ to scalar performance variables z_∞ satisfies the bound

$$\|\tilde{H}\|_1^2 \leqslant \frac{1}{\alpha}\sigma_{\max}(\tilde{E}_\infty\hat{Q}\tilde{E}_\infty^\mathrm{T}) = \frac{1}{\alpha}\operatorname{tr}\tilde{E}_\infty\hat{Q}\tilde{E}_\infty^\mathrm{T}. \tag{28}$$

Remark 4.1. Note that in order to provide the tightest upper bound for the L_1 norm of the closed-loop system we can replace Eq. (28) by

$$\|\tilde{H}\|_1^2 \leqslant \inf_{\alpha > 0} \frac{1}{\alpha} \operatorname{tr} \tilde{E}_\infty \hat{Q} \tilde{E}_\infty^{\mathrm{T}}, \tag{29}$$

where $\hat{Q}$ satisfies Eq. (27).

Lemma 4.1 shows that the L_1 norm constraint is enforced when a positive-definite solution to Eq. (27) is known to exist and $\tilde{A}$ is Hurwitz. Furthermore, H_2 performance can be captured by $\operatorname{tr} \tilde{Q}\tilde{R}$ where $\tilde{Q}$ is the nonnegative definite solution to Eq. (26). Furthermore, note that the equality in Eq. (28) holds since for multi-input/single-output systems, rank $\tilde{E}_\infty^{\mathrm{T}} \tilde{E}_\infty = 1$ and hence $\operatorname{tr} \tilde{E}_\infty \hat{Q} \tilde{E}_\infty^{\mathrm{T}} = \sigma_{\max}(\tilde{E}_\infty \hat{Q} \tilde{E}_\infty^{\mathrm{T}})$. Next, the combined H_2/L_1 dynamic output feedback control problem can be recast as the following auxiliary optimization problem.

4.1. Auxiliary optimization problem

For $\mu \in [0, 1]$ determine $\mathscr{K} \in \mathbb{R}^{(2n_c+m)\times(2n_c+l)}$ that minimizes

$$\mathscr{J}(\mathscr{K}) = \mu \operatorname{tr} \tilde{Q}\tilde{R} + \frac{1-\mu}{\alpha} \operatorname{tr} \tilde{E}_\infty \hat{Q} \tilde{E}_\infty^{\mathrm{T}}, \tag{30}$$

where $\tilde{Q} \geqslant 0$ and $\hat{Q} > 0$ satisfy Eqs. (26) and (27), respectively.

Remark 4.2. In the case where $D_1 = D_{1\infty}$ and $D_2 = D_{2\infty}$ the solution to Eq. (26) satisfies the bound

$$\tilde{Q} \leqslant \hat{Q}. \tag{31}$$

Furthermore, if $E_1 = E_{1\infty}$ and $E_2 = E_{2\infty}$ then

$$\|G\|_2^2 \leqslant \operatorname{tr} \tilde{E}_\infty \hat{Q} \tilde{E}_\infty^{\mathrm{T}}. \tag{32}$$

Hence, in this case taking $\mu = 0$ in Eq. (30) minimizes an upper bound on L_1 performance while providing an upper bound on H_2 performance.

5. Optimality conditions for mixed-norm H_2/L_1 fixed-order dynamic compensation

In this section we state optimality conditions for characterizing dynamic output feedback controllers guaranteeing closed-loop stability and mixed H_2/L_1 performance. For convenience in stating the main result define $\tilde{R}_\infty \triangleq \tilde{E}_\infty^{\mathrm{T}} \tilde{E}_\infty$.

Theorem 5.1. *Let* $\alpha > 0$ *and let* $\mathscr{K} \in \mathbb{R}^{(2n_c+m)\times(2n_c+l)}$ *be such that* $\tilde{A}$ *is asymptotically stable and* $\mathscr{J}(\mathscr{K})$ *is minimized. Then there exists* $\tilde{n} \times \tilde{n}$ *nonnegative-definite matrices*

$\tilde{Q}$ *and* $\tilde{P}$ *and* $\tilde{n}\times\tilde{n}$ *positive-definite matrices* $\hat{Q}$ *and* $\hat{P}$ *satisfying*

