Optimal nonlinear–nonquadratic feedback control for nonlinear discrete-time systems with $l_2$ and $l_\infty$ disturbances

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1. Introduction

In a recent paper [6] an optimality-based disturbance rejection nonlinear control framework for nonlinear continuous-time systems with bounded exogenous disturbances was developed. In this paper, analogous results are developed for the discrete-time case. Specifically, to address the optimality-based discrete-time disturbance rejection nonlinear control problem we extend the nonlinear–nonquadratic controller analysis and synthesis framework developed in [5]. The basic underlying ideas of the results in [5] are based on the fact that the steady-state solution to the Bellman equation is a control Lyapunov function for the nonlinear controlled system thus guaranteeing both optimality and stability. In this paper we extend the framework developed in [5] to address the problem of optimal nonlinear–nonquadratic discrete-time control that additionally guarantees disturbance rejection to bounded input disturbances. Specifically, using nonlinear discrete-time dissipation theory [1] with appropriate storage functions and supply rates we transform the nonlinear disturbance rejection problem into an optimal control problem. This is accomplished by properly modifying the cost functional to account for exogenous disturbances so that the solution of the modified optimal nonlinear control problem serves as the solution to the disturbance rejection problem.

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Our framework guarantees that the closed-loop nonlinear input–output map is dissipative with respect to general supply rates. Specializing to quadratic supply rates involving net system energy flow and weighted input and output energy our results guarantee passive and non-expansive (gain bounded) closed-loop input–output maps, respectively. In the special case where the controlled system is linear our results, with appropriate quadratic supply rates, specialize to the mixed-norm $H_2/H_\infty$ framework developed in [3] and the discrete-time analog of the continuous-time mixed $H_2$/positivity framework developed in [4].

The contents of the paper are as follows. In Section 2 we establish notation and mathematical preliminaries. In Section 3 we consider a nonlinear discrete-time system with bounded input disturbances and a nonlinear–nonquadratic performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees stability and dissipativity with respect to general supply rates. This result is then specialized to linear systems with quadratic supply rates and connections are drawn to the mixed-norm $H_2/H_\infty$ and mixed $H_2$/positivity linear-quadratic problems. Furthermore, as a special case of the framework developed in Section 3 we provide a Riccati equation characterization for capturing the $l_1$ norm ($l_\infty$ equi-induced norm) of a discrete-time linear time-invariant system. It is further shown that this framework can be used to enforce point-wise-in-time performance constraints ($l_1$ performance) providing worst-case amplification attenuation from $l_\infty$ input–output signals along with $H_2$ performance. In Section 4 we state a nonlinear discrete-time disturbance rejection control problem and transform the problem to an optimal control problem. Theorem 4.1 presents sufficient conditions that characterize an optimal nonlinear feedback controller that guarantees stability and dissipativity of the closed-loop system with respect to general supply rates. This result is then specialized to the linear-quadratic case to draw connections with the mixed-norm $H_2/H_\infty$ linear-quadratic regulator problem [3] and the mixed $H_2$/positivity control problem [4]. In Section 5 we specialize the results of Section 4 to discrete-time nonlinear affine systems controlled by nonlinear controllers subjected to bounded exogenous disturbances that minimize a nonlinear–nonquadratic performance criterion involving a nonlinear–nonquadratic function of the state and a linear and a quadratic function of the feedback control. In Section 6 we close the paper with conclusions.

2. Notation and mathematical preliminaries

In this section we establish definitions and notation. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^{n \times m}$ denote the set of real $n \times m$ matrices, let $\mathbb{H}^{n \times n}$ (resp., $\mathbb{P}^{n \times n}$) denote the set of $n \times n$ nonnegative (resp., positive) definite matrices, and let $\mathcal{N}^e$ denote the set of nonnegative integers. Furthermore, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix $A$ is nonnegative (resp., positive) definite and $A \geq B$ (resp., $A > B$) denotes the fact that $A - B \geq 0$ (resp., $A - B > 0$). Let $l_2$ (resp., $l_\infty$) denote square-summable (resp., bounded) sequences on $\mathcal{N}^e$ and let $H_2$ (resp., $H_\infty$) denote the Hardy space of real-rational transfer function matrices square-integrable (resp., bounded) on the unit disk with analytic continuation outside the unit disk.
In this paper we consider nonlinear discrete-time systems \( G \) of the form

\[
\begin{align*}
\dot{x}(k+1) &= f(x(k)) + J_1(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \\
\dot{z}(k) &= h(x(k)) + J_2(x(k))w(k),
\end{align*}
\]

where \( x \in \mathbb{R}^n, \ w \in \mathbb{R}^d, \ z \in \mathbb{R}^p, \ f : \mathbb{R}^n \to \mathbb{R}^n, \ J_1 : \mathbb{R}^n \to \mathbb{R}^{n \times d}, \ h : \mathbb{R}^n \to \mathbb{R}^p, \) and \( J_2 : \mathbb{R}^n \to \mathbb{R}^{p \times d}. \) We assume \( f(\cdot) \) has at least one equilibrium so that, without loss of generality, \( f(0) = 0 \) and \( h(0) = 0. \) For the dynamical system \( G \) given by Eqs. (1) and (2) a function \( r : \mathbb{R}^p \to \mathbb{R}^d \) is called a supply rate \([1]\) if it is locally summable, that is, for all input–output pairs \( w \in \mathbb{R}^d, \ z \in \mathbb{R}^p, \) it satisfies

\[
\sum_{k=k_1}^{k_2} |r(z(s), w(s))| < \infty, \quad k_1, k_2 \in \mathcal{N}.
\]

Definition 2.1 (Caines [1]). A system \( G \) of the form (1) and (2) is dissipative with respect to the supply rate \( r \) if there exists a nonnegative-definite function \( V : \mathbb{R}^n \to \mathbb{R} \), called a storage function, such that the dissipation inequality

\[
V(x(k)) \leq V(x(k_1)) + \sum_{i=k_1}^{k-1} r(z(i), w(i)),
\]

is satisfied for all \( k, k_1 \in \mathcal{N}, \) \( k_1 < k, \) and where \( x(k), k \in \mathcal{N}, \) is the solution to Eq. (1) with \( w(\cdot) \in \mathbb{R}^d. \)

Definition 2.2 (Goodwin and Sin [2]). A system \( G \) of the form (1) and (2) with \( x(0) = x_0 \) is nonexpansive if the solution \( x(k), k \in \mathcal{N}, \) to Eq. (1) satisfies

\[
\sum_{i=0}^{k} z^T(i)z(i) \leq \gamma^2 \sum_{i=0}^{k} w^T(i)w(i) + V(x_0),
\]

for all \( k \in \mathcal{N}, \ w(\cdot) \in l_2, \) and where \( \gamma > 0 \) is given.

Definition 2.3 (Goodwin and Sin [2]). A system \( G \) of the form (1) and (2) with \( x(0) = x_0 \) and \( p = d \) is passive if the solution \( x(k), k \in \mathcal{N}, \) to Eq. (1) satisfies

\[
2 \sum_{i=0}^{k} z^T(i)w(i) + V(x_0) \geq 0,
\]

for all \( k \in \mathcal{N} \) and \( w(\cdot) \in l_2. \)

Note that appropriate supply rates for testing nonexpansivity and passivity of \( G \) are \( r(z, w) = \gamma z^Tw - z^Tz \) and \( r(z, w) = 2z^Tw, \) respectively.

3. Dissipative nonlinear systems with bounded disturbances

In this section we present sufficient conditions for dissipativity for a class of nonlinear discrete-time systems with bounded energy and bounded amplitude disturbances. In
addition we consider the problem of evaluating a performance bound for a nonlinear–
nonquadratic cost functional. The cost bound is evaluated in closed form by relating the
cost functional to an underlying Lyapunov function that guarantees asymptotic stability
of the nonlinear system.

In this paper we restrict our attention to time-invariant infinite horizon systems.
For the following result, let \( \mathcal{D} \subset \mathbb{R}^n \) be an open set, assume \( 0 \in \mathcal{D} \), let \( L : \mathcal{D} \to \mathbb{R} \), let
\( f : \mathcal{D} \to \mathbb{R}^n \) such that \( f(0) = 0 \), and let \( h : \mathcal{D} \to \mathbb{R}^d \), \( J_1 : \mathcal{D} \to \mathbb{R}^{n \times d} \), and \( J_2 : \mathcal{D} \to \mathbb{R}^{p \times d} \)
such that \( h(0) = 0 \). Finally, let \( \mathcal{W} \subset \mathbb{R}^d \) be a subset of measurable functions and let
\( r : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R} \) be a given function.

First, we present sufficient conditions under which a nonlinear discrete-time system is
dissipative with respect to the supply rate \( r(z,w) \).

**Lemma 3.1.** Consider the nonlinear discrete-time system

\[
x(k + 1) = f(x(k)) + J_1(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \quad w(\cdot) \in \mathcal{W},
\]

\[
z(k) = h(x(k)) + J_2(x(k))w(k).
\]

Furthermore, assume there exist functions \( \Gamma : \mathcal{D} \to \mathbb{R}, \quad P_{1w} : \mathcal{D} \to \mathbb{R}^{1 \times d}, \quad P_{2w} : \mathcal{D} \to \mathbb{R}^{d \times d} \), and a nonnegative-definite function \( V : \mathcal{D} \to \mathbb{R} \) such that

\[
P_{1w}(0) = 0,
\]

\[
V(0) = 0,
\]

\[
P_{1w}(x)w + w^TP_{2w}(x)w \leq r(z,w) + \Gamma(x), \quad x \in \mathcal{D}, \quad w \in \mathcal{W},
\]

\[
V(f(x) + J_1(x)w) = V(f(x)) + P_{1w}(x)w + w^TP_{2w}(x)w, \quad x \in \mathcal{D}, \quad w \in \mathcal{W},
\]

\[
V(f(x)) - V(x) + \Gamma(x) \leq 0, \quad x \in \mathcal{D}.
\]

Then the solution \( x(k), \quad k \in \mathcal{N}, \) of Eq. (6) satisfies

\[
V(x(k + 1)) \leq \sum_{i=0}^{k} r(z(i),w(i)) + V(x_0), \quad k \in \mathcal{N}, \quad w(\cdot) \in \mathcal{W}.
\]

**Proof.** Let \( x(k), \quad k \in \mathcal{N}, \) satisfy Eq. (6) and let \( w(\cdot) \in \mathcal{W}. \) Then it follows from Eqs. (10)–(12) that for all \( k \in \mathcal{N} \),

\[
\Delta V(x(k)) = V(x(k + 1)) - V(x(k))
\]

\[
= V(f(x(k)) + J_1(x(k))w(k)) - V(x(k))
\]

\[
= V(f(x(k))) + P_{1w}(x(k))w(k) + w^TP_{2w}(x(k))w - V(x(k))
\]

\[
= V(f(x(k))) - V(x(k)) + \Gamma(x(k)) + P_{1w}(x(k))w(k)
\]

\[
+ w^TP_{2w}(x(k))w(k) - \Gamma(x(k))
\]

\[
\leq r(z(k),w(k)),
\]
which yields

$$V(x(k + 1)) - V(x_0) \leq \sum_{i=0}^{k} r(z(i), w(i)), \quad k \in \mathcal{N}. \quad \square$$

For the next result let \( L : \mathcal{D} \to \mathbb{R} \) be given.

