

Chapter 25

The Role of Systems Biology, Neuroscience, and Thermodynamics in Network Control and Learning



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Abstract Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, battle space management, and congestion control in communication networks. To address the problem of autonomy, complexity, and adaptation for control and coordination of multiagent network systems, in this paper we look to systems biology, neurophysiology, and thermodynamics for inspiration in developing innovative architectures for control and learning.

25.1 Introduction

Due to advances in embedded computational resources over the last several years, a considerable research effort has been devoted to the control of networks and control over networks [1–19]. Network systems involve distributed decision-making for coordination of networks of dynamic agents involving information flow enabling enhanced operational effectiveness via cooperative control in autonomous systems. These dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance [20], distributed reconfigurable sensor networks for managing power levels of wireless networks [21], air and ground transportation systems for air traffic control and payload transport and traffic management [22], swarms of air and space vehicle formations for command and control between heterogeneous air and space assets [10, 14], and congestion control in communication networks for routing the flow of information through a network [5].

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To enable the applications for these complex large-scale dynamical systems, cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, cohesion, separation, alignment, and consensus need to be developed [9, 13, 15, 18, 23–26]. To realize these tasks, individual agents need to share information of the system objectives as well as the dynamical network. In particular, in many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. Information consensus over dynamic information-exchange topologies guarantees agreement between agents for a given coordination task. Distributed consensus algorithms involve neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well.

In systems possessing a continuum of equilibria, *semistability*, and not asymptotic stability is the relevant notion of stability [27–29]. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability thus implies Lyapunov stability and is implied by asymptotic stability. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable. Finally, since multiagent network systems can alternate between multiple semistable states under changing system inputs and system parameters, it is also necessary to address system *multistability* for network systems.

Multistability is the property whereby the solutions of a dynamical system can alternate between two or more mutually exclusive Lyapunov stable and convergent states under slowly changing inputs or system parameters. In particular, multistable systems give rise to the existence of multiple (isolated and/or a continuum of) Lyapunov stable equilibria involving a quasistatic-like behavior between these multiple semistable steady states [30–32].

Modern complex large-scale dynamical systems can additionally give rise to systems which have nonsmooth dynamics. Examples include variable structure systems where control inputs switch discontinuously between extreme input values to generate minimum-time or minimum-fuel system trajectories. Discontinuities are also intentionally designed to achieve hierarchical system stabilization [33, 34]. In particular, the complexity of modern controlled large-scale dynamical systems is further exacerbated by the use of hierarchical embedded control subsystems within the feedback control system, that is, abstract decision-making units performing logical checks that identify system mode operation and specify the continuous-variable sub-controller to be activated. Such systems typically possess a multiechelon hierarchical hybrid decentralized control architecture characterized by continuous-time dynamics

at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy. The lower level units directly interact with the dynamical system to be controlled while the higher level units receive information from the lower level units as inputs and provide (possibly discrete) output commands which serve to coordinate and reconcile the (sometimes competing) actions of the lower level units [35].

The hierarchical controller organization reduces processor cost and controller complexity by breaking up the processing task into relatively small pieces and decomposing the fast and slow control functions. Typically, the higher level units perform logical checks that determine system mode operation, whereas the lower level units execute continuous-variable commands for a given system mode of operation. Due to their multiechelon hierarchical structure, hybrid dynamical systems are capable of simultaneously exhibiting continuous-time dynamics, discrete-time dynamics, logic commands, discrete events, and resetting events. Such systems include dynamical switching systems [34, 36, 37], nonsmooth impact systems [38, 39], biological systems [40], sampled-data systems [41], discrete-event systems [42], intelligent vehicle/highway systems [43], constrained mechanical systems [38], flight control systems [44], and command and control for future battlefields, to cite but a few examples. The mathematical descriptions of many of these systems can be characterized by hybrid systems and impulsive differential equations with discontinuous right-hand sides [35, 40, 45–51].

To enable the autonomous operation for multiagent systems, the development of functional algorithms for agent coordination and control is needed. In particular, control algorithms need to address agent interactions, cooperative and non-cooperative control, task assignments, and resource allocations. To realize these tasks, appropriate sensory and cognitive capabilities such as adaptation, learning, decision-making, and agreement (or consensus) on the agent and multiagent levels are required. The common approach for addressing the autonomous operation of multiagent systems is using distributed control algorithms involving neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents. Since most multiagent network systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks, these systems are characterized by high-dimensional, large-scale interconnected dynamical systems.

To develop distributed methods for control and coordination of autonomous multiagent systems, many researchers have looked to autonomous *swarm* systems appearing in nature for inspiration [1, 2, 6, 52–54]. These systems necessitate the development of relatively simple autonomous agents that are inherently distributed, self-organized, and truly scalable. Scalability follows from the fact that such systems do not involve centralized control and communication architectures. In addition, these systems should be inherently robust to individual agent failures, unplanned task assignment changes, and environmental changes.

In this paper, we highlight the role systems biology, neuroscience, and thermodynamics can play in the control and learning of network systems. Specifically, we explore synergies of cooperative control, learning, dynamical thermodynamics, and neuroscience in developing highly adaptive bioinspired autonomous control meth-

ods for self-organizing teams of multiagent vehicles under sparse sensing. In self-organizing systems, the connection between the local subsystem interactions and the globally complex system behavior is often elusive. In nature, these systems are known as dissipative systems and consume energy and matter while maintaining their stable structure by dissipating entropy to the environment. A common phenomenon among these systems is that they evolve in accordance with the laws of (nonequilibrium) thermodynamics [55, 56]. Dynamical thermodynamics involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics.

The network-based control schemes implemented by the central nervous system for controlling biological aggregations of complex autonomous systems differs drastically from standard control system approaches appearing in the control engineering literature. Specifically, the control scheme is event-triggered, wherein a neuron sends a voltage spike signal (action potential) when the voltage across the cell membrane reaches a certain threshold. In addition, the form of the action potential carries no information. These neurocontrol architectures comprise inherent parallel processing, distributed cellular mechanisms of adaptation and robustness, distributed memory, and low energy consumption making them very attractive for controlling complex, large-scale network autonomous systems.

In light of the above, it seems both natural and appropriate to postulate the following paradigm for nonlinear analysis and control law design of complex large-scale dynamical network systems and multiagent autonomous systems: Develop a unified neuroinspired thermodynamic analysis and control systems design framework for hybrid hierarchical nonlinear large-scale dynamical network systems and multiagent systems in the face of a specified level of modeling uncertainty. This paradigm can provide a rigorous foundation for developing a unified network thermodynamic system analysis and synthesis framework for systems that combine continuous-time and discrete-time dynamics possessing hybrid, hierarchical, and feedback structures. Correspondingly, the main goal of this paper is to present ideas toward the development of analysis and hierarchical hybrid nonlinear control design tools for nonlinear hybrid dynamical systems and multiagent systems which support this paradigm. The proposed paradigm provides the basis for control system partitioning/embedding and develops concepts of thermodynamic-based control and neuroinspired stabilization for complex, large-scale multiagent networked systems.

The contents of this paper are as follows. In Sect. 25.2, we address the problem of network system complexity for multiagent systems with changing system parameters and switching topologies. Specifically, we propose a generalization of the concepts of energy, entropy, and temperature to undirected and directed networks and use hybrid thermodynamic principles to explore the design of distributed consensus control algorithms for static and dynamic networked systems in the face of system uncertainty and intermittent communication. Here the zeroth, first, and second laws for hybrid thermodynamics [57] are used to design distributed consensus controllers that cause networked dynamical systems to emulate natural thermodynamic behavior involving inherent robustness network system guarantees. In particular,

the proposed hybrid controller architecture involves intermittent exchange of state information between agents guaranteeing that the closed-loop dynamical network is semistable to an equipartitioned equilibrium state representing a state of information consensus consistent with hybrid thermodynamic principles [56].

To capture network system uncertainty and communication uncertainty between the agents in a network, wherein the evolution of each link of the network communication topology follows a Markov process for modeling unknown communication noise and attenuations, in Sect. 25.3 we highlight an extension of deterministic semistability and finite time semistability to the stochastic setting in order to develop almost sure consensus protocols for multiagent systems with nonlinear stochastic dynamics. Specifically, using the stochastic semistability and stochastic finite time semistability frameworks developed in [58] we formulate a design paradigm for distributed asymptotic and finite time consensus control protocols for nonlinear undirected and directed dynamical networks with stochastic communication uncertainty. The proposed controller architectures are predicated on the recently developed notion of stochastic dynamical thermodynamics [56] and result in controller architectures involving the exchange of generalized charge or energy state information between agents that guarantee that the closed-loop dynamical network is consistent with stochastic thermodynamic principles.

In Sect. 25.4, we highlight the role systems biology, neuroscience, and thermodynamics can play in the control of dynamical network systems. In particular, we draw on the fundamental principles from dynamical thermodynamics and neuroscience to propose control system architectures for complex large-scale, interconnected systems that are capable of adaptive learning. More specifically, we discuss how dynamical neuroscience and dynamical thermodynamics can be integrated by embedding thermodynamic state notions (e.g., entropy, energy, free energy, chemical potential) within a distributed control system design framework to directly address large-scale swarm networks. This can allow for the development of the next generation of agile, highly adaptive autonomous micro-air vehicles that can utilize mechanisms from systems biology and dynamical thermodynamics to process information from distributed sensors.

Since multiagent systems can involve information laws governed by nodal dynamics and rerouting strategies that can be modified to minimize waiting times and optimize system throughput, in Sect. 25.5 we outline a stochastic nonlinear optimal control framework for large-scale complex networks. Nonlinear controllers may be required for a variety of reasons. These include system nonlinearities, state and control variable constraints (such as saturation and quantization), nonquadratic cost functionals, and structured plant uncertainty. In accordance with the goal of designing practically implementable nonlinear controllers, we extend the classical stochastic Hamilton–Jacobi Bellman optimal control framework to address optimal finite time coordination in multiagent systems as well as develop connections between the newly developed notion of stochastic dissipativity [59], optimal control, and stability margins for stochastic nonlinear feedback regulators. In addition, we discuss the role of universal (asymptotic and finite time) optimal feedback controllers for nonlinear stochastic systems that possess guaranteed gain, sector, and disk margins for

addressing robust stabilization problems involving both stochastic and deterministic uncertainty with averaged and worst-case performance criteria. Furthermore, we discuss how reinforcement learning can provide model-free learning controllers for optimal, thermodynamic-based control architectures of network systems.

In Sect. 25.6, we discuss a network thermodynamic framework for addressing consensus problems for swarm dynamics. Specifically, we explore the development of distributed boundary controller architectures involving the exchange of information between uniformly distributed swarms over an n -dimensional (not necessarily Euclidian) space that guarantees that the closed-loop system is consistent with continuum thermodynamic principles [56]. Since the closed-loop system satisfies thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior is automatically guaranteed. Finally, in Sect. 25.7, we show how the theories of dynamical thermodynamics, information theory, and nonlinear dynamical systems can be merged to develop a unified nonlinear stabilization framework with a priori achievable system performance guarantees using connections between Bode integrals, Shannon entropy, and Clausius entropy.

25.2 Large-Scale Networks and Hybrid Thermodynamics

As discussed in the introduction, modern complex multiagent dynamical systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. Distributed decision-making for coordination of networks of dynamic agents involving information flow can be naturally captured by graph-theoretic notions. These dynamical network systems cover a very broad spectrum of applications. A key application area is cooperative control of air and space vehicle formations using distributed and decentralized controller architectures. Distributed control refers to a control architecture wherein the control is distributed via multiple computational units that are interconnected through information and communication networks, whereas decentralized control refers to a control architecture wherein local decisions are based only on local information. Vehicle formations are typically dynamically decoupled, that is, the motion of a given agent or vehicle does not directly affect the motion of the other agents or vehicles. The multiagent system is coupled via the task which the agents or vehicles are required to perform.

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. For example, in a group of autonomous vehicles this property might be a common heading or a shared communication frequency. Moreover, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics but depends on the initial system state as well. As noted earlier, for

such systems possessing a continuum of equilibria, semistability [27–29], and not asymptotic stability, is the relevant notion of stability.

Alternatively, in other applications of multiagent systems, groups of agents are required to achieve and maintain a prescribed geometric shape. This *formation* problem includes *flocking* [11, 18] and *cyclic pursuit* [15], wherein parallel and circular formations of vehicles are sought. For formation control of multiple vehicles, *cohesion*, *separation*, and *alignment* constraints are typically required for individual agent steering, which describe how a given vehicle maneuvers based on the positions and velocities of nearby agents. Specifically, cohesion refers to a steering rule wherein a given vehicle attempts to move toward the average position of local vehicles, separation refers to collision avoidance with nearby vehicles, whereas alignment refers to velocity matching with nearby vehicles.

Using graph-theoretic notions, in [60] the authors developed control algorithms for addressing consensus problems for nonlinear multiagent dynamical systems with fixed communication topologies. The proposed controller architectures were predicated on the recently developed notion of dynamical thermodynamics [56, 61] resulting in controller architectures involving the exchange of information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles.

A key extension to the framework presented in [60] is to develop a unified *hybrid* thermodynamic control framework for addressing consensus, flocking, and cyclic pursuit problems for multiagent dynamical systems with fixed and dynamic (i.e., switching) topologies. Specifically, building on the work of [60], hybrid distributed and decentralized controller architectures for multiagent coordination can be developed based on the hybrid thermodynamic principles presented in [56, 57]. A unique feature of the proposed framework is that the controller architectures are hybrid, and hence, the overall closed-loop dynamics under these controller algorithms achieving consensus, flocking, or cyclic pursuit will possess discontinuous flows since they will combine logical switchings with continuous dynamics, leading to impulsive differential equations [35, 40, 45, 48, 62–64]. The proposed controllers can involve undirected and directed communication graphs to accommodate for a full range of possible graph information topologies without limitations of bidirectional communication.

To present the key ideas for developing a thermodynamic-based hybrid control architecture for information consensus consider the hybrid dynamical system

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \quad (25.1)$$

$$\Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{Z}, \quad (25.2)$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\Delta x(t) \triangleq x(t^+) - x(t)$, where $x(t^+) \triangleq x(t) + f_d(x(t)) = \lim_{\varepsilon \rightarrow 0^+} x(t + \varepsilon)$, $f_c : \mathcal{D} \rightarrow \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_c(0) = 0$, $f_d : \mathcal{D} \rightarrow \mathbb{R}^n$ is continuous, and $\mathcal{Z} \subset \mathcal{D}$ is the *resetting set*. Note that $x_e \in \mathcal{D}$ is an equilibrium point of (25.1) and (25.2) if and only if $f_c(x_e) = 0$ and $f_d(x_e) = 0$. We refer to the differential equation (25.1)

as the *continuous-time dynamics*, and we refer to the difference Eq. (25.2) as the *resetting law*.

A function $x : \mathcal{I}_{x_0} \rightarrow \mathcal{D}$ is a *solution* to the impulsive dynamical system (25.1) and (25.2) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$, where $\mathcal{I}_{x_0}$ denotes the maximal interval of existence of a solution to (25.1) and (25.2), if $x(\cdot)$ is left-continuous and $x(t)$ satisfies (25.1) and (25.2) for all $t \in \mathcal{I}_{x_0}$. For further discussion on solutions to impulsive differential equations, see [35, 40, 47, 48, 64, 65]. For convenience, we use the notation $s(t, x_0)$ to denote the solution $x(t)$ of (25.1) and (25.2) at time $t \geq 0$ with initial condition $x(0) = x_0$.

To ensure well-posedness of the solutions to (25.1) and (25.2), we make the following additional assumptions [35].

Assumption 25.1 If $x \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $\psi(\delta, x) \notin \mathcal{Z}$, where $\psi(\cdot, \cdot)$ denotes the solution to the continuous-time dynamics (25.1).

Assumption 25.2 If $x \in \mathcal{Z}$, then $x + f_d(x) \notin \mathcal{Z}$.

Assumption 25.1 ensures that if a trajectory reaches the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$; that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, Assumption 25.2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) \notin \mathcal{Z}$, which serves as the initial condition for the continuous-time dynamics (25.1).

For a particular closed-loop trajectory $x(t)$, we let $t_k \triangleq \tau_k(x_0)$ denote the k th instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (25.1) and (25.2) from the initial condition $x(0) = x_0$ is given by $\psi(t, x_0)$ for $0 < t \leq t_1$. If the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_d(x_1)$ according to the resetting law (25.2). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t - t_1, x_1^+)$, and so on. Our convention here is that the solution $x(t)$ of (25.1) and (25.2) is left continuous, that is, it is continuous everywhere except at the resetting times t_k , and $x_k \triangleq x(t_k) = \lim_{\varepsilon \rightarrow 0^+} x(t_k - \varepsilon)$ and $x_k^+ \triangleq x(t_k) + f_d(x(t_k)) = \lim_{\varepsilon \rightarrow 0^+} x(t_k + \varepsilon)$ for $k = 1, 2, \dots$

It follows from Assumptions 25.1 and 25.2 that for a particular initial condition, the resetting times $t_k = \tau_k(x_0)$ are distinct and well defined [35]. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, impulsive dynamical systems of the form (25.1) and (25.2) are time-invariant systems. These systems are called *state-dependent impulsive dynamical systems* [35]. Since the resetting times are well defined and distinct, and since the solution to (25.1) exists and is unique, it follows that the solution of the impulsive dynamical system (25.1) and (25.2) also exists and is unique over a forward time interval. For details on the existence and uniqueness of solutions of impulsive dynamical systems in forward time see [40, 47, 48].

The information consensus problem we consider is a dynamic graph involving the trajectories of the dynamical network characterized by the multiagent impulsive

dynamical system $\mathcal{G}_i$ given by

$$\dot{x}_i(t) = u_{ci}(t), \quad x_i(0) = x_{i0}, \quad x_i(t) \notin \mathcal{Z}_i, \quad i = 1, \dots, q, \quad (25.3)$$

$$\Delta x_i(t) = u_{di}(t), \quad x_i(t) \in \mathcal{Z}_i, \quad (25.4)$$

where, for every $t \geq 0$, $x_i(t) \in \mathbb{R}$ denotes the information state of the i th agent and $u_{ci}(t) \in \mathbb{R}$ and $u_{di}(t) \in \mathbb{R}$, respectively, denote the continuous and discrete information control inputs of the i th agent associated with the controller resetting set $\mathcal{Z}_i \subset \mathbb{R}$, $i \in \{1, \dots, q\}$. For simplicity of exposition, we limit our attention to static controller architectures; dynamic controller architectures are discussed in the end of this section.

The hybrid consensus protocol is given by

$$u_{ci}(t) = \sum_{j=1, i \neq j}^q \phi_{cij}(x_i(t), x_j(t)), \quad (25.5)$$

$$u_{di}(t) = \sum_{j=1, i \neq j}^q \phi_{dij}(x_i(t), x_j(t)), \quad (25.6)$$

where $\phi_{cij}(\cdot, \cdot)$ is locally Lipschitz continuous, $\phi_{dij}(\cdot, \cdot)$ is continuous, $\phi_{cij}(x_i, x_j) = -\phi_{cji}(x_j, x_i)$, and $\phi_{dij}(x_i, x_j) = -\phi_{dji}(x_j, x_i)$ for all $i, j = 1, \dots, q$, $i \neq j$. In this case, the closed-loop system (25.3)–(25.6) is given by

$$\dot{x}_i(t) = \sum_{j=1, i \neq j}^q \phi_{cij}(x_i(t), x_j(t)), \quad x_i(0) = x_{i0}, \quad x_i(t) \notin \mathcal{Z}_i, \quad i = 1, \dots, q, \quad (25.7)$$

$$\Delta x_i(t) = \sum_{j=1, i \neq j}^q \phi_{dij}(x_i(t), x_j(t)), \quad x_i(t) \in \mathcal{Z}_i, \quad (25.8)$$

or, equivalently, in vector form

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \quad (25.9)$$

$$\Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{Z}, \quad (25.10)$$

where $x(t) \triangleq [x_1(t), \dots, x_q(t)]^T \in \mathbb{R}^q$, $f_c(x(t)) \triangleq [f_{c,1}(x(t)), \dots, f_{c,q}(x(t))]^T \in \mathbb{R}^q$, $f_d(x(t)) \triangleq [f_{d,1}(x(t)), \dots, f_{d,q}(x(t))]^T \in \mathbb{R}^q$, $\mathcal{Z} \triangleq \bigcup_{i=1}^q \{x \in \mathbb{R}^q : x_i \in \mathcal{Z}_i\}$, and, for $i, j = 1, \dots, q$,

$$f_{ci}(x(t)) = \sum_{j=1, i \neq j}^q \phi_{cij}(x_i(t), x_j(t)), \quad (25.11)$$

$$f_{di}(x(t)) = \sum_{j=1, i \neq j}^q \phi_{dij}(x_i(t), x_j(t)). \quad (25.12)$$

Note that $\mathcal{G}$ given by (25.7) and (25.8) describe an interconnected network where information states are updated using a distributed hybrid controller involving neighbor-to-neighbor interaction between agents. Furthermore, note that although our results can be directly extended to the case where (25.3) and (25.4) describe the hybrid dynamics of an aggregate system with an aggregate state vector $x(t) = [x_1(t), \dots, x_q(t)]^T \in \mathbb{R}^{Nq}$, where $x_i(t) \in \mathbb{R}^N$, $u_{ci}(t) \in \mathbb{R}^N$ and $u_{di}(t) \in \mathbb{R}^N$, $i = 1, \dots, q$, by using Kronecker algebra, for simplicity of exposition we focus on individual agent states evolving in $\mathbb{R}$ (i.e., $N = 1$)

In order to define the resetting set $\mathcal{Z}$ associated with $\mathcal{G}$, we require some additional notation. Let $\mathcal{O}_i$ denote the set of all agents with information flowing out of the i th agent, let $\mathcal{I}_i$ denote the set of all agents receiving information from the i th agent, and let $\mathcal{V} = \{1, \dots, q\}$ denote the set of vertices representing the connections of the hybrid dynamical network. Furthermore, define the local resetting set $\mathcal{Z}_i$ by

$$\mathcal{Z}_i \triangleq \left\{ x_i \in \mathbb{R} : \sum_{j \in \mathcal{O}_i} \phi_{cij}(x_i, x_j)(x_i - x_j) - \sum_{j \in \mathcal{I}_i} \phi_{cij}(x_i, x_j)(x_i - x_j) = 0 \right. \\ \left. \text{and } x_i \neq x_j, \quad j \in \mathcal{O}_i \cup \mathcal{I}_i \right\}, \quad i = 1, \dots, q \quad (25.13)$$

with

$$\mathcal{Z} \triangleq \cup_{i=1}^q \{x \in \mathbb{R}^q : x_i \in \mathcal{Z}_i\}. \quad (25.14)$$

It follows from Theorem 5.2 of [57] that (25.13) implies that if the time rate of change of the difference in entropies of the connected input information flow and output information flow between the agents is zero and consensus is not reached, then a resetting occurs.

To further ensure a thermodynamically consistent information flow model, we make the following assumptions on the information flow functions $\phi_{cij}(\cdot, \cdot)$, $i, j = 1, \dots, q$, between state resettings:

Assumption 25.3 The connectivity matrix $C \in \mathbb{R}^{q \times q}$ associated with the hybrid multiagent dynamical system $\mathcal{G}$ is defined by

$$C_{(i,j)} = \begin{cases} 0, & \text{if } \phi_{cij}(x_i(t), x_j(t)) \equiv 0, \\ 1, & \text{otherwise,} \end{cases} \quad i \neq j, \quad i, j = 1, \dots, q, \quad t \geq 0 \quad (25.15)$$

and

$$C_{(i,i)} = - \sum_{k=1, k \neq i}^q C_{(k,i)}, \quad i = j, \quad i = 1, \dots, q, \quad (25.16)$$

with rank $C = q - 1$, and for $C_{(i,j)} = 1$, $i \neq j$, $\phi_{cij}(x_i(t), x_j(t)) = 0$ if and only if $x_i(t) = x_j(t)$ for all $x(t) \notin \mathcal{Z}$, $t \geq 0$.

Assumption 25.4 For $i, j = 1, \dots, q$, $[x_i(t) - x_j(t)]\phi_{cij}(x_i(t), x_j(t)) \leq 0$, $x(t) \notin \mathcal{Z}$, $t \geq 0$.

Furthermore, across resettings the information difference must satisfy the following assumption:

Assumption 25.5 For $i, j = 1, \dots, q$, $[x_i(t_{k+1}) - x_j(t_{k+1})][x_i(t_k) - x_j(t_k)] \geq 0$ for all $x_i(t_k) \neq x_j(t_k)$, $x(t_k) \in \mathcal{Z}$, $k \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of non-negative integers.

The condition $\phi_{cij}(x_i(t), x_j(t)) = 0$ if and only if $x_i(t) = x_j(t)$, $i \neq j$, for all $x(t) \notin \mathcal{Z}$ implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *connected*, and hence, can share information; alternatively $\phi_{cij}(x_i, x_j) \equiv 0$ implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *disconnected*, and hence, cannot share information.

Assumption 25.3 implies that if the energies or information in the connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then energy or information exchange between these agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $C = C^T$ and rank $C = q - 1$, then it follows that the connectivity matrix C is irreducible, which implies that for any pair of agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, of $\mathcal{G}$ there exists a sequence of information connectors (information arcs) of $\mathcal{G}$ that connect $\mathcal{G}_i$ and $\mathcal{G}_j$.

Assumption 25.4 implies that energy or information flows from more energetic or information rich agents to less energetic or information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (i.e., energy in transition) must flow in the direction of lower temperatures. Finally, Assumption 25.5 implies that for any pair of connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, the energy or information difference between consecutive jumps is monotonic.

The following key assumption and definition are needed for the main result of this section:

Assumption 25.6 Consider the impulsive dynamical system (25.1) and (25.2), and let $s(t, x_0)$, $t \geq 0$, denote the solution to (25.1) and (25.2) with initial condition x_0 . Then, for every $x_0 \notin \mathcal{Z}$ and every $\varepsilon > 0$ and $t \neq t_k$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $\|x_0 - z\| < \delta(\varepsilon, x_0, t)$, $z \in \mathcal{D}$, then $\|s(t, x_0) - s(t, z)\| < \varepsilon$.

Assumption 25.6 is a weakened version of the quasi-continuous dependence assumption given in [35, 64] and is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. For the hybrid consensus protocol (25.7) and

(25.8) with $\mathcal{Z}_i$ given by (25.13), a straightforward, but lengthy, calculation shows that Assumptions 25.1, 25.2, and 25.6 hold.

The following definition for semistability is needed. Recall that for addressing the stability of an impulsive dynamical system the usual stability definitions are valid [35]. For the statement of the next definition and theorem $\mathcal{B}_\varepsilon(x)$ denotes the open ball centered at x with radius ε and $\mathbf{e} \in \mathbb{R}^q$ denotes the ones vector of order q , that is, $\mathbf{e} \triangleq [1, \dots, 1]^T$.

Definition 25.1 An equilibrium solution $x(t) \equiv x_e \in \mathbb{R}^n$ to (25.1) and (25.2) is *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e)$, then $\lim_{t \rightarrow \infty} x(t)$ exists and corresponds to a Lyapunov stable equilibrium point. An equilibrium point $x_e \in \mathbb{R}^n$ is a *globally semistable equilibrium* if it is Lyapunov stable and, for every $x_0 \in \mathbb{R}^n$, $\lim_{t \rightarrow \infty} x(t)$ exists and corresponds to a Lyapunov stable equilibrium point.

Theorem 25.1 ([66]) Consider the closed-loop hybrid multiagent dynamical system $\mathcal{G}$ given by (25.9) and (25.10) with resetting set $\mathcal{Z}$ given by (25.14). Then, for every $\alpha \geq 0$, $\alpha \mathbf{e}$ is a semistable equilibrium state of $\mathcal{G}$. Furthermore, $x(t) \rightarrow \frac{1}{q} \mathbf{e} \mathbf{e}^T x(0)$ as $t \rightarrow \infty$ and $\frac{1}{q} \mathbf{e} \mathbf{e}^T x(0)$ is a semistable equilibrium state.

Next, we provide explicit connections of the proposed thermodynamic-based control architecture developed in this section with the recently developed notion of hybrid thermodynamics [57]. To develop these connections the following hybrid definition of entropy is needed.

Definition 25.2 For the distributed hybrid consensus control protocol $\mathcal{G}$ given by (25.9) and (25.10), a function $\mathcal{S} : \mathbb{R}^q \rightarrow \mathbb{R}$ satisfying

$$\mathcal{S}(x(T)) \geq \mathcal{S}(x(t_1)), \quad t_1 \leq t_k < T, \quad k \in \mathbb{Z}_+, \quad (25.17)$$

is called an *entropy* function of $\mathcal{G}$.

The next result gives necessary and sufficient conditions for establishing the existence of a hybrid entropy function of $\mathcal{G}$ over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1} , $k \in \mathbb{Z}_+$.

Theorem 25.2 ([66]) Consider the closed-loop hybrid multiagent dynamical system $\mathcal{G}$ given by (25.9) and (25.10), and assume Assumptions 25.3, 25.4, and 25.5 hold. Then a function $\mathcal{S} : \mathbb{R}^q \rightarrow \mathbb{R}$ is an entropy function of $\mathcal{G}$ if and only if

$$\mathcal{S}(x(\hat{t})) \geq \mathcal{S}(x(t)), \quad t_k < t \leq \hat{t} \leq t_{k+1}, \quad (25.18)$$

$$\mathcal{S}(x(t_k^+)) \geq \mathcal{S}(x(t_k)), \quad k \in \mathbb{Z}_+. \quad (25.19)$$

The next theorem establishes the existence of a continuously differentiable entropy function for the closed-loop hybrid multiagent dynamical system $\mathcal{G}$ given by (25.9) and (25.10).

Theorem 25.3 ([66]) *Consider the closed-loop hybrid multiagent dynamical system $\mathcal{G}$ given by (25.9) and (25.10), and assume Assumptions 25.4 and 25.5 hold. Then the function $\mathcal{S} : \mathbb{R}^q \rightarrow \mathbb{R}$ given by*

$$\mathcal{S}(x) = \mathbf{e}^T \mathbf{log}_e(\mathbf{c}\mathbf{e} + x) - q \log_e c, \quad (25.20)$$

where $\mathbf{log}_e(\mathbf{c}\mathbf{e} + x)$ denotes the vector natural logarithm given by $[\log_e(c + x_1), \dots, \log_e(c + x_q)]^T$ and $c > \|x\|_\infty$, is a continuously differentiable entropy function of $\mathcal{G}$. In addition,

$$\dot{\mathcal{S}}(x(t)) \geq 0, \quad x(t) \notin \mathcal{Z}, \quad t_k < t < t_{k+1}, \quad (25.21)$$

$$\Delta \mathcal{S}(x(t_k)) \geq 0, \quad x(t_k) \in \mathcal{Z}, \quad k \in \mathbb{Z}_+. \quad (25.22)$$

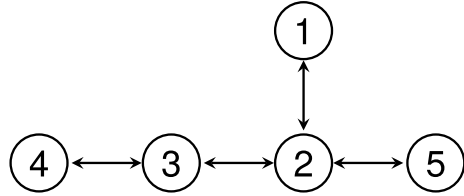
Note that it follows from Theorem 25.2 that the entropy function given by (25.20) satisfies (25.17) as an equality for an equilibrium process and as a strict inequality for a nonequilibrium process. The entropy expression given by (25.20) is identical in form to the Boltzmann entropy for statistical thermodynamics [56]. Due to the fact that the entropy is indeterminate to the extent of an additive constant, we can place the constant $q \log_e c$ to zero by taking $c = 1$. Note that $\mathcal{S}(x)$ given by (25.20) achieves a maximum when all the information states $x_i, i = 1, \dots, q$, are equal [55, 56]. Inequality (25.17) is a generalization of Clausius' inequality for equilibrium and nonequilibrium thermodynamics as well as reversible and irreversible thermodynamics as applied to adiabatically isolated hybrid large-scale thermodynamic systems involving discontinuous phase transitions. For details, see [57].

Next, we demonstrate the proposed distributed hybrid consensus framework on a set of aircrafts achieving asymptotic pitch rate consensus. Specifically, consider the multiagent system comprised of the controlled longitudinal motion of five Boeing 747 airplanes [67] linearized at an altitude of 40 kft and a velocity of 774 ft/s given by

$$\dot{z}_i(t) = A z_i(t) + B \delta_i(t), \quad z_i(0) = z_{i_0}, \quad i = 1, \dots, 5, \quad t \geq 0, \quad (25.23)$$

where $z_i(t) = [v_{x_i}(t), v_{z_i}(t), q_i(t), \theta_{e_i}(t)]^T \in \mathbb{R}^4$, $t \geq 0$, is state vector of agent $i \in \{1, \dots, 5\}$, with $v_{x_i}(t)$, $t \geq 0$, representing the x -body-axis component of the velocity of the airplane center of mass with respect to the reference axes (in ft/s), $v_{z_i}(t)$, $t \geq 0$, representing the z -body-axis component of the velocity of the airplane center of mass with respect to the reference axes (in ft/s), $q_i(t)$, $t \geq 0$, representing the y -body-axis component of the angular velocity of the airplane (pitch rate) with respect to the reference axes (in crad/s), $\theta_{e_i}(t)$, $t \geq 0$, representing the pitch Euler angle of the airplane body axes with respect to the reference axes (in crad), $\delta(t)$, $t \geq 0$, representing the elevator control input (in crad), and

Fig. 25.1 Aircraft communication topology



$$A = \begin{bmatrix} -0.003 & 0.039 & 0 & -0.332 \\ -0.065 & -0.319 & 7.74 & 0 \\ 0.020 & -0.101 & -0.429 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.010 \\ -0.180 \\ -1.16 \\ 0 \end{bmatrix}. \quad (25.24)$$

Here we use a two-level control hierarchy comprised of a lower level controller for command following and a higher level hybrid consensus controller for pitch rate consensus with a communication topology as shown in Fig. 25.1. To address the lower level controller design, let $x_i(t)$, $i = 1, \dots, 5$, $t \geq 0$, denote a command generated by (25.7) and (25.8) (i.e., the guidance command) and let $s_i(t)$, $i = 1, \dots, 5$, $t \geq 0$, denote the integrator state satisfying

$$\dot{s}_i(t) = Ez_i(t) - x_i(t), \quad s_i(0) = s_{i0}, \quad i = 1, \dots, 5, \quad t \geq 0, \quad (25.25)$$

where $E = [0, 0, 1, 0]$. Now, defining the augmented state $\hat{z}(t) \triangleq [z^T(t), s_i(t)]^T$, (25.23) and (25.25) give

$$\dot{\hat{z}}_i(t) = \hat{A}\hat{z}_i(t) + \hat{B}_1\delta_i(t) + \hat{B}_2x_i(t), \quad \hat{z}_i(0) = \hat{z}_{i0}, \quad i = 1, \dots, 5, \quad t \geq 0, \quad (25.26)$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & 0 \\ E & 0 \end{bmatrix}, \quad \hat{B}_1 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{B}_2 \triangleq \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \quad (25.27)$$

Furthermore, let the elevator control input be given by

$$\delta(t) = -K\hat{z}(t), \quad (25.28)$$

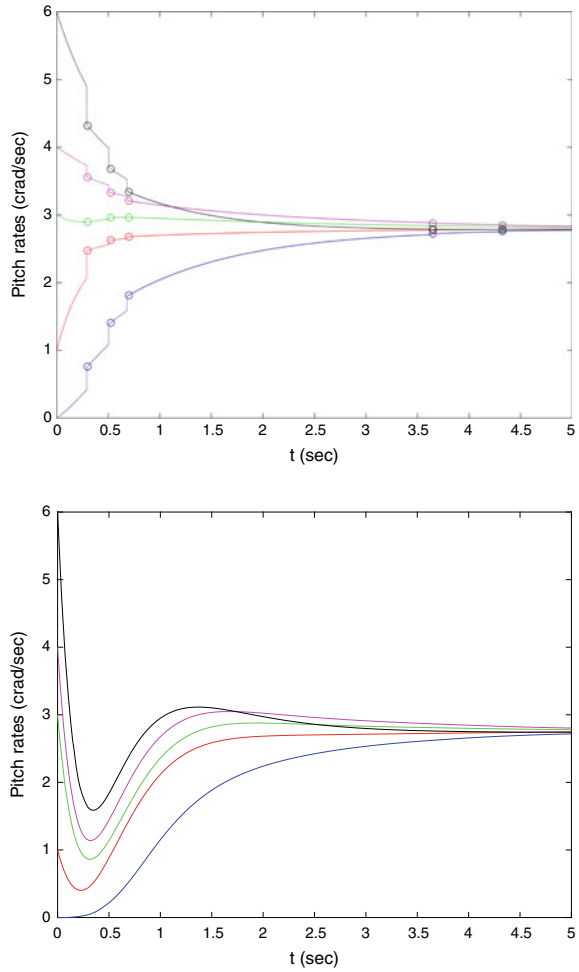
where

$$K = [-0.0157, 0.0831, -4.7557, -0.1400, -9.8603],$$

which is designed based on an optimal linear-quadratic regulator.

For the higher level hybrid consensus controller design, we use (25.7) with $\phi_{cij}(x(t)) = x_j(t) - x_j(t)$, $i, j \in \{1, \dots, 5\}$, $i \neq j$, and (25.8) with $\phi_{dij}(x(t)) = (x_j(t) - x_j(t))/5$, $i, j \in \{1, \dots, 5\}$, $i \neq j$, to generate $x(t)$, $t \geq 0$, that has a direct effect on the lower level controller design to achieve pitch rate consensus. For our simulation we set $x_1(0) = q_1(0) = 0$, $x_2(0) = q_2(0) = 1$, $x_3(0) = q_3(0) = 3$, $x_4(0) =$

Fig. 25.2 Closed-loop command signal $x_i(t)$ (top) and pitch rate $q_i(t)$ (bottom) trajectories for the hybrid consensus protocol; $x_1(t)$ in blue, $x_2(t)$ in red, $x_3(t)$ in green, $x_4(t)$ in magenta, and $x_5(t)$ in black



$q_4(0) = 4$, $x_5(0) = q_5(0) = 6$ and all other initial conditions to zero. Figure 25.2 presents the command signal and pitch rate trajectories of each aircraft versus time. Figure 25.3 shows the system entropy versus time.

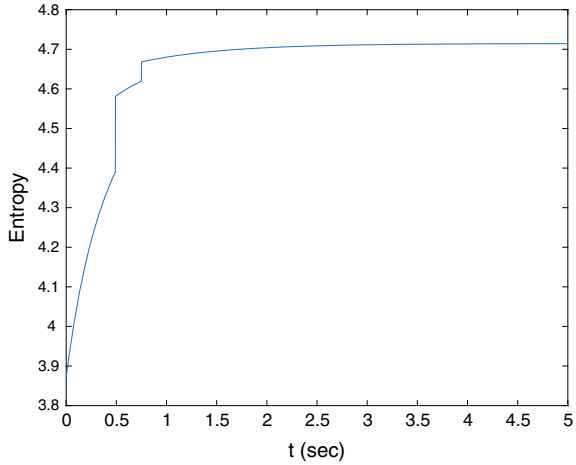
Finally, to present the key ideas for extending this framework to hybrid *dynamic* protocols, consider the continuous-time nonlinear dynamical system given by

$$\dot{x}_p(t) = f_p(x_p(t), u(t)), \quad x_p(0) = x_{p0}, \quad t \geq 0, \quad (25.29)$$

$$y(t) = h_p(x_p(t)), \quad (25.30)$$

where, for every $t \geq 0$, $x_p(t) \in \mathcal{D}_p \subseteq \mathbb{R}^{n_p}$, $\mathcal{D}_p$ is an open set, $u(t) \in \mathbb{R}^m$, $f_p : \mathcal{D}_p \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_p}$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}_p \times \mathbb{R}^m$, and $h_p : \mathcal{D}_p \rightarrow \mathbb{R}^l$ is smooth. Furthermore, consider the hybrid (i.e., resetting) dynamic controller given

Fig. 25.3 System entropy versus time



by

$$\dot{x}_c(t) = f_{cc}(x_c(t), y(t)), \quad x_c(0) = x_{c0}, \quad (x_c(t), y(t)) \notin \mathcal{Z}_c, \quad (25.31)$$

$$\Delta x_c(t) = f_{dc}(x_c(t), y(t)), \quad (x_c(t), y(t)) \in \mathcal{Z}_c, \quad (25.32)$$

$$u(t) = h_{cc}(x_c(t), y(t)), \quad (25.33)$$

where, for every $t \geq 0$, $x_c(t) \in \mathcal{D}_c \subseteq \mathbb{R}^{n_c}$, $\mathcal{D}_c$ is an open set, $\Delta x_c(t) \triangleq x_c(t^+) - x_c(t)$, where $x_c(t^+) \triangleq x_c(t) + f_{dc}(x_c(t), y(t)) = \lim_{\varepsilon \rightarrow 0^+} x_c(t + \varepsilon)$, $(x_c(t), y(t)) \in \mathcal{Z}_c$, $f_{cc} : \mathcal{D}_c \times \mathbb{R}^l \rightarrow \mathbb{R}^{n_c}$ is smooth on $\mathcal{D}_c \times \mathbb{R}^l$, $h_{cc} : \mathcal{D}_c \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ is smooth, $f_{dc} : \mathcal{D}_c \times \mathbb{R}^l \rightarrow \mathbb{R}^{n_c}$ is continuous, and $\mathcal{Z}_c \subset \mathcal{D}_c \times \mathbb{R}^l$ is the *resetting set*. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_c , which may be less than the plant order n_p .

The equations of motion for the closed-loop dynamical system (25.29)–(25.33) have the form (25.1) and (25.2) with

$$x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix} \in \mathbb{R}^n, \quad f_c(x) \triangleq \begin{bmatrix} f_p(x_p, h_{cc}(x_c, h_p(x_p))) \\ f_{cc}(x_c, h_p(x_p)) \end{bmatrix}, \quad f_d(x) \triangleq \begin{bmatrix} 0 \\ f_{dc}(x_c, h_p(x_p)) \end{bmatrix}, \quad (25.34)$$

and $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_c, h_p(x_p)) \in \mathcal{Z}_c\}$, with $n \triangleq n_p + n_c$ and $\mathcal{D} \triangleq \mathcal{D}_p \times \mathcal{D}_c$. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from (25.34) that only those states associated with the controller are reset.

As shown in [68], this hybrid dynamic controller architecture can be used to develop a unified framework for addressing *general* nonlinear dynamical systems of the form (25.29) and (25.30) for achieving consensus, flocking, and cyclic pursuit. Specifically, hybrid distributed and decentralized controller architectures for multiagent coordination can be developed. In contrast to virtually all of the existing results in the literature on control of networks, the proposed controllers will be dynamic compensators. Another unique feature of this framework is that the con-

troller architectures are hybrid, and hence, the overall closed-loop dynamics under these controller algorithms achieving consensus, flocking, or cyclic pursuit will possess discontinuous flows since they will combine logical switchings with continuous dynamics, leading to impulsive differential equations [35, 40, 45, 48, 62–64].

The proposed controllers can use undirected and directed graphs to accommodate for a full range of possible graph information topologies without limitations of bidirectional communication. Furthermore, finite time consensus protocols for addressing finite time coordination, as well as addressing disturbance rejection and robustness extensions can also be developed (see Sect. 25.3). In addition, asynchronism, system time-delays, and dynamic (i.e., switching) network topologies for addressing possible information asynchrony between agents, message transmission and processing delays, and communication link failures and communication dropouts can be addressed.

25.3 Multiagent Systems with Uncertain Interagent Communication

To capture network system uncertainty and communication uncertainty between the agents in a network, almost sure consensus protocols for multiagent systems with nonlinear stochastic dynamics can be formulated using a stochastic thermodynamics formulation [56, 69]. Specifically, the notions of deterministic semistability and finite time semistability to nonlinear stochastic dynamical systems that possess a continuum of equilibrium solutions can be extended to develop almost sure *asymptotic* and *finite time* convergence as well as stochastic Lyapunov stability properties.

For deterministic dynamical systems, the authors in [25, 27–29] developed a unified stability analysis framework for systems having a continuum of equilibria. Since, as noted earlier, every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable nor finite time stable. Hence, asymptotic and finite time stability are not the appropriate notions of stability for systems having a continuum of equilibria. Two notions that are of particular relevance to such systems are convergence and semistability. Convergence is the property whereby every system solution converges (asymptotically or in finite time) to a limit point that may depend on the system initial condition. Semistability (resp., finite time semistability) is the additional requirement that all solutions converge asymