Nearly Nonminimal Linear-Quadratic-Gaussian Compensators for Reduced-Order Control Design Initialization

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I. Introduction

The development of linear-quadratic-Gaussian (LQG) theory was a major breakthrough in modern control theory since it provides a systematic way to synthesize high-performance controllers for nominal models of complex, multi-input multi-output systems. One of the well-known deficiencies of an LQG compensator, however, is that its minimal dimension is usually equal to the dimension of the design plant. This has led to the development of techniques to synthesize reduced-order approximations of the optimal full-order compensator (see Refs. 1–3 and the references therein).

The controller reduction methods almost always yield suboptimal (and sometimes destabilizing) reduced-order control laws since an optimal reduced-order controller is not usually a direct function of the parameters used to compute or describe the optimal full-order controller. Nevertheless, these methods are computationally inexpensive and sometimes do yield high performing and even nearly optimal control laws. An observation that holds true about most of these methods is that they tend to work best at low control authority. To date, however, no algebraic conditions have been established that guarantee that a given suboptimal controller reduction method will work well at low authority.

This Note considers the balanced controller reduction algorithm of Ref. 1 and provides a constructive way of choosing the weights in an LQG control problem of dimension n such that for a given n0 < n, the corresponding nth-order controller obtained by this suboptimal reduction method has essentially the same performance as the LQG controller at low control authority. The usefulness of this result is for initializing homotopy algorithms for optimal reduced-order control design that requires a nearly optimal controller for an initial set of design parameters.

The discussion here focuses on stable systems. It is shown that if the state weighting matrix R1 or disturbance weighting matrix V1 has a specific structure in a basis in which the plant dynamics A is upper or lower block triangular, respectively, then at low control authority the corresponding LQG compensator is nearly nonminimal with minimal dimension n0. It follows that the LQG compensator can be easily reduced to a n0th-order controller having nearly the same performance. These results are directly applicable in initializing continuation and homotopy algorithms that require a reduced-order controller that is nearly optimal for some set of initial design parameters corresponding to a low authority controller.

A special case of the conditions presented for R1 and V1 has a strong physical interpretation for structural control problems. In particular, assuming that all of the eigenvalues in the plant are complex (lightly damped structures) and n is an even number, then either R1 is allowed to weight only n/2 structural modes or V1 is allowed to disturb only n/2 structural modes.

II. Construction of Nearly Nonminimal LQG Compensators

Consider the nth-order linear time-invariant plant

\[ \dot{x}(t) = Ax(t) + Bu(t) + D_1 w(t), \]
\[ y(t) = Cx(t) + D_2 w(t) \]

(1)

where (A, B) is stabilizable, (A, C) is detectable, x ∈ R^n, u ∈ R^m, y ∈ R^q, and w ∈ R^p is a standard white noise disturbance with intensity I_q and rank D_2 = l. The intensities of D_1 w(t) and D_2 w(t) are thus given by V_1 = D_1 D_1^T ≥ 0 and V_2 = D_2 D_2^T > 0, respectively. Then, the LQG compensator

\[ \tilde{x}(t) = A_c x_c(t) + B_c y(t), \]
\[ u(t) = -C_c x_c(t) \]

(2)

for the plant (1) maximizing the steady-state quadratic performance criterion

\[ J(A_c, B_c, C_c) \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ x^T(s) R x(s) + u^T(s) R u(s) \right] ds \]

(3)

where R_1 ≥ 0 and R_2 > 0 are the weighting matrices for the controlled states and controller input, respectively, the plant is given by

\[ A_c = A - \Sigma P - Q \hat{\Sigma}, \]
\[ B_c = QC^T V_2^{-1} \]
\[ C_c = R_2^{-1} B^T P \]

(4)

\[ \Sigma \triangleq BR_2^{-1} B^T \text{ and } \hat{\Sigma} \triangleq C V_2^{-1} C, \] and P and Q are the unique, nonnegative-definite solutions of

\[ A^T P + PA + R_1 - P \Sigma P = 0 \]

(5)

and

\[ AQ + QA^T + V_1 - Q \hat{\Sigma} Q = 0 \]