**Theorem 3.1.** Consider the nonlinear system given by Eqs. (6) and (7) with performance functional

$$J(x_0) \triangleq \sum_{k=0}^{\infty} L(x(k)), \quad (15)$$

where \( x(k), \quad k \in \mathcal{N}, \) solves Eq. (6) with \( w(k) \equiv 0 \). Assume there exist functions \( \Gamma : \mathcal{D} \to \mathbb{R}, \quad P_{1w} : \mathcal{D} \to \mathbb{R}^{1 \times d}, \quad P_{2w} : \mathcal{D} \to \mathbb{R}^{d \times d}, \) and \( V : \mathcal{D} \to \mathbb{R} \) such that

$$P_{1w}(0) = 0, \quad (16)$$

$$V(0) = 0, \quad (17)$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \quad (18)$$

$$P_{1w}(x)w + w^T P_{2w}(x)w \leq r(z, w) + L(x) + \Gamma(x), \quad x \in \mathcal{D}, \quad w \in l_2, \quad (19)$$

$$V(f(x)) - V(x) < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \quad (20)$$

$$V(f(x) + J_1(x)w) = V(f(x)) + P_{1w}(x)w + w^T P_{2w}(x)w, \quad x \in \mathcal{D}, \quad w \in l_2, \quad (21)$$

$$L(x) + V(f(x)) - V(x) + \Gamma(x) = 0, \quad x \in \mathcal{D}. \quad (22)$$

Then there exists a neighborhood \( \mathcal{D}_0 \subset \mathcal{D} \) of the origin such that if \( x_0 \in \mathcal{D}_0, \) then \( x(k) \equiv 0 \) is a locally asymptotically stable solution of Eq. (6) with \( w(k) \equiv 0 \). If, in addition, \( \Gamma(x) \geq 0, \quad x \in \mathcal{D}, \) then

$$J(x_0) \leq J(x_0) = V(x_0), \quad (23)$$

where

$$J(x_0) \triangleq \sum_{k=0}^{\infty} [L(x(k)) + \Gamma(x(k))] \quad (24)$$

and where \( x(k), \quad k \in \mathcal{N}, \) is a solution to Eq. (6) with \( w(k) \equiv 0 \). Furthermore, the solution \( x(k), \quad k \in \mathcal{N}, \) of Eq. (6) satisfies the dissipativity constraint

$$\sum_{i=0}^{k} r(z(i), w(i)) + V(x_0) \geq 0, \quad k \in \mathcal{N}, \quad w(\cdot) \in l_2. \quad (25)$$

Finally, if \( \mathcal{D} = \mathbb{R}^n, \) \( w(k) \equiv 0, \) and

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \quad (26)$$

then the solution \( x(k) = 0, \quad k \in \mathcal{N}, \) of Eq. (6) is globally asymptotically stable.
Proof. Let \( x(k), k \in \mathcal{N} \), satisfy Eq. (6). Then

\[
\Delta V(x(k)) = V(x(k+1)) - V(x(k)) = V(f(x(k))) - V(x(k)) + P_{1w}(x(k))w(k) + w^T(k)P_{2w}(x(k))w(k), \quad k \in \mathcal{N}.
\]

Hence, with \( w(k) \equiv 0 \), it follows from Eq. (20) that

\[
\Delta V(x(k)) < 0, \quad k \in \mathcal{N}, \quad x(k) \neq 0.
\]

Thus, from Eqs. (16), (18), and (28) it follows that \( V(\cdot) \) is a Lyapunov function [9] for Eq. (6), which proves local asymptotic stability of the solution \( x(k) \equiv 0 \) with \( w(k) \equiv 0 \). Consequently, \( x(k) \to 0 \) as \( k \to \infty \) for all initial conditions \( x_0 \in \mathcal{D}_0 \) for some neighborhood \( \mathcal{D}_0 \subseteq \mathcal{D} \) of the origin. Now, for \( w(k) \equiv 0 \), Eq. (27) implies that

\[
0 = -\Delta V(x(k)) + V(f(x(k))) - V(x(k)), \quad k \in \mathcal{N}.
\]

Next, if \( \Gamma(x) \geq 0 \), \( x \in \mathcal{D} \), Eq. (22) implies

\[
L(x(k)) = -\Delta V(x(k)) + L(x(k)) + V(f(x(k))) - V(x(k)) \\
\leq -\Delta V(x(k)) + L(x(k)) + V(f(x(k))) - V(x(k)) + \Gamma(x(k)) \\
= -\Delta V(x(k)).
\]

Now, summing over \([0, k]\) yields

\[
\sum_{i=0}^{k} L(x(i)) \leq -V(x(k+1)) + V(x_0).
\]

Letting \( k \to \infty \) and noting that \( V(x(k)) \to 0 \) as \( k \to \infty \) for all \( x_0 \in \mathcal{D}_0 \) yields \( J(x_0) \leq V(x_0) \). Next, let \( x(k), k \in \mathcal{N} \), satisfy Eq. (6) with \( w(k) \equiv 0 \). Then it follows from Eq. (22) that

\[
L(x(k)) + \Gamma(x(k)) = -\Delta V(x(k)) + L(x(k)) + V(f(x(k))) - V(x(k)) + \Gamma(x(k)) \\
= -\Delta V(x(k)).
\]

Summing over \([0, k]\) yields

\[
\sum_{i=0}^{k} [L(x(i)) + \Gamma(x(i))] = -V(x(k+1)) + V(x_0).
\]

Now letting \( k \to \infty \) yields \( J(x_0) = V(x_0) \). Finally, it follows that Eqs. (16), (17), (19), (21), (22) imply Eqs. (18)–(12) and hence, with \( \Gamma(x) \) replaced by \( L(x) + \Gamma(x) \) and \( \mathcal{N} = \ell_2 \), Lemma 3.1 yields

\[
V(x(k+1)) \leq \sum_{i=0}^{k} r(z(i),w(i)) + V(x_0), \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N}.
\]
Now Eq. (25) follows by noting that $V(x(k+1)) \geq 0$, $k \in \mathcal{N}$. Finally, for $\mathcal{D} = \mathbb{R}^n$ global asymptotic stability of the solution $x(k) = 0$, $k \in \mathcal{N}$, is a direct consequence of the radially unbounded condition (26) on $V(x)$, $x \in \mathbb{R}^n$. □

3.1. Specialization to dissipative systems with quadratic supply rates

In this section we consider the special case in which $r(z,w)$ is a quadratic functional. Specifically, let $h: \mathcal{D} \to \mathbb{R}^p$, $J_2: \mathcal{D} \to \mathbb{R}^{p \times d}$, $\hat{Q} \in \mathbb{R}^{p \times p}$, $\hat{S} \in \mathbb{R}^{p \times d}$, $\hat{R} \in \mathbb{R}^{d \times d}$, and

$$r(z,w) = z^T \hat{Q}z + 2z^T \hat{S}w + w^T \hat{R}w,$$

such that

$$N(x) \overset{\triangle}{=} J_2^T(x) \hat{Q} J_2(x) + J_2^T(x) \hat{S} + \hat{S}^T J_2(x) + \hat{R} > P_{2w}(x), \quad x \in \mathcal{D}.$$  

Furthermore, let $L(x) \geq 0$, $x \in \mathcal{D}$. Then

$$\Gamma(x) = \left[ \frac{1}{2} P_{1w}(x) - J_2^T(x) \hat{Q} h(x) - \hat{S}^T h(x) \right]^T \left( N(x) - P_{2w}(x) \right)^{-1} \left[ \frac{1}{2} P_{1w}(x) - J_2^T(x) \hat{Q} h(x) - \hat{S}^T h(x) \right] - h^T(x) \hat{Q} h(x),$$

satisfies Eq. (19) since in this case

$$L(x) + \Gamma(x) + r(z,w) - P_{1w}(x)w - w^T P_{2w}(x)w$$

$$= L(x) + \left[ \frac{1}{2} P_{1w}(x) - J_2^T(x) \hat{Q} h(x) - \hat{S}^T h(x) \right]^T \left( N(x) - P_{2w}(x) \right)^{-1} \left[ \frac{1}{2} P_{1w}(x) - J_2^T(x) \hat{Q} h(x) - \hat{S}^T h(x) \right] - h^T(x) \hat{Q} h(x)$$

$$- \hat{S}^T(x) h(x) - (N(x) - P_{2w}(x))w \geq 0.$$  

Corollary 3.1. Let $L(x) \geq 0$, $x \in \mathcal{D}$, and consider the nonlinear system given by Eqs. (6) and (7) with performance functional

$$J(x_0) \overset{\triangle}{=} \sum_{k=0}^{\infty} L(x(k)),$$

where $x(k)$, $k \in \mathcal{N}$, solves Eq. (6) with $w(k) \equiv 0$. Assume there exists functions $P_{1w}: \mathcal{D} \to \mathbb{R}^{d \times d}$, $P_{2w}: \mathcal{D} \to \mathbb{R}^{d \times d}$, and $V: \mathcal{D} \to \mathbb{R}$ such that

$$P_{1w}(0) = 0,$$

$$V(0) = 0,$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0,$$

$$V(f(x)) - V(x) < 0, \quad x \in \mathcal{D}, \quad x \neq 0,$$

$$V(f(x) + J_1(x)w) = V(f(x)) + P_{1w}(x)w + w^T P_{2w}(x)w, \quad x \in \mathcal{D}, \quad w \in l_2,$$
\[ \gamma^2 I_d - J_2^T(x)J_2(x) - P_{2u}(x) > 0, \quad x \in \mathcal{D}, \]  
\[ L(x) + V(f(x)) - V(x) + \Gamma(x) = 0, \quad x \in \mathcal{D}, \]  
where
\[ \Gamma(x) = \left[ \frac{1}{2} P_{1w}^T(x) + J_2^T(x)h(x) \right] \left[ \gamma^2 I_d - J_2^T(x)J_2(x) - P_{2u}(x) \right]^{-1} \]
\[ \times \left[ \frac{1}{2} P_{1w}^T(x) + J_2^T(x)h(x) \right] + h^T(x)h(x). \]

Then there exists a neighborhood \( \mathcal{D}_0 \subset \mathcal{D} \) of the origin such that if \( x_0 \in \mathcal{D}_0 \), then \( x(k) \equiv 0 \) is a locally asymptotically stable solution of Eq. (6) with \( w(k) \equiv 0 \). Furthermore,
\[ J(x_0) \leq J(x_0) = V(x_0), \]  
where
\[ J(x_0) \overset{\triangle}{=} \sum_{k=0}^{\infty} \left[ L(x(k)) + \Gamma(x(k)) \right] \]
and where \( x(k), \quad k \in \mathcal{N}, \) solves Eq. (6) with \( w(k) \equiv 0 \). Furthermore, the solution \( x(k), \quad k \in \mathcal{N}, \) of Eq. (6) satisfies the nonexpansivity constraint
\[ \sum_{i=0}^{k} z^T(i)z(i) \leq \gamma^2 \sum_{i=0}^{k} w^T(i)w(i) + V(x_0), \quad k \in \mathcal{N}, \; w(\cdot) \in l_2. \]

Finally, if \( \mathcal{D} = \mathbb{R}^n \), \( w(k) \equiv 0 \), and
\[ V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty, \]
then the solution \( x(k) = 0, \quad k \in \mathcal{N}, \) of Eq. (6) is globally asymptotically stable.

**Proof.** With \( \hat{Q} = -I, \quad \hat{S} = 0, \) and \( \hat{R} = \gamma^2 I \), it follows from Eq. (30) that \( \Gamma(x) \) given by Eq. (39) satisfies Eq. (19). The result now follows as a direct consequence of Theorem 3.1. \( \Box \)

**Remark 3.1.** Note that if \( L(x) = h^T(x)h(x) \) in Corollary 3.1 then \( \Gamma(x) \) can be chosen as
\[ \Gamma(x) = \left[ \frac{1}{2} P_{1w}^T(x) + J_2^T(x)h(x) \right] \left[ \gamma^2 I_d - J_2^T(x)J_2(x) - P_{2u}(x) \right]^{-1} \]
\[ \times \left[ \frac{1}{2} P_{1w}^T(x) + J_2^T(x)h(x) \right]. \]

The framework presented in Corollary 3.1 is an extension of the mixed-norm \( H_2/H_\infty \) framework of Haddad et al. [3] to nonlinear systems. Specifically, letting \( f(x) = Ax, \)
\( J_1(x) = D, \quad h(x) = Ex, \quad J_2(x) = 0, \quad L(x) = x^T Rx, \) and \( V(x) = x^T Px, \) where \( A \in \mathbb{R}^{n \times n}, \quad D \in \mathbb{R}^{n \times d}, \quad E \in \mathbb{R}^{p \times n}, \quad R \geq E^T E > 0, \) and \( P \in \mathbb{P}^{n \times n} \) satisfies
\[ P = A^TPA + A^TPD(\gamma^2 I_d - D^TPD)^{-1}D^TPA + R, \]  
\[ (44) \]
it follows from Remark 3.1 that \( \Gamma(x) \) can be chosen as \( \Gamma(x) = x^T A^T P D (\gamma^2 I_d - D^T P D)^{-1} D^T P A x \), where \( \gamma^2 I_d - D^T P D > 0 \). Hence, with \( x_0 = 0 \), Corollary 3.1 implies that

\[
\sum_{i=0}^{k} x^T(i) R x(i) \leq \gamma^2 \sum_{i=0}^{k} w^T(i) w(i), \quad k \in \mathcal{N}, \; w(\cdot) \in l_2,
\]

or, equivalently, the \( H_\infty \) norm of

\[
G(z) \sim \begin{bmatrix} A & D \\ E & 0 \end{bmatrix}
\]

satisfies

\[
\|G\|_\infty \triangleq \sup_{\theta \in [0,2\pi]} \sigma_{\text{max}}(G(e^{j\theta})) \leq \gamma,
\]

where \( \sigma_{\text{max}}(\cdot) \) denotes the maximum singular value. Now, Eq. (40) implies

\[
\sum_{k=0}^{\infty} x^T(k) R x(k) \leq \sum_{k=0}^{\infty} x^T(k) [R + A^T P D (\gamma^2 I_d - D^T P D)^{-1} D^T P A] x(k)
\]

\[
= \sum_{k=0}^{\infty} x_0^T(\omega) A^T [R + A^T P D (\gamma^2 I_d - D^T P D)^{-1} D^T P A] A^T x_0,
\]

where \( x(k), \; k \in \mathcal{N} \), solves Eq. (6) with \( w(k) \equiv 0 \). Now, as is common practice [8], we eliminate explicit dependence on the initial condition \( x_0 \) by assuming \( x_0 x_0^T \) has expected value \( V \), that is, \( E[x_0 x_0^T] = V \), where \( E \) denotes expectation. Invoking this step leads to

\[
E \left[ \sum_{k=0}^{\infty} x^T(k) R x(k) \right] = E \left[ \sum_{k=0}^{\infty} x_0^T(\omega) A^T R A^T x_0 \right] = E[x_0^T \hat{P} x_0] = \operatorname{tr} \hat{P} V,
\]

where

\[
\hat{P} = A^T \hat{P} A + R
\]

and

\[
E \left[ \sum_{k=0}^{\infty} x^T(k) [R + A^T P D (\gamma^2 I_d - D^T P D)^{-1} D^T P A] x(k) \right]
\]

\[
= E \left[ \sum_{k=0}^{\infty} x_0^T(\omega) [R + A^T P D (\gamma^2 I_d - D^T P D)^{-1} D^T P A] A^T x_0 \right] = E[x_0^T P x_0] = \operatorname{tr} PV,
\]

where \( P \) satisfies Eq. (44). Hence \( \|G\|_2^2 = \operatorname{tr} \hat{P} V \leq \operatorname{tr} PV \) which implies that \( \|G(x_0)\| \) given by Eq. (41) provides an upper bound to the \( H_2 \) norm of \( G(z) \).
Corollary 3.2. Let \( L(x) \geq 0, x \in \mathcal{D}, \ p = d, \) and consider the nonlinear system given by Eqs. (6) and (7) with performance functional

\[
J(x_0) = \sum_{k=0}^{\infty} L(x(k)),
\]

(47)

where \( x(k), \ k \in \mathcal{N}, \) solves Eq. (6) with \( w(k) = 0. \) Assume there exist functions \( P_{1w}: \mathcal{D} \to \mathbb{R}^{1 \times d}, \ P_{2w}: \mathcal{D} \to \mathbb{R}^{d \times d}, \) and \( V: \mathcal{D} \to \mathbb{R} \) such that

\[
P_{1w}(0) = 0, \quad V(0) = 0, \quad V(x) > 0, \quad x \in \mathcal{D}, \ x \neq 0, \quad V(f(x)) - V(x) < 0, \quad x \in \mathcal{D}, \ x \neq 0, \quad V(f(x) + J_1(x)w) = V(f(x)) + P_{1w}(x)w + w^T P_{2w}(x)w, \quad x \in \mathcal{D}, \ w \in I_2, \quad J_2(x) + J_2^T(x) - P_{2w}(x) > 0, \quad x \in \mathcal{D}, \quad L(x) + V(f(x)) - V(x) + \Gamma(x) = 0, \quad x \in \mathcal{D},
\]

(50)

(51)

(52)

(53)

(54)

(55)

where

\[
\Gamma(x) = \left[ \frac{1}{2} P_{1w}(x) - h(x) \right]^T \left[ J_2(x) + J_2^T(x) - P_{2w}(x) \right]^{-1} \left[ \frac{1}{2} P_{1w}(x) - h(x) \right].
\]

(55)

Then there exists a neighborhood \( \mathcal{D}_0 \subset \mathcal{D} \) of the origin such that if \( x_0 \in \mathcal{D}_0, \) then \( x(k) \equiv 0 \) is a locally asymptotically stable solution of Eq. (6) with \( w(k) \equiv 0. \) Furthermore,

\[
J(x_0) \leq \mathcal{J}(x_0) = V(x_0),
\]

(56)

where

\[
\mathcal{J}(x_0) = \sum_{k=0}^{\infty} [L(x(k)) + \Gamma(x(k))]
\]

(57)

and where \( x(k), \ k \in \mathcal{N}, \) solves Eq. (6) with \( w(k) \equiv 0. \) Furthermore, the solution \( x(k), \ k \in \mathcal{N}, \) of Eq. (6) satisfies the passivity constraint

\[
2 \sum_{i=0}^{k} z^T(i)w(i) + V(x_0) \geq 0, \quad k \in \mathcal{N}, \ w(\cdot) \in I_2.
\]

(58)

Finally, if \( \mathcal{D} = \mathbb{R}^n, \ w(k) \equiv 0, \) and

\[
V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty,
\]

(59)

then the solution \( x(k) = 0, \ k \in \mathcal{N}, \) of Eq. (6) is globally asymptotically stable.
Proof. With $\dot{Q} = 0$, $\dot{S} = I$, and $\dot{R} = 0$, it follows from Eq. (30) that $\Gamma(x)$ given by Eq. (55) satisfies Eq. (19). The result now follows as a direct consequence of Theorem 3.1.

Note that the framework presented in Corollary 3.2 is an extension of the continuous-time $H_2$-positivity framework of Haddad and Bernstein [4] to nonlinear discrete-time systems. Specifically, letting $f(x) = Ax$, $J_1(x) = D$, $h(x) = Ex$, $J_2(x) = E_\infty$, $L(x) = x^T Rx$, $V(x) = x^T Px$, and $\Gamma(x) = x^T (D^T PA - E)^T (E_\infty + E_\infty^T - D^T PD)^{-1} (D^T PA - E)x$, where $E_\infty + E_\infty^T - D^T PD > 0$ and where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{n \times d}$, $E \in \mathbb{R}^{d \times n}$, $E_\infty \in \mathbb{R}^{d \times d}$, $R \in \mathbb{R}^{n \times n}$, and $P \in \mathbb{P}^{n \times n}$ satisfies

$$P = A^T PA + (D^T PA - E)^T (E_\infty + E_\infty^T - D^T PD)^{-1} (D^T PA - E) + R, \quad (60)$$

it follows from Corollary 3.2, with $x_0 = 0$, that

$$\sum_{i=0}^{k} w^T(i)z(i) \geq 0, \quad k \in \mathcal{N}, \quad w(\cdot) \in l_2, \quad (61)$$

or, equivalently,

$$G_\infty(z) + G_\infty^*(z) \geq 0, \quad |z| > 1, \quad (62)$$

where

$$G_\infty(z) \sim \begin{bmatrix} A & D \\ E & E_\infty \end{bmatrix}. \quad$$

Now, using similar arguments as in the $H_\infty$ case, Eq. (56) implies

$$\text{tr} \dot{P} V = E \left[ \sum_{k=0}^{\infty} x^T(k) R x(k) \right]$$

$$\leq E \left[ \sum_{k=0}^{\infty} x^T(k) [R + (D^T PA - E)^T (E_\infty + E_\infty^T - D^T PD)^{-1} (D^T PA - E)] x(k) \right]$$

or, equivalently, since

$$E \left[ \sum_{k=0}^{\infty} x^T(k) [R + (D^T PA - E)^T (E_\infty + E_\infty^T - D^T PD)^{-1} (D^T PA - E)] x(k) \right]$$

$$= E \left[ \sum_{k=0}^{\infty} x_0^T A^k [R + (D^T PA - E)^T (E_\infty + E_\infty^T - D^T PD)^{-1} (D^T PA - E)] A^k x_0 \right]$$

$$= E [x_0^T P x_0] = \text{tr} PV,$$

$$\|G\|_2^2 = \text{tr} \dot{P} V \leq \text{tr} PV$$

which implies that $\mathcal{F}(x_0)$ given by Eq. (57) provides an upper bound to the $H_2$ norm of $G(z)$. 

Next, we present a result in which \( \mathcal{W} \) is assumed to be a bounded subset of \( l_2 \). Specifically, define the subset of square-summable bounded disturbances

\[
\mathcal{W}_\beta = \left\{ w(\cdot) : \sum_{i=0}^{k-1} w^T(i)w(i) \leq \beta, \ k \in \mathcal{N} \right\},
\]

where \( \beta > 0 \). Furthermore, let \( L : \mathcal{D} \to \mathbb{R} \) be given such that \( L(x) \geq 0, \ x \in \mathcal{D} \).

**Theorem 3.2.** Let \( \gamma > 0, L(x) \geq 0, x \in \mathcal{D} \), and consider the nonlinear system (6) with performance functional (15). Assume there exist functions \( P_{1w} : \mathcal{D} \to \mathbb{R}^{1 \times d}, P_{2w} : \mathcal{D} \to \mathbb{R}^{d \times d} \), and \( V : \mathcal{D} \to \mathbb{R} \) such that

\[
P_{1w}(0) = 0,
\]

\[
V(0) = 0,
\]

\[
V(x) > 0, \ x \in \mathcal{D}, \ x \neq 0,
\]

\[
V(f(x)) - V(x) < 0, \ x \in \mathcal{D}, \ x \neq 0,
\]

\[
V(f(x) + J_1(x)w) = V(f(x)) + P_{1w}(x)w + w^TP_{2w}(x)w, \ x \in \mathcal{D}, \ w \in \mathcal{W}_\beta,
\]

\[
\frac{\gamma}{\beta}I_d - P_{2w}(x) > 0, \ x \in \mathcal{D},
\]

\[
L(x) + V(f(x)) - V(x) + \frac{1}{4}P_{1w}(x)(\frac{\gamma}{\beta}I_d - P_{2w}(x))^{-1}P_{1w}(x) = 0, \ x \in \mathcal{D}.
\]

Then there exists a neighborhood \( \mathcal{D}_0 \subset \mathcal{D} \) of the origin such that if \( x_0 \in \mathcal{D}_0 \), then \( x(k) \equiv 0 \) is a locally asymptotically stable solution of Eq. (6) with \( w(k) \equiv 0 \). Furthermore,

\[
J(x_0) \leq J(x_0) = V(x_0),
\]

where

\[
J(x_0) = \sum_{k=0}^{\infty} \left[ L(x(k)) + \Gamma(x(k)) \right],
\]

\[
\Gamma(x) = \frac{1}{4}P_{1w}(x) \left( \frac{\gamma}{\beta}I_d - P_{2w}(x) \right)^{-1}P_{1w}(x)
\]

and where \( x(k), k \in \mathcal{N} \), solves Eq. (6) with \( w(k) \equiv 0 \). Furthermore, if \( x_0 = 0 \) then the solution \( x(k), k \in \mathcal{N} \), of Eq. (6) satisfies

\[
V(x(k)) \leq \gamma, \ k \in \mathcal{N}, \ w(\cdot) \in \mathcal{W}_\beta.
\]

Finally, if \( \mathcal{D} = \mathbb{R}^n \), \( w(k) \equiv 0 \), and

\[
V(x) \to \infty \text{ as } \|x\| \to \infty,
\]

then the solution \( x(k) = 0, k \in \mathcal{N} \), of Eq. (6) is globally asymptotically stable.
Proof. The proofs for local and global asymptotic stability and the performance bound (71) are identical to the proofs of local and global asymptotic stability given in Theorem 3.1 and the performance bound (23). Next, with \( r(z,w) = (\gamma/\beta)w^Tw \) and \( \Gamma(x) \) given by Eq. (73), it follows from Lemma 3.1 that

\[
V(x(k)) \leq \frac{\gamma}{\beta} \sum_{i=0}^{k-1} w^T(i)w(i), \quad k > 0, \ k \in \mathcal{N}, \ w(\cdot) \in \mathcal{W},
\]

which yields (74).

3.2. A Riccati equation characterization for mixed \( H_2/H_1 \) performance

In this section we consider yet another linearization of the nonlinear system (6) and (7) with bounded amplitude persistent \( l_1 \) disturbances. Specifically, let \( f(x) = Ax, \ J_1(x) = D, \ h(x) = Ex, \) and \( J_2(x) = E_\infty \), where \( A \in \mathbb{R}^{n \times n}, \ D \in \mathbb{R}^{n \times d}, \ E \in \mathbb{R}^{p \times n}, \) \( E_\infty \in \mathbb{R}^{p \times d}, \) and \( A \) is asymptotically stable so that

\[
x(k+1) = Ax(k) + Dw(k), \quad x(0) = 0, \ k \in \mathcal{N}, \ w(\cdot) \in \mathcal{W}, \quad z(k) = Ex(k) + E_\infty w(k),
\]

where \( \mathcal{W} \) consists of unit-peak input signals defined by

\[
\mathcal{W} = \{ w(\cdot) : w^T(k)w(k) \leq 1, \ k \in \mathcal{N} \}.
\]

The following result provides an upper bound to the \( l_1 \) norm (\( l_\infty \) equi-induced norm) of the convolution operator \( G \) of the linear time-invariant system (76) and (77) given by

\[
\|G\|_1 \triangleq \sup_{w(\cdot) \in \mathcal{W}} \left\{ \sup_{k \in \mathcal{N}} \|z(k)\| \right\},
\]

where \( \|\cdot\| \) denotes the Euclidean vector norm. From an input–output point of view the \( l_1 \) norm captures the worst-case amplification from input disturbance signals to output signals, where the signal size is taken to be the supremum over time of the signal’s point-wise-in-time Euclidean norm.

**Theorem 3.3.** Let \( \alpha > 1 \) and consider the linear system (76) and (77). Then

\[
\|G\|_1 \leq \sigma_{\max}^{1/2}(EP^{-1}E^T) + \sigma_{\max}^{1/2}(E_\infty E_\infty^T),
\]

where \( P > 0 \) satisfies \( (\alpha - 1)I_d - \alpha D^TPD > 0 \) and

\[
P \geq \alpha A^TPA + \alpha^2 A^TPD[(\alpha - 1)I_d - \alpha D^TPD]^{-1}D^TPA.
\]

**Proof.** Let \( N > 0, \ N \in \mathcal{N}, \) and consider the dilated linear system

\[
\ddot{x}(k+1) = A_\alpha \ddot{x}(k) + D_\alpha v(k), \quad \dot{x}(0) = 0, \ k \in \mathcal{N},
\]
\[ z(k) = x^{-(k-N)/2}(E\tilde{x}(k) + E_{\infty}v(k)), \quad (82) \]

where \( A_2 \triangleq \sqrt{A} \), \( D_2 \triangleq \sqrt{D} \), \( \tilde{x}(k) \triangleq x^{(k-N)/2}x(k) \), and \( v(k) \triangleq x^{(k-N)/2}w(k) \). Note that Eqs. (81) and (82) are equivalent to Eqs. (76) and (77). Furthermore, note that if \( w(\cdot) \in \mathcal{W} \) then \( v(\cdot) \in \mathcal{V} \), where \( \mathcal{V} \triangleq \{ v(\cdot) : \sum_{k=0}^{N-1} v^T(k)v(k) \leq 1/(x-1) \} \). Hence,

\[ \| G \| \leq \sup_{N \geq 0} \sup_{v(\cdot) \in \mathcal{V}} \| z(N) \|. \]