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}, \tag{33}$$

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \mu\tilde{R}, \tag{34}$$

$$0 = \tilde{A}\hat{Q} + \hat{Q}\tilde{A}^{\mathrm{T}} + \alpha\hat{Q} + \tilde{V}_{\infty}, \tag{35}$$

$$0 = \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A} + \alpha\hat{P} + \tfrac{1}{\alpha}(1-\mu)\tilde{R}_{\infty}, \tag{36}$$

such that A_c, B_c, C_c *satisfy*

$$0 = \mathscr{B}_1^{\mathrm{T}}(\tilde{P}\tilde{Q} + \hat{P}\hat{Q})\mathscr{C}_1^{\mathrm{T}}, \tag{37}$$

$$0 = \mathscr{B}_2^{\mathrm{T}}(\tilde{P}\tilde{Q} + \hat{P}\hat{Q})\mathscr{C}_2^{\mathrm{T}} + \mathscr{B}_2^{\mathrm{T}}\tilde{P}\tilde{D}\mathscr{D}_{2_2}^{\mathrm{T}} + \mathscr{B}_2^{\mathrm{T}}\hat{P}\tilde{D}_{\infty}\mathscr{D}_{2\infty_2}^{\mathrm{T}}, \tag{38}$$

$$0 = \mathscr{B}_3^{\mathrm{T}}(\tilde{P}\tilde{Q} + \hat{P}\hat{Q})\mathscr{C}_3^{\mathrm{T}} + \mu\mathscr{E}_{2_3}^{\mathrm{T}}\tilde{E}\tilde{Q}\mathscr{C}_3^{\mathrm{T}} + \frac{(1-\mu)}{\alpha}\,\mathscr{E}_{2\infty_3}^{\mathrm{T}}\tilde{E}_{\infty}\hat{Q}\mathscr{C}_3^{\mathrm{T}}. \tag{39}$$

Furthermore,

$$\|\tilde{G}\|_2^2 = \operatorname{tr}\tilde{Q}\tilde{R}, \tag{40}$$

$$\|\tilde{H}\|_1^2 \leqslant \frac{1}{\alpha}\,\sigma_{\max}(\tilde{E}_{\infty}\hat{Q}\tilde{E}_{\infty}^{\mathrm{T}}). \tag{41}$$

Proof. The result follows from standard Lagrange multiplier arguments. Specifically, to optimize Eq. (30) subject to Eqs. (26) and (27), form the Lagrangian

$$\begin{aligned}\mathscr{L}(\tilde{Q},\hat{Q},\tilde{P},\hat{P},A_c,B_c,C_c,\lambda) = \operatorname{tr}\Big\{\lambda\Big[\mu\tilde{Q}\tilde{R} + \frac{1-\mu}{\alpha}\tilde{E}_{\infty}\hat{Q}\tilde{E}_{\infty}^{\mathrm{T}}\Big]\\ + [(\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V})\tilde{P}] + [(\tilde{A}\hat{Q} + \hat{Q}\tilde{A}^{\mathrm{T}} + \alpha\hat{Q} + \tilde{V}_{\infty})\hat{P}]\Big\},\end{aligned} \tag{42}$$

where the Lagrange multipliers $\lambda \geqslant 0$ and $\tilde{P}, \hat{P} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ are all not zero. By viewing $\tilde{Q}, \hat{Q}, A_c, B_c,$ and C_c as independent variables in Eq. (42), we can now obtain Eqs. (34), (36), and (37)–(39). For details of a similar proof see [16]. □

Equations (33)–(39) provide optimality conditions that yield dynamic controllers for fixed-order mixed H_2/L_1 output feedback compensation. In the design eqs. (33)–(39) one can view α as a free parameter and optimize the combined H_2/L_1 performance criterion (30) with respect to α. In particular, setting $\partial\mathscr{J}/\partial\alpha = 0$ yields

$$\alpha = \sqrt{\frac{(1-\mu)\operatorname{tr}\tilde{E}_{\infty}\hat{Q}\tilde{E}_{\infty}^{\mathrm{T}}}{\operatorname{tr}\hat{Q}\hat{P}}}. \tag{43}$$

In this paper we employed a quasi-Newton optimization algorithm to solve the optimality conditions in Eqs. (33)–(39). For full-order controller design the algorithm was initialized with an LQG controller while for reduced-order control the algorithm was initialized with a balanced truncated LQG controller. For feasible values of α and μ the quasi-Newton optimization algorithm was used to find A_c, B_c, and C_c satisfying the necessary conditions. After each iteration α was increased (or μ was decreased) and the current values of the controller gains (A_c, B_c, C_c) were then used as the starting point for the next iteration. For details of a similar algorithm see [19].

6. Illustrative numerical examples

In this section we provide several numerical examples to demonstrate the proposed mixed-norm H_2/L_1 fixed-order dynamic compensation framework.

Example 6.1. Consider the dynamic system (3), (4) with performance variables (5), (6) where [13]

$$A = \frac{1}{3}\begin{bmatrix} 2 & -8 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad C = [1 \quad 0],$$

$$D_1 = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad D_{1\infty} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad D_2 = [0 \quad 0 \quad 1], \quad D_{2\infty} = [0 \quad 0],$$

$$E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad E_{1\infty} = [4 \quad 3], \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_{2\infty} = 0.$$

Several full-order ($n_c = 2$) controllers were designed to examine the trade-off between H_2 and L_1 performance objectives. Figure 1 shows H_2 and L_1 norm variations with respect to α for $\mu = 0.1$. Figure 1 also provides a trade-off between the H_2 and L_1 norm for $\mu = 0.1$ which clearly shows an inverse proportionality trend between the two norms. Figure 2 provides similar trade-offs for $\mu = 0.95$. Table 1 shows the values of the H_2 norm, α corresponding to the lowest L_1 norm bound, and the L_1 norm bound.

Example 6.2. Consider the spring–mass–damper system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_\infty(t),$$

$$y(t) = [1 \quad 0]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0 \quad 1]\, w(t),$$

where x_1 and x_2 are the position and velocity of the mass, respectively and $\omega_n = 1$ rad/s and $\zeta = 0.4$. Furthermore, let the H_2 and L_1 performance variables be given by

$$z_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$z_\infty(t) = [1 \quad 0]\, x(t).$$

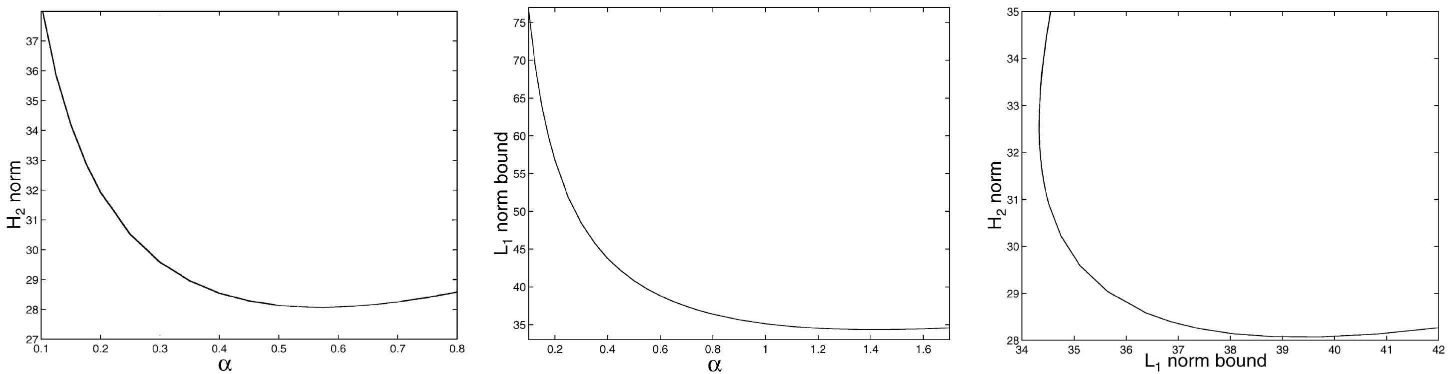

Fig. 1. Trade-off between H_2 and L_1 performance ($\mu = 0.1$): Example 6.1.

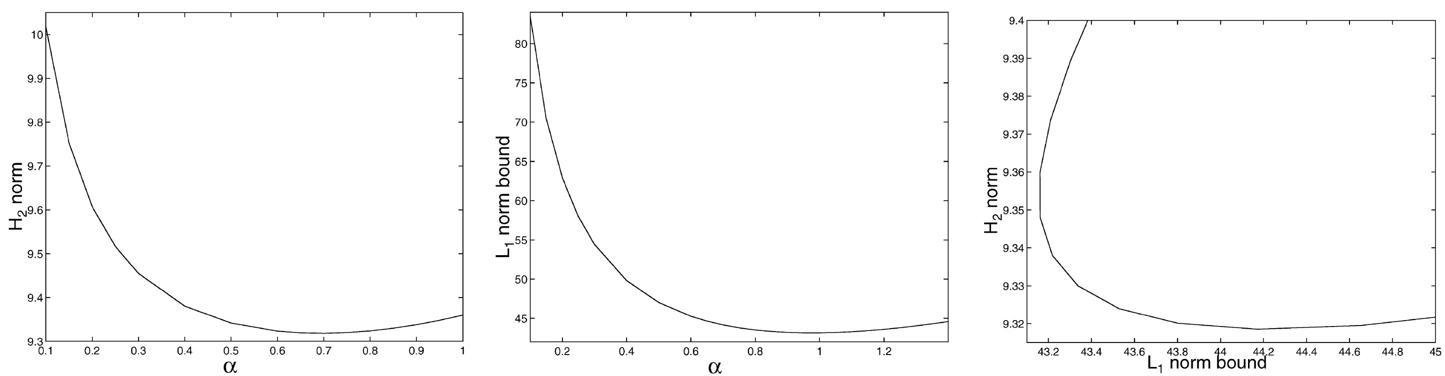

Fig. 2. Trade-off between H_2 and L_1 performance ($\mu = 0.95$): Example 6.1.

Table 1
Summary of design study: Example 6.1

μ	α	H_2 norm	L_1 norm bound
0.1	1.4	32.429	34.331
0.95	1.0	9.3599	43.161

Table 2
Summary of design study: Example 6.2

μ	α	H_2 norm	L_1 norm bound	Actual L_1 norm
0.1	0.8	1.8956	1.0647	0.8994

For $\mu = 0.1$ and $n_c = 2$, Table 2 shows the values of the H_2 norm, α corresponding to the lowest L_1 norm bound, the L_1 norm bound, and the actual L_1 norm. Note that the gap between the actual L_1 norm and the L_1 norm bound is 19.04%. Figure 3 compares the position response of the mixed H_2/L_1 full-order dynamic output feedback controller ($\mu = 0.1, \alpha = 0.8$) to an H_2 optimal linear-quadratic Gaussian controller with an L_∞ disturbance signal $w_\infty(t) = \sin 0.8t$. Note that the mixed H_2/L_1 controller reduces the maximum excursion of the position response by 37.2% over the H_2 optimal controller.

Example 6.3. In this example we consider a simply supported uniform beam with force actuation and position sensing [20]. The beam deflection $w(x, t)$ is governed by

$$m\frac{\partial^2 w(x,t)}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left[EI\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] + f(x,t), \tag{44}$$

with boundary conditions

$$w(x,t)|_{x=0,L} = 0, \quad EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=0,L} = 0,$$

where m is the beam mass and EI is the flexural rigidity. Using standard modal decomposition,

$$w(x,t) = \sum_{r=1}^{\infty} W_r(x)\,q_r(t),$$

where

$$\int_0^L mW_r^2(x)\,\mathrm{d}x = 1, \qquad W_r(x) = \sqrt{\frac{2}{ml}}\sin\frac{r\pi x}{L}.$$

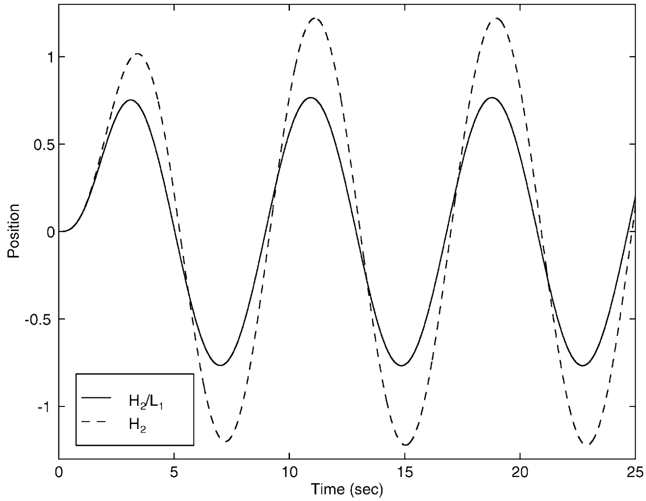

Fig. 3. Comparison of H_2 and mixed H_2/L_1 controllers: Example 6.2.

Hence, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \int_0^L f(x,t)\, W_r(x)\,\mathrm{d}x, \quad r = 1, 2, \ldots . \tag{45}$$

In this example, we place a position sensor at $x = 0.45L$ and a point force actuator at $x = 0.65L$. The disturbance is located at $x = 0.7L$ while the performance variable corresponds to the transverse beam velocity at $x = 0.53L$. Setting $L = \pi$, $m = EI = 2/\pi$, and $\zeta = 0.01$ the resulting state space model for five modes and problem data are given by

$$A = \underset{i=1,\ldots,5}{\text{block-diag}} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega i \end{bmatrix}, \quad \omega_i = i^2, \quad i = 1, \ldots, 5,$$

$$B = [0 \quad 0.891 \quad 0 \quad -0.809 \quad 0 \quad -0.156 \quad 0 \quad 0.951 \quad 0 \quad -0.7071]^{\mathrm{T}},$$

$$D_1 = \begin{bmatrix} 0 & 0.809 & 0 & -0.951 & 0 & 0.309 & 0 & 0.5878 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{1\infty} = [0 \quad 0.809 \quad 0 \quad -0.951 \quad 0 \quad 0.309 \quad 0 \quad 0.5878 \quad 0 \quad -1],$$

$$D_2 = [0 \quad 1], \quad D_{2\infty} = 0,$$

$$C = [0.9877 \quad 0 \quad -0.3090 \quad 0 \quad -0.8910 \quad 0 \quad 0.5878 \quad 0 \quad 0.7071 \quad 0],$$

$$E_1 = \begin{bmatrix} 0 & 0.9956 & 0 & -0.1873 & 0 & -0.9603 & 0 & 0.3681 & 0 & 0.8910 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{1\infty} = [0 \quad 0.9956 \quad 0 \quad -0.1873 \quad 0 \quad -0.9603 \quad 0 \quad 0.3681 \quad 0 \quad 0.8910],$$

$$E_2 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad E_{2\infty} = 0.$$

For this example we used the fixed-order mixed-norm H_2/L_1 dynamic compensation framework presented in this paper to design second-order ($n_c = 2$) controllers. Table 3 shows the values of the H_2 norm, α corresponding to the lowest L_1 norm bound, the L_1 norm bound, and the actual L_1 norm. Note that the gap between the actual L_1 norm and the L_1 norm bound is 36.15%. Figure (4) compares the transverse beam velocity at $x = 0.53L$ of the mixed-norm H_2/L_1 second-order dynamic output feedback controller ($\mu = 0.1, \alpha = 0.07$) to a second-order balanced truncated LQG controller with an L_∞ disturbance signal $w_\infty(t) = \sin t$. Note that the second-order mixed H_2/L_1 controller reduces the maximum excursion of the position response by 79.945% over the second-order balanced truncated LQG controller.

Table 3
Summary of design study: Example 6.3

μ	α	H_2 norm	L_1 norm bound	Actual L_1 norm
0.1	0.07	8.8141	7.1833	5.2760

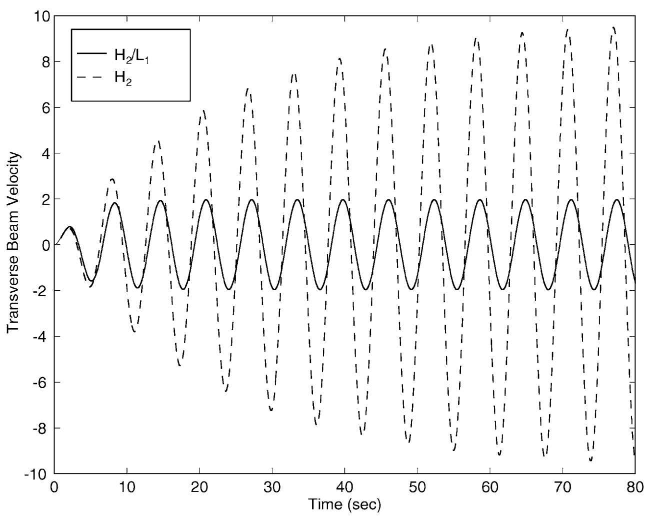

Fig. 4. Comparison of second-order balanced truncated LQG and second-order mixed H_2/L_1 controllers: Example 6.3.

7. Conclusion

This paper extended the Riccati equation approach for mixed H_2/L_1 static output feedback regulation [16] to dynamic output feedback compensation. Specifically, the fixed-structure controller synthesis framework was used to address the problem of mixed-norm H_2/L_1 controller synthesis via fixed-order (i.e., full- and reduced-order) dynamic compensation for multi-input/single-output systems. A quasi-Newton optimization algorithm was used to obtain disturbance rejection controllers for several illustrative examples.

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