(6)

respectively. Furthermore, the shifted observability and controllability Gramians\(^1\) of the compensator, P and Q, are the unique, nonnegative-definite solutions of

\[ (A - Q \hat{\Sigma})^T \hat{P} + \hat{P} (A - Q \hat{\Sigma}) + P \Sigma P = 0 \]

(7)

and

\[ (A - \Sigma P)^T \hat{Q} + \hat{Q} (A - \Sigma P)^T + Q \hat{\Sigma} Q = 0 \]

(8)

respectively. The magnitudes of R_1 and V_2 relative to the state weighting matrix R_1 and plant disturbance intensity V_1 govern the regulator and estimator authorities, respectively. The selection of R_1 and V_2 such that ||R_1|| ≥ ||R_2|| or ||V_1|| ≥ ||V_2|| yields a low authority compensator. This section shows that when the open-loop plant is stable and (A, R_1) or (A, V_1) have a particular structure, the LQG controller approaches nonminimalism as the controller authority decreases. To prove this result, we exploit structural properties of the solutions of Riccati and Lyapunov equations assuming the coefficient matrix A and the constant driving term R_1 have certain partitioned forms.

Lemma 2.1. Suppose

\[ A = \begin{bmatrix} A_1 & 0 \\ A_{12} & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} R_{11} & 0 \\ 0 & 0_{n-n_1} \end{bmatrix} \]

(9)

where A_1, R_{11} ∈ R^{n_1 × n_1}, B_1 ∈ R^{n_1 × m_1}, and R_{11} > 0, and assume A is asymptotically stable. Then the following statements hold.

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1) The unique, nonnegative-definite solution of the Riccati equation

\[ A^T P + PA + R_1 - PBB^T P = 0 \]  

is given by

\[ P = \begin{bmatrix} P_1 & 0 \\ 0 & 0_{n-r} \end{bmatrix} \]

where the \( n_r \times n_r \) matrix \( P_1 \) is the unique, positive-definite solution of

\[ A_1^T P_1 + P_1 A_1 + R_{1,1} - P_1 B_1 B_1^T P_1 = 0 \]

2) The unique, nonnegative-definite solution of the Lyapunov equation

\[ A^T P + PA + R_1 = 0 \]

is given by

\[ P = \begin{bmatrix} P_1 & 0 \\ 0 & 0_{n-r} \end{bmatrix} \]

where the \( n_r \times n_r \) matrix \( P_1 \) is the unique, positive-definite solution of

\[ A_1^T P_1 + P_1 A_1 + R_{1,1} = 0 \]

Proof: The proof follows from Theorem 12.2 of Ref. 5. □

The following lemma states the dual of Lemma 2.1 if the coefficient matrix \( A \) is upper block triangular and \( V_1 \) is block diagonal.

**Lemma 2.2.** Suppose

\[ A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad C = [C_1 \ C_2], \quad V_1 = \begin{bmatrix} V_{1,1} & 0 \\ 0 & 0_{n_r-n_r} \end{bmatrix} \]

where \( A_1, V_{1,1} \in R^{n_r \times n_r}, C_1 \in R^{r \times n_r}, \) and \( V_{1,1} > 0, \) and assume \( A \) is asymptotically stable. Then the following statements hold.

1) The unique, nonnegative-definite solution of the Riccati equation

\[ A Q + Q A^T + V_1 - Q C^T C Q = 0 \]

is given by

\[ Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0_{n_r-n_r} \end{bmatrix} \]

where the \( n_r \times n_r \) matrix \( Q_1 \) is the unique, positive-definite solution of

\[ A_1 Q_1 + Q_1 A_1^T + V_{1,1} - Q_1 C_1^T C_1 Q_1 = 0 \]

2) The unique, nonnegative-definite solution of the Lyapunov equation

\[ A Q + Q A^T + V_1 = 0 \]

is given by

\[ Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0_{n_r-n_r} \end{bmatrix} \]

where the \( n_r \times n_r \) matrix \( Q_1 \) is the unique, positive-definite solution of

\[ A_1 Q_1 + Q_1 A_1^T + V_{1,1} = 0 \]

The following theorem shows that with the proper choice of the weighting and disturbance matrices, a low authority LQG controller for a stable plant is nearly nonminimal.

**Theorem 2.1.** Consider the plant given by Eq. (1) and assume \( A \) is asymptotically stable.

1) Suppose \( A \) and \( R_1 \) are partitioned as in Eq. (11) and let \( V_2 \triangleq \beta V_2 \) where \( V_2 \) is a positive-definite matrix and \( \beta \in R \) is a positive scalar. Then for all \( \delta > 0, \) there exists \( N \) such that for all \( \beta > N, (\lambda_{n_r+1}/\lambda_{n_r}) < \delta, \) where \( \lambda_i \) represents the \( i \)th eigenvalue of \( \tilde{Q} \tilde{P}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_1 \geq \lambda_{n_r+1} \geq \cdots \geq 0, \) and where \( \tilde{Q} \) and \( \tilde{P} \) are the shifted controllability and observability Gramians of the corresponding LQG compensator, satisfying Eqs. (8) and (7), respectively.

2) Suppose \( A \) and \( V_1 \) are partitioned as in Eq. (16) and let \( R_2 \triangleq \alpha R_2, \) where \( R_2 \) is a positive-definite matrix and \( \alpha \in R \) is a positive scalar. Then for all \( \delta > 0, \) there exists \( N \) such that for all \( \alpha > N, (\lambda_{n_r+1}/\lambda_{n_r}) < \delta, \) where \( \lambda_i \) represents the \( i \)th eigenvalue of \( \tilde{Q} \tilde{P}. \)

Proof: 1) Partition

\[ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix} \]

conformal to \( A \) in Eq. (9), \( A \) is asymptotically stable if and only if \((A, B)\) and \((A_1, B_1)\) are both stabilizable. Thus, it follows from property 1 of Lemma 2.1 that the unique, nonnegative-definite solution \( P \) of the Riccati equation (5) has the structure given by Eq. (11), which implies that

\[ P \Sigma P = \begin{bmatrix} P_1 \Sigma_1 P_1 & 0 \\ 0 & 0 \end{bmatrix} \]

Next, using the partitioned structures in Eqs. (21) and (9) and the asymptotic stability of \( A, \) it follows from property 2 of Lemma 2.1 that there exists

\[ \tilde{P} = \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & 0_{n_r-n_r} \end{bmatrix} \]

which is the unique, nonnegative-definite solution of

\[ A^T \tilde{P}_0 + \tilde{P}_0 A + P \Sigma P = 0 \]

where the \( n_r \times n_r \) matrix \( \tilde{P}_0 \) is the unique, nonnegative-definite solution of Eq. (11). After computing Eq. (22) and using \( V_2 = \beta V_2, \) yields

\[ A^T \delta \tilde{P}_0 + \delta \tilde{P}_0 A + \beta^{-1} \left[ (\Sigma_0 Q \tilde{P}_0 + (\Sigma_0 Q \tilde{P}_0)^T \right] = 0 \]

The proof of 2 is dual to the proof of 1. □

Remark 2.1. Theorem 2.1 provides two constructions for weighting matrix selection resulting in a nearly nonminimal, low authority LQG compensator for a stable plant. The first approach involves the transformation of the plant \( A \) into coordinates such that \( A \) has the
representation given by Eq. (9). In this case, the weighting matrix \( R_1 \) is selected as in Eq. (9) and with rank \( R_1 = n_s \). By decreasing the authority of the compensator or, equivalently, increasing \( ||V_2|| \) or \( \beta \), the eigenvalue ratio, \( (\lambda_{n+1})/\lambda_{n} \), of the LQG compensator decreases and the LQG compensator approaches nonminimality with minimal dimension of \( n_s \). Using a dual approach, with \( A \) and \( V_1 \) partitioned as in Eq. (16), by increasing \( ||R_1|| \) or \( \alpha \), the resulting LQG compensator approaches nonminimality. In the limiting case, however, as \( \alpha \to \infty \) or \( \beta \to \infty \), then it follows from Eqs. (7) and (8) that \( \hat{P} \to 0 \) and \( \hat{Q} \to 0 \), respectively.

Remark 2.2. Note that if \( A \) is in modal form, then Eqs. (9) and (16) are automatically satisfied. In this case, \( R_1 \) given by Eq. (9) describes a state weighting matrix in which only the states pertaining to selected modes are weighted. Similarly, \( V_1 \), given by Eq. (16) describes a disturbance that excites only certain modes. It is not uncommon for these conditions to be satisfied or nearly satisfied in practice.

III. Numerical Illustrative Example

To illustrate the proper choices of the weighting matrices resulting in a nearly nonminimal, low authority LQG compensator for a stable continuous-time plant, consider a simply supported beam with two collocated sensor/actuator pairs. Assuming the beam has length 2 and that the sensor/actuator pairs are placed at coordinates \( a = 55/172 \) and \( b = 46/43 \), a five-mode continuous-time model, along with the problem data is given in Ref. 6. The noise intensities are \( V_1 = 0.1I_{10} \) and \( V_2 = \beta I_2 \). The design objective is to minimize the continuous-time cost (3) with \( R_2 = \alpha I_2 \). Note that the magnitude of the positive real numbers \( \alpha \) and \( \beta \) are the indicators of the controller authority level. For this system, \( A \) has the representation given by Eqs. (9) and (16) with \( A_{12} = 0 \) and \( A_{21} = 0 \). Here, we illustrate the results of property 1 of Theorem 2.1 for the case of \( n_s = 2 \). Setting \( \alpha = 0.1 \), by selecting the weighting matrix

\[
R_1 = \begin{bmatrix} I_{10} & 0 \\ 0 & 0 \end{bmatrix}
\]

and increasing \( \beta \) (hence, decreasing the compensator authority), the resulting LQG compensator approaches nonminimality with minimal dimension of \( n_s \) or, equivalently, \( (\lambda_{n+1})/(\lambda_{n} \hat{P}) \to 0 \) where \( \lambda_i \) is the (in descending order) \( i \)th eigenvalue of \( \hat{Q}-\hat{P} \).

Figure 1 shows the ratio curve for \( n_s = 2 \) with \( \beta \in (0.01, 0.1, 10, 10^2, 10^3, 10^5, 10^6) \). The curve clearly indicates that the ratio decreases as \( \beta \) increases. To illustrate that suboptimal controller reduction methods yield nearly optimal reduced-order compensators for low authority control problems, Fig. 1 also shows the norm of the cost gradient of the second-order controller obtained by balancing.

The cost gradient is defined as

\[
\begin{bmatrix} \text{vec}(J / \partial A_i)^	op \\ \text{vec}(J / \partial B_i)^	op \\ \text{vec}(J / \partial C_i)^	op \end{bmatrix}^	op
\]

where \( \text{vec}(\cdot) \) denotes the column stacking operator. The cost gradient indicates the balanced controller approaches the optimal reduced-order compensator as \( \beta \) increases, or as the control authority decreases.

Conversely, if the weighting term \( R_1 \) for the preceding example does not have the structure given by Eq. (9), decreasing the controller authority (i.e., increasing \( \beta \)) may not necessarily yield a nearly nonminimal LQG compensator. As an apparent consequence, the norm of the cost gradient of the corresponding second-order balanced controller does not approach zero as the control authority decreases. This is illustrated in Fig. 2 for \( n_s = 2 \) and \( R_1 = I_{10} \). Note that for this particular example, at \( \beta = 0.01 \) the balanced controller destabilizes the closed-loop system and, hence, the norm of the cost gradient becomes infinite.

IV. Conclusion

By exploiting structural properties of the solutions of the regulator and observer LQG Riccati equations this paper shows that, for continuous-time stable systems, if the coefficient matrix \( A \) and driving weighting term \( R_1 \) (or \( V_1 \)) have specific structures, the corresponding LQG compensator becomes nonminimal as the control authority is decreased. The results are directly applicable for initializing and homotopy algorithms that require a reduced-order controller that is nearly optimal for some set of initial design constraints.

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