Next, with \( f(x) = A_2\tilde{x}(k), J_1(x) = D_2, V(x) = \tilde{x}^T \mathcal{F} \tilde{x}, P_{1w}(x) = 2\tilde{x}^T A_2^T P D_2, P_{2w}(x) = D_2^T P D_2, \mathcal{W}_\beta = \mathcal{V}, \beta = 1/(x-1), \gamma = 1, \) and \( L(x) = \tilde{x}^T R \tilde{x} \), where \( R \in \mathbb{R}^{n \times n} \) is an arbitrary positive definite matrix, it follows from Theorem 3.2 that if there exists \( P > 0 \) such that

\[ P - x A^T P A + x^2 A^T P D [(x - 1)I_d - x D^T P D]^{-1} D^T P A + R, \quad (83) \]

then

\[ \tilde{x}^T(N) \mathcal{F} \tilde{x}(N) \leq 1, \]

and hence, since \( x(N) = \tilde{x}(N) \), for all \( v(\cdot) \in \mathcal{V} \)

\[ \| z(N) \| = \| E x(N) + E_{\infty} w(N) \| \leq \sigma_{\max}^{1/2}(E^T P^{-1} E^T) + \sigma_{\max}^{1/2}(E_{\infty} E_{\infty}^T). \]

The result is now immediate by noting that Eq. (83) is equivalent to Eq. (80). \( \square \)

Note that the Riccati inequality (80) is equivalent to the linear matrix inequality

\[ \begin{bmatrix} P & 0 \\ 0 & (x-1)I_d \end{bmatrix} \geq x \begin{bmatrix} A^T \\ D^T \end{bmatrix} P \begin{bmatrix} A & D \end{bmatrix}, \quad (84) \]

or, equivalently,

\[ \begin{bmatrix} P & 0 \\ 0 & (x-1)I_d \end{bmatrix} \geq x \begin{bmatrix} A & D \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & (x-1)I_d \end{bmatrix} \begin{bmatrix} A & D \\ 0 & 0 \end{bmatrix}. \quad (85) \]

Next, since \( x > 1 \), using Schur compliments, Eq. (85) is equivalent to

\[ \begin{bmatrix} P^{-1} & 0 \\ 0 & \frac{1}{(x-1)}I_d \end{bmatrix} \geq x \begin{bmatrix} A & D \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & \frac{1}{(x-1)}I_d \end{bmatrix} \begin{bmatrix} A & D \\ 0 & 0 \end{bmatrix}^T. \quad (86) \]

Now, letting \( \mathcal{Z} = P^{-1} \) in Eq. (86) yields

\[ \mathcal{Z} \geq x A \mathcal{Z} A^T + \frac{x}{x-1} D D^T. \quad (87) \]

Hence, Theorem 3.3 yields

\[ \| G \| \leq \sigma_{\max}^{1/2}(E \mathcal{Z} E^T) + \sigma_{\max}^{1/2}(E_{\infty} E_{\infty}^T), \quad (88) \]
where $\varphi$ satisfies Eq. (87). It is interesting to note that in the case where $E_\infty = 0$ the solution $\varphi$ to Eq. (87) satisfies the bound

$$Q \leq \varphi,$$

where $Q$ satisfies

$$Q = AQ^T + DD^T$$

and hence

$$\|G\|^2 = \text{tr} E Q E^T \leq \text{tr} E \varphi E^T.$$  

Thus, Eq. (87) can be used to provide a trade-off between $H_2$ and mixed $H_2/\ell_1$ performance. For further details see [7].

4. Nonlinear–nonquadratic controllers for systems with bounded disturbances

In this section we consider a control problem involving a notion of optimality with respect to an auxiliary cost which guarantees a bound on the worst-case value of a nonlinear–nonquadratic cost functional over a prescribed set of bounded input disturbances. The optimal feedback controllers are derived as a direct consequence of Theorem 3.1 and provide a transparent generalization of the Bellman conditions for time-invariant, infinite horizon problems for addressing nonlinear feedback controllers for nonlinear discrete-time systems with bounded energy disturbances that additionally minimize a nonlinear–nonquadratic cost functional. To address the optimal control problem let $\mathcal{D} \subset \mathbb{R}^n$ be an open set and let $0 \in \mathcal{D}$ and $0 \in \mathcal{C}$. Furthermore, let $\mathcal{W} \subset \mathbb{R}^d$ be a subset of measurable functions and let $r: \mathbb{R}^p \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a given function. Next, consider the controlled system

$$x(k + 1) = \hat{f}(x(k), u(k)) + J_1(x(k)) w(k) x(0) = x_0, \quad w(\cdot) \in \mathcal{W}, \quad k \in \mathcal{N},$$

with performance variables

$$z(k) = \hat{h}(x(k), u(k)) + J_2(x(k)) w(k),$$

where $\hat{f}: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $\hat{f}(0, 0) = 0$, $J_1: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$, $\hat{h}: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^p$, $J_2: \mathbb{R}^n \rightarrow \mathbb{R}^{p \times d}$ and the control $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(k) \in \mathcal{U}$ for all $k \in \mathcal{N}$, where the control constraint set $\mathcal{U} \subset \mathcal{C}$ is given. We assume $0 \in \mathcal{U}$. A measurable mapping $\phi: \mathcal{D} \rightarrow \mathcal{U}$ satisfying $\phi(0) = 0$ is called a control law. If $u(k) = \phi(x(k))$, where $\phi(\cdot)$ is a control law and $x(k), k \in \mathcal{N}$, satisfies Eq. (92), then $u(\cdot)$ is called a feedback control. Given a control law $\phi(\cdot)$ and a feedback control $u(k) = \phi(x(k))$, the closed-loop system has the form

$$x(k + 1) = \hat{f}(x(k), \phi(x(k))) + J_1(x(k)) w(k), x(0) = x_0, \quad w(\cdot) \in \mathcal{W}, \quad k \in \mathcal{N},$$

$$z(k) = \hat{h}(x(k), \phi(x(k))) + J_2(x(k)) w(k).$$
Next, we present an extension of Theorem 4.1 of [5] for characterizing feedback controllers that guarantee stability, minimize an auxiliary performance functional, and guarantee that the input–output map of the closed-loop system is dissipative, non-expansive, and passive for bounded input disturbances. For the statement of this result let \( \hat{L}: \mathcal{D} \times \mathcal{U} \to \mathbb{R} \) and define the set of asymptotically stabilizing controllers for the nonlinear system with \( w(k) \equiv 0 \) by

\[
\mathcal{S}(x_0) \triangleq \{ u(\cdot) : u(\cdot) \text{ is admissible and } \}
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x(\cdot) \text{ given by Eq. (92) satisfies } x(k) \to 0 \text{ as } k \to \infty \text{ with } w(k) \equiv 0 \}.
\]

**Theorem 4.1.** Consider the controlled system (92) and (93) with performance functional

\[
\tilde{J}(x_0, u(\cdot)) \triangleq \sum_{k=0}^{\infty} \tilde{L}(x(k), u(k)),
\]

(96)

where \( u(\cdot) \) is an admissible control. Assume there exist functions \( V : \mathcal{D} \to \mathbb{R}, \tilde{\Gamma} : \mathcal{D} \times \mathcal{U} \to \mathbb{R}, P_{1w} : \mathcal{D} \times \mathcal{U} \to \mathbb{R}^{1 \times d}, P_{2w} : \mathcal{D} \times \mathcal{U} \to \mathbb{R}^{d \times d} \), and a control law \( \phi : \mathcal{D} \to \mathcal{U} \) such that

\[
P_{1w}(0, 0) = 0,
\]

(97)

\[
V(0) = 0,
\]

(98)

\[
V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0,
\]

(99)

\[
\phi(0) = 0,
\]

(100)

\[
V(\tilde{f}(x, \phi(x))) - V(x) < 0, \quad x \in \mathcal{D}, \quad x \neq 0,
\]

(101)

\[
P_{1w}(x, \phi(x))w + w^TP_{2w}(x, \phi(x))w \leq r(z, w) + \tilde{L}(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x)),
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
system (94) is locally asymptotically stable. If, in addition, $\Gamma(x, \phi(x)) \geq 0$, $x \in \mathcal{D}$, then

$$J(x_0, \phi(x(\cdot))) \leq \tilde{J}(x_0, \phi(x(\cdot))) = V(x_0),$$

(107)

where

$$\tilde{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [\tilde{L}(x(k), u(k)) + \tilde{\Gamma}(x(k), u(k))],$$

(108)

and where $u(\cdot)$ is admissible and $x(k)$, $k \in \mathcal{N}$, solves Eq. (92) with $w(k) \equiv 0$. In addition, if $x_0 \in \mathcal{D}$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $\tilde{J}(x_0, u(\cdot))$ in the sense that

$$\tilde{J}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{F}(x_0)} \tilde{J}(x_0, u(\cdot)).$$

(109)

Furthermore, the solution $x(k)$, $k \in \mathcal{N}$, of Eq. (94) satisfies the dissipativity constraint

$$\sum_{i=0}^{k} r(z(i), w(i)) + V(x_0) \geq 0, \quad k \in \mathcal{N}, \quad w(\cdot) \in \mathcal{W}.$$  

(110)

Finally, if $\mathcal{D} = \mathbb{R}^n$, $\mathcal{W} = \mathbb{R}^m$, $w(k) \equiv 0$, and

$$V(x) \to \infty \quad \text{as} \quad x \to \infty,$$

(111)

then the solution $x(k) = 0$, $k \in \mathcal{N}$, of the closed-loop system (94) is globally asymptotically stable.

**Proof.** Local and global asymptotic stability is a direct consequence of Eqs. (97)–(101) by applying Theorem 3.1 to the closed-loop system (94). Furthermore, using Eq. (104), the performance bound (107) is a restatement of Eq. (23) as applied to the closed-loop system. Next, let $u(\cdot) \in \mathcal{F}(x_0)$ and let $x(k)$, $k \in \mathcal{N}$, be the solution of Eq. (92) with $w(k) \equiv 0$. Then it follows that

$$0 = -\Delta V(x(k)) + V(\tilde{f}(x(k), u(k))) - V(x(k)).$$

Hence,

$$\tilde{L}(x(k), u(k)) + \tilde{\Gamma}(x(k), u(k))$$

$$= -\Delta V(x(k)) + \tilde{L}(x(k), u(k)) + V(\tilde{f}(x(k), u(k))) - V(x(k)) + \tilde{\Gamma}(x(k), u(k))$$

$$= -\Delta V(x(k)) + H(x(k), u(k)).$$

Now using Eqs. (105) and (108) and the fact that $u(\cdot) \in \mathcal{F}(x_0)$, it follows that

$$\tilde{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [-\Delta V(x(k)) + H(x(k), u(k))]$$

$$= -\lim_{k \to \infty} V(x(k+1)) + V(x_0) + \sum_{k=0}^{\infty} H(x(k), u(k)).$$
\[ V(x_0) + \sum_{k=0}^{\infty} H(x(k), u(k)) \geq V(x_0) = \mathcal{F}(x_0, \phi(x(\cdot))) \]

which yields Eq. (109). Finally, using Eqs. (102) and (103), condition (110) is a restatement of Eq. (25) as applied to the closed-loop system.

Remark 4.1. Note that Eq. (104) is the steady state (time-independent) Bellman equation for the discrete-time nonlinear system \( f(\cdot, \cdot) \) with auxiliary cost \( \mathcal{J}(x_0, u(\cdot)) \).

Remark 4.2. Theorem 4.1 guarantees optimality with respect to the set of admissible stabilizing controllers \( \mathcal{S}(x_0) \). However, it is important to note that an explicit characterization of \( \mathcal{S}(x_0) \) is not required. Furthermore, the optimal stabilizing control law \( u = \phi(x) \) is a feedback controller and its optimality is independent of the initial condition \( x_0 \).

Next, we specialize Theorem 4.1 to linear systems with bounded energy disturbances and provide connections to the mixed-norm \( H_2/H_\infty \) framework developed in [3]. Specifically, we consider the case in which \( f(x, u) = Ax + Bu \), \( h(x, u) = E_1x + E_2u \), and \( J_2(x) = 0 \), where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( D \in \mathbb{R}^{n \times d} \), \( E_1 \in \mathbb{R}^{p \times n} \), and \( E_2 \in \mathbb{R}^{p \times m} \).

In this case, \( \mathcal{H} = \ell_2 \) and \( r(z, w) = \gamma^2 w^T w - z^T z \), where \( \gamma > 0 \) is given. For the following result define the notation

\[
\begin{align*}
R_1 & \triangleq E_1^T E_1 > 0, \\
R_2 & \triangleq E_2^T E_2 > 0, \\
R_{12} & \triangleq E_1^T E_2 = 0, \\
R_{2a} & \triangleq R_2 + B^T PB + B^T PD (\gamma^2 I_d - D^T PD)^{-1} D^T PB, \\
P_a & \triangleq B^T PA + B^T PD (\gamma^2 I_d - D^T PD)^{-1} D^T PA,
\end{align*}
\]

for arbitrary \( P \in \mathbb{R}^{n \times n} \) when the indicated inverse exists.

Corollary 4.1. Consider the linear discrete-time system

\[
\begin{align*}
x(k+1) &= Ax(k) + Bu(k) + Dw(k), & x(0) &= x_0, & k \in \mathbb{N}, & w(\cdot) \in \ell_2, \\
z(k) &= E_1x(k) + E_2u(k),
\end{align*}
\]

with performance functional

\[
\mathcal{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [x^T(k) R_1 x(k) + u^T(k) R_{2a} u(k)],
\]

where \( u(\cdot) \) is admissible. Assume there exists \( P \in \mathbb{P}^{n \times n} \) such that \( \gamma^2 I_d - D^T PD > 0 \) and

\[
P = A^T P A + R_1 + A^T P D (\gamma^2 I_d - D^T PD)^{-1} D^T P A - P_{a}^T R_{2a}^{-1} P_a.
\]
Then, with the feedback control law \( u = \phi(x) = -R_2^{-1}P_\omega x \), the closed-loop undisturbed (\( w(k) \equiv 0 \)) system (112) is globally asymptotically stable for all \( x_0 \in \mathbb{R}^n \). Furthermore, \( \dot{J}(x_0, \phi(x(\cdot))) \leq \dot{J}(x_0, \phi(x(\cdot))) = x_0^T P x_0 \),

(116)

where

\[
\dot{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \left( (D^T p(Ax(k) + Bu(k)))^T \gamma I_d - D^T P D \right)^{-1} (D^T p(Ax(k) + Bu(k)))
\]

\[ + x^T(k) R_1 x(k) + u^T(k) R_2 u(k) \]

(117)

and where \( u(\cdot) \) is admissible and \( x(k), k \in \mathcal{N} \), solves Eq. (112) with \( w(k) \equiv 0 \). In addition,

\[
\dot{J}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{J}(x_0)} \dot{J}(x_0, u(\cdot)),
\]

(118)

where \( \mathcal{J}(x_0) \) is the set of asymptotically stabilizing controllers for the system (112) with \( w(k) \equiv 0 \) and \( x_0 \in \mathbb{R}^n \). Finally, if \( x_0 = 0 \) then, with \( u = \phi(x) \), the solution \( x(k), k \in \mathcal{N} \), of Eq. (112) satisfies the nonexpansivity constraint

\[
\sum_{i=0}^{k} z^T(i) z(i) \leq \gamma \sum_{i=0}^{k} w^T(i) w(i), \quad k \in \mathcal{N}, \quad w(\cdot) \in l_2,
\]

(119)

or, equivalently, \( \| \hat{G}(z) \|_\infty \leq \gamma \), where

\[
\hat{G}(z) \sim \left[ \begin{array}{ccc}
A + BK & D^T


\end{array} \right]
\]

and \( K \triangleq -R_2^{-1}P_\omega \).

**Proof.** The result is a direct consequence of Theorem 4.1 with \( \dot{J}(x, u) = Ax + Bu \), \( J_1(x) = D \), \( r(z, w) = z^T w w - z^T z \), \( \tilde{L}(x, u) = x^T R_1 x + u^T R_2 u \), \( V(x) = x^T P x \), \( \tilde{\Gamma}(x, u) = \left[ \begin{array}{c}
P_1 \phi(x) u


P_2 \phi(x) u

\end{array} \right] \), \( P_1 \phi(x) u = 2(Ax + Bu)^T P D, P_2 \phi(x) u = D^T P D \), \( \mathcal{D} = \mathbb{R}^n \), and \( \mathcal{W} = \mathbb{R}^m \). Specifically, conditions (97)–(101) and (103) are trivially satisfied. Now, forming \( x^T(115)x \) it follows that, after some algebraic manipulations, \( P_1 \phi(x) u w + w^T P_2 \phi(x) u w \leq r(z, w) + \tilde{L}(x, \phi(x)) + \tilde{\Gamma}(x, \phi(x)) \), for all \( x \in \mathbb{R}^n \) and \( w(\cdot) \in \mathcal{W} \). Furthermore, it follows from Eq. (115) that \( H(x, \phi(x)) \equiv 0 \) and \( H(x, u) = H(x, u) - H(x, \phi(x)) = [u - \phi(x)]^T R_2 u [u - \phi(x)] \geq 0 \), \( x, u \in \mathbb{R}^n \). Hence, all conditions of Theorem 4.1 are satisfied. Finally, since \( V(x), x \in \mathbb{R}^n \), is radially unbounded, Eq. (112) with \( u(k) = \phi(x(k)) = -R_2^{-1}P_\omega x(k) \), is globally asymptotically stable. \( \square \)

**Remark 4.3.** Note that in the case where \( \mathcal{W} = \mathbb{R}^m \) our feedback control \( u = \phi(x) \) is globally optimal since it minimizes \( H(x, u) \) and satisfies Eq. (104). Specifically, setting

\[
\frac{\partial}{\partial u} H(x, u) = 0,
\]

(120)
yields the feedback control
\[ \phi(x) = -R^{-1}_{2a}P_{2a}x. \]  
\tag{121}

Now, since
\[ \frac{\partial^2}{\partial u^2} H(x,u) = R_{2a} > 0, \]  
\tag{122}
it follows that for all \( x \in \mathbb{R}^n \) the feedback control law given by Eq. (121) minimizes \( H(x,u) \). In particular, the optimal feedback control law \( \phi(x) \) in Corollary 4.1 is derived using the properties of \( H(x,u) \) as defined in Theorem 4.1. Specifically, since \( H(x,u) = x^T(A^T P A - P + R_1 + A^T P D (\gamma^2 I - D^T P D)^{-1} D^T P A)x + u^T R_{2a} u + 2u^T P_{2a} x \), it follows that \( \partial^2 H/\partial u^2 > 0 \). Now, \( \partial H/\partial u = 2R_{2a} u + 2P_{2a} x = 0 \) gives the unique global minimum of \( H(x,u) \). Hence, since \( \phi(x) \) minimizes \( H(x,u) \) it follows that \( \phi(x) \) satisfies \( \partial H/\partial u = 0 \) or, equivalently, \( R_{2a} \phi(x) + P_{2a} x = 0 \) so that \( \phi(x) \) is given by Eq. (121). Similar remarks hold for the controllers developed in Corollary 4.2.

Next, we specialize Theorem 4.1 to develop mixed \( H_2 \)/positivity controllers for discrete-time linear systems. For the statement of this result recall the definitions of \( R_1 \) and \( R_2 \) and for arbitrary \( P \in \mathbb{R}^{n \times n} \), define the notation
\[ R_0 \triangleq E_{\infty} + E_{\infty}^T - D^T P D, \]
\[ R_{2a} \triangleq B^T P B + R_2 + (D^T P B - E_2)^T R_{0}^{-1} (D^T P B - E_2), \]
\[ P_1 \triangleq B^T P A + (D^T P B - E_2)^T R_{0}^{-1} (D^T P A - E_1), \]
when the indicated inverse exists.

**Corollary 4.2.** Let \( p = d \). Consider the linear system (112) with performance variables
\[ z(k) = E_1 x(k) + E_2 u(k) + E_{\infty} w(k) \]  
\tag{123}
and performance functional (114). Assume there exists \( P \in \mathbb{R}^{n \times n} \) such that \( R_0 > 0 \) and
\[ P = A^T P A + R_1 + (D^T P A - E_1)^T R_{0}^{-1} (D^T P A - E_1) - P_1^T R_{2a}^{-1} P_1. \]  
\tag{124}
Then, with the feedback control law \( u = \phi(x) = -R_{2a}^{-1}P_{2a} x \), the closed-loop undisturbed \((w(k) = 0)\) system (112) is globally asymptotically stable for all \( x_0 \in \mathbb{R}^n \). Furthermore,
\[ \hat{J}(x_0, \phi(x(\cdot))) \leq \hat{J}(x_0, \phi(x(\cdot))) = x_0^T P x_0, \]  
\tag{125}
where
\[ \hat{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [x^T(k) (R_1 + (D^T P A - E_1)^T R_{0}^{-1} (D^T P A - E_1)) x(k) + u^T(k)] 
\times (R_{2a} - B^T P B) u(k) + 2x^T(k) (D^T P A - E_1)^T R_{0}^{-1} (D^T P B - E_2) u(k)] \]  
\tag{126}
and where \( u(\cdot) \) is admissible and \( x(k), k \in \mathcal{N} \), solves Eq. (112) with \( w(k) = 0 \). In addition,

\[
\hat{f}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{U}(x_0)} \hat{f}(x_0, u(\cdot)),
\]

where \( \mathcal{U}(x_0) \) is the set of asymptotically stabilizing controllers for system (112) with \( w(k) = 0 \) and \( x_0 \in \mathbb{R}^n \). Finally, if \( x_0 = 0 \) then, with \( u = \phi(x) \), the solution \( x(k) \), \( k \in \mathcal{N} \), of Eq. (112) satisfies the passivity constraint

\[
\sum_{i=0}^{k} z^T(i)w(i) \geq 0, \quad k \in \mathcal{N}, \quad w(\cdot) \in l_2,
\]

or, equivalently, \( \mathcal{G}_\infty(z) + \mathcal{G}_\infty^*(z) \geq 0, \quad |z| > 1, \) where

\[
\mathcal{G}_\infty(z) \sim \begin{bmatrix} A + BK & D \\ E_1 + E_2K & E_\infty \end{bmatrix}
\]

and \( K = -R^{-1}_{2s}P_s \).

**Proof.** The result is a direct consequence of Theorem 4.1 with \( \hat{f}(x, u) = Ax + Bu \), \( J_1(x) = D, J_2(x) = E_\infty, r(z, w) = 2z^Tsw, \hat{L}(x, u) = x^TR_1x + w^TR_2u, V(x) = x^TPx, \hat{\Gamma}(x, u) = [(D^TPA - E_1)x + (D^TPB - E_2)u]R^{-1}_0[(D^TPA - E_1)x + (D^TPB - E_2)u], P_{1w}(x, u) = 2(AX + Bu)PD, P_{2w}(x, u) = D^TPD, \mathcal{D} = \mathbb{R}^n, \) and \( \mathcal{W} = \mathbb{R}^m \). Specifically, conditions (97)–(101) and (103) are trivially satisfied. Now, forming \( x^T(124)x \) it follows that, after some algebraic manipulations, \( P_{1w}(x, \phi(x))w + w^TP_{2w}(x, \phi(x))w = r(z, w) + \hat{L}(x, \phi(x)) + \hat{\Gamma}(x, \phi(x)), \) for all \( x \in \mathbb{R}^n \) and \( w(\cdot) \in \mathcal{W} = l_2 \). Furthermore, it follows from Eq. (124) that \( H(x, \phi(x)) = 0 \) and \( H(x, u) = H(x, u) - H(x, \phi(x)) = [u - \phi(x)]^TR_2(x)[u - \phi(x)] \geq 0, \) \( (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \). Hence, all conditions of Theorem 4.1 are satisfied. Finally, since \( V(x) \) is radially unbounded, Eq. (112) with \( u(k) = \phi(x(k)) = -R^{-1}_{2s}P_sx(k) \), is globally asymptotically stable. \( \square \)

5. **Nonlinear–nonquadratic controllers for affine systems**

In this section we specialize the results of Section 4 to affine systems of the form

\[
x(k+1) = f(x(k)) + g(x(k))u(k) + J_1(x(k))w(k), \quad x(0) = x_0, \quad w(\cdot) \in l_2, \quad k \in \mathcal{N},
\]

with the performance variables

\[
z(k) = h(x(k)) + G(x(k))u(k) + J_2(x(k))w(k),
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f(0) = 0 \), \( g : \mathbb{R}^n \to \mathbb{R}^{n \times m} \), \( G : \mathbb{R}^n \to \mathbb{R}^{p \times m} \), and \( h : \mathbb{R}^n \to \mathbb{R}^p \) such that \( h(0) = 0 \). Furthermore, we consider performance integrands \( \hat{L}(x, u) \) of the form

\[
\hat{L}(x, u) = L_1(x) + L_2(x)u + u^TR_2(x)u,
\]
where $L_1: \mathbb{R}^n \to \mathbb{R}$, $L_2: \mathbb{R}^n \to \mathbb{R}^{1 \times m}$, and $R_2: \mathbb{R}^n \to \mathbb{R}^{m \times m}$ so that Eq. (96) becomes

$$
\tilde{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [L_1(x(k)) + L_2(x(k))u(k) + u^T(k)R_2(x(k))u(k)].
$$  \tag{132}

The next theorem provides state feedback controllers that guarantee that the closed-loop system from disturbances $w$ to performance variables $z$ is nonexpansive.

**Theorem 5.1.** Consider the controlled system (129) and (130) with performance functional (132). Assume there exist functions $V: \mathbb{R}^n \to \mathbb{R}$, $P_{1w}: \mathbb{R}^n \to \mathbb{R}^{1 \times d}$, $P_{2w}: \mathbb{R}^n \to \mathbb{N}^{m \times d}$, $P_{aw}: \mathbb{R}^n \to \mathbb{R}^{m \times d}$, and $L_2: \mathbb{R}^n \to \mathbb{R}^{1 \times m}$ such that

$$
P_{1w}(0) = 0, \tag{133}
P_{aw}(0) = 0, \tag{134}
L_2(0) = 0, \tag{135}
V(0) = 0, \tag{136}
V(x) > 0, \quad x \in \mathbb{R}^n, \; x \neq 0, \tag{137}
V(f(x)) + P_{1w}(x)\phi(x) + \phi^T(x)P_{2w}(x)\phi(x) - V(x) + \tilde{\Gamma}(x, \phi(x)) < 0,
$$

$$
\quad x \in \mathbb{R}^n, \; x \neq 0, \tag{138}
V(f(x) + g(x)u + J_1(x)w)
\quad = V(f(x)) + P_{1w}(x)u + u^T P_{2w}(x)u + P_{1w}(x)w + u^T P_{aw}(x)w + w^T P_{2w}(x)w,
$$

$$
\quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^n, \; w \in l_2, \tag{139}
\gamma^2 I_d - P_{2w}(x) - J_2(x)J_2(x) > 0, \quad x \in \mathbb{R}^n, \tag{140}
$$

where

$$
\phi(x) = -R_{2w}^{-1}(x)P_{aw}(x), \tag{141}
$$

$$
R_{2w}(x) \triangleq R_2(x) + \frac{1}{2}(P_{aw}(x) + 2G^T(x)J_2(x))(\gamma^2 I_d - P_{2w}(x) - J_2^T(x)J_2(x))^{-1}
\quad \times (P_{aw}(x) + 2G^T(x)J_2(x))^T + P_{2w}(x) + G^T(x)G(x), \tag{142}
$$

$$
P_a(x) \triangleq \frac{1}{2}[L_2^2(x) + P_{1w}(x) + \frac{1}{4}(P_{aw}(x) + 2G^T(x)J_2(x))(\gamma^2 I_d - P_{2w}(x) - J_2^T(x)J_2(x))^{-1}
\quad \times (P_{aw}(x) + 2G^T(x)J_2(x))^T + 2G^T(x)h(x)], \tag{143}
$$

$$
\tilde{\Gamma}(x, u) \triangleq \frac{1}{4}[(P_{1w}(x) + 2h^T(x)J_2(x)) + u^T(P_{aw}(x) + 2G^T(x)J_2(x))]
\quad \times (\gamma^2 I_d - P_{2w}(x) - J_2^T(x)J_2(x))^{-1}[(P_{1w}(x) + 2h^T(x)J_2(x))
\quad + u^T(P_{aw}(x) + 2G^T(x)J_2(x))]^T + (h(x) + G(x)u)^T(h(x) + G(x)u).
$$  \tag{144}
Then, with the feedback control \( u(\cdot) = \phi(x(\cdot)) \), the solution \( x(k) = 0 \) of the closed-loop undisturbed \((w(k) = 0)\) system (129) is globally asymptotically stable for all \( x_0 \in \mathbb{R}^n \). Furthermore, the performance functional (132) satisfies

\[
\mathcal{J}(x_0, \phi(x(\cdot))) \leq \hat{\mathcal{J}}(x_0, \phi(x(\cdot))) = V(x_0),
\]

where

\[
\hat{\mathcal{J}}(x_0, u(\cdot)) \doteq \sum_{k=0}^{\infty} [\hat{L}(x(k), u(k)) + \hat{\Gamma}(x(k), u(k))].
\]

In addition, the performance functional (132), with

\[
L_1(x) = P_a(x)R_a^{-1}(x)P_a(x) - [V(f(x)) - V(x) + \frac{1}{4}(P_{iw}(x) + 2h^T(x)J_2(x))
\times (\gamma^2 L_d - P_{iw}(x) - J_2^T(x)J_2(x))^{-1}(P_{iw}(x) + 2h^T(x)J_2(x))^T + h^T(x)h(x)],
\]

is minimized in the sense that

\[
\hat{\mathcal{J}}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{U}(x_0)} \hat{\mathcal{J}}(x_0, u(\cdot)).
\]

Finally, with \( u(\cdot) = \phi(x(\cdot)) \), the solution \( x(k), k \in \mathcal{N} \), of the closed-loop system (129) satisfies the nonexpansivity constraint

\[
\sum_{i=0}^{k} z^T(i)z(i) \leq \gamma^2 \sum_{i=0}^{k} w^T(i)w(i) + V(x_0), \quad k \in \mathcal{N}, \quad w(\cdot) \in I_2.
\]

**Proof.** The result is a direct consequence of Theorem 4.1 with \( \hat{f}(x, u) = f(x) + g(x)u, z = h(x) + G(x)u, r(z, w) = \gamma^2 w^T w - z^T z, \hat{L}(x, u) = L_1(x) + L_2(x)u + u^T R_2(x)u, \hat{\Gamma}(x, u) \) given by Eq. (144), \( \mathcal{D} = \mathbb{R}^n \), and \( \mathcal{U} = \mathbb{R}^m \). Specifically, conditions (97)–(99) and (101)–(103) are trivially satisfied by replacing \( P_{iw}(x, u) \) by \( P_{iw}(x) + u^T P_{uw}(x) \). Furthermore, with Eqs. (129), (131), and (144), the Hamiltonian has the form

\[
H(x, u) = L_1(x) + L_2(x)u + u^T R_2(x)u + V(f(x) + g(x)u) - V(x) + \frac{1}{4}[P_{iw}(x)
+ 2h^T(x)J_2(x)] + u^T (P_{uw}(x) + 2G^T(x)J_2(x))\{[\gamma^2 L_d - P_{iw}(x) - J_2^T(x)
\times J_2(x)]^{-1}[(P_{iw}(x) + 2h^T(x)J_2(x)) + u^T (P_{uw}(x) + 2G^T(x)J_2(x))]\}^T
+ (h(x) + G(x)u)^T h(x) + G(x)u).
\]

Now, the feedback control law (141) is obtained by setting \( \hat{c}(\hat{c}u)H(x, u) = 0 \). Next, since by assumption \( L_2(0) = 0, P_{iw}(0) = 0, P_{uw}(0) = 0, \) and \( h(0) = 0 \), it follows that \( \phi(0) = 0 \) which proves Eq. (100). Next, with \( L_1(x) \) given by Eq. (147) and \( \phi(x) \) given by Eqs. (141) and (104) holds. Finally, since \( H(x, u) = H(x, u) - H(x, \phi(x)) = \)}
\[ u - \phi(x) \] is positive definite for all \( x \in \mathbb{R}^n \), condition (105) holds. The result now follows as a direct consequence of Theorem 4.1. \( \square \)

Next, we provide state feedback controllers that guarantee that the closed-loop system from disturbances \( w \) to performance variables \( z \) is passive.

**Theorem 5.2.** Let \( p = d \). Consider the controlled system (129) and (130) with performance functional (132). Assume there exist functions \( V : \mathbb{R}^n \to \mathbb{R} \), \( P_{1u} : \mathbb{R}^n \to \mathbb{R}^{1 \times m} \), \( P_{2u} : \mathbb{R}^n \to \mathbb{R}^{m \times m} \), \( P_{1w} : \mathbb{R}^n \to \mathbb{R}^{1 \times d} \), \( P_{2w} : \mathbb{R}^n \to \mathbb{R}^{d \times d} \), \( P_{aw} : \mathbb{R}^n \to \mathbb{R}^{m \times d} \), and \( L_2 : \mathbb{R}^n \to \mathbb{R}^{1 \times m} \) such that

\[
P_{1u}(0) = 0, \quad P_{1u}(0) = 0, \quad L_2(0) = 0, \quad V(0) = 0, \quad V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad V(f(x)) + P_{1u}(x)\phi(x) + \phi^T(x)P_{2u}(x)\phi(x) - V(x) + \tilde{\Gamma}(x, \phi(x)) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad V(f(x) + g(x)u + J_1(x)w) = V(f(x)) + P_{1u}(x)u + u^TP_{2u}(x)u + P_{1w}(x)w + u^TP_{aw}(x)w + w^TP_{2w}(x)w, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad w \in L_2, \quad J_2(x) + J_2^T(x) - P_{2w}(x) > 0, \quad x \in \mathbb{R}^n,
\]

where

\[
\phi(x) = -R_{2u}^{-1}(x)P_{1u}(x),
\]

\[
R_{2u} \triangleq R_2(x) + \left( \frac{1}{2}P_{aw}(x) - G^T(x) \right) \left( J_2(x) + J_2^T(x) - P_{2w}(x) \right)^{-1} \times \left( \frac{1}{2}P_{aw}(x) - G^T(x) \right)^T + P_{2u}(x),
\]

\[
P_z \triangleq \frac{1}{2}(L_2^T(x) + P_{1w}(x) + 2(\frac{1}{2}P_{aw}(x) - G(x))^T(J_2(x) + J_2^T(x) - P_{2w}(x))^{-1} \times (\frac{1}{2}P_{aw}(x) - G(x))],
\]

\[
\tilde{\Gamma}(x, u) \triangleq \left[ \frac{1}{2}P_{1w}(x) - h(x) + (\frac{1}{2}P_{aw}(x) - G(x))u \right] \left( J_2(x) + J_2^T(x) - P_{2w}(x) \right)^{-1} \times \left[ \frac{1}{2}P_{1w}(x) - h(x) + (\frac{1}{2}P_{aw}(x) - G(x))u \right].
\]

Then, with the feedback control \( u(\cdot) = \phi(x(\cdot)) \), the solution \( x(k) \equiv 0 \) of the undisturbed \( (w(k) \equiv 0) \) closed-loop system (129) is globally asymptotically stable for all \( x_0 \in \mathbb{R}^n \). Furthermore, the performance functional (132) satisfies

\[
\tilde{J}(x_0, \phi(x(\cdot))) \leq \tilde{J}(x_0, \phi(x(\cdot))) = V(x_0),
\]
where

\[ \hat{J}(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [\hat{L}(x(k), u(k)) + \hat{\Gamma}(x(k), u(k))]. \]  

(164)

In addition, the performance functional (132), with

\[ L_1(x) = P_3^T(x)R_2^{-1}(x)P_3(x) - [V(f(x)) - V(x) + \frac{1}{2}P_{1w}^T(x) \nonumber \]
\[ -h(x)]^T(J_2(x) + J_2^T(x) - P_{2w}(x))^{-1}[\frac{1}{2}P_{1w}^T(x) - h(x)], \]

(165)
is minimized in the sense that

\[ \hat{J}(x_0, \phi(x(\cdot))) = \min_{w(\cdot) \in \mathcal{W}(x_0)} \hat{J}(x_0, u(\cdot)). \]  

(166)

Finally, with \( u(\cdot) = \phi(x(\cdot)) \), the solution \( x(k), k \in \mathcal{N} \), of Eq. (129) satisfies the passivity constraint

\[ \sum_{i=0}^{k} 2z^T(i)w(i) + V(x_0) \geq 0, \quad k \in \mathcal{N}, \quad w(\cdot) \in \mathcal{W}. \]

(167)

**Proof.** The result is a direct consequence of Theorem 4.1 with \( \hat{J}(x, u) = f(x) + g(x)u, \)
\( z = h(x) + G(x)u, \quad r(z, w) = 2z^Tw, \)
\( \hat{L}(x, u) = L_1(x) + L_2(x)u + u^TR_2(x)u, \quad \hat{\Gamma}(x, u) \) given by Eq. (162), \( \mathcal{D} = \mathbb{R}^n \), and \( \mathcal{W} = \mathbb{R}^m \). Specifically, conditions (97)–(99) and (101)–(103) are trivially satisfied by replacing \( P_{1w}(x, u) \) by \( P_{1w}(x) + u^TP_{aw}(x) \). Furthermore, with Eqs. (129), (131) and (162), the Hamiltonian has the form

\[ H(x, u) = L_1(x) + L_2(x)u + u^TR_2(x)u + V(f(x) + g(x)u) - V(x) + \frac{1}{2}P_{1w}^T(x) - h(x) \nonumber \]
\[ + \left( \frac{1}{2}P_{aw}^T(x) - G(x)u \right)^T(J_2(x) + J_2^T(x) - P_{2w}(x))^{-1} \nonumber \]
\[ \times \left[ \frac{1}{2}P_{1w}^T(x) - h(x) + \left( \frac{1}{2}P_{aw}^T(x) - G(x)u \right) \right]. \]

(168)

Now, the feedback control law (159) is obtained by setting \( (\partial/\partial u)H(x, u) = 0 \). Next, since by assumption \( L_2(0) = 0 \), \( P_{1w}(0) = 0 \), and \( h(0) = 0 \), it follows that \( \phi(0) = 0 \) which proves (100). Next, with \( L_1(x) \) given by Eq. (165) and \( \phi(x) \) given by Eqs. (159) and (104) holds. Finally, since \( H(x, u) = H(x, u) - H(x, \phi(x)) = [u - \phi(x)]^T R_{2z}(x)[u - \phi(x)] \) and \( R_{2z}(x) \) is positive definite for all \( x \in \mathbb{R}^n \), condition (105) holds. The result now follows as a direct consequence of Theorem 4.1. \( \square \)

6. Conclusion

A unified optimality-based framework was developed to address the problem of nonlinear–nonquadratic control for disturbance rejection of nonlinear discrete-time systems with bounded exogenous disturbances. Specifically, by properly modifying a nonlinear–nonquadratic performance criterion to account for system disturbances the
The disturbance rejection problem was translated into an optimal control problem. Furthermore, the resulting optimal control law was shown to render the closed-loop nonlinear input-output map dissipative with respect to general supply rates. In addition, the Lyapunov function guaranteeing closed-loop stability was shown to be a solution to the steady-state Bellman equation and thus guaranteeing optimality. Finally, in the special case where the controlled system is linear our results, with appropriate quadratic supply rates, specialize to the mixed-norm $H_2/H_\infty$ framework developed in [3] and the discrete-time analog of the continuous-time mixed $H_2/positivity framework developed in [4].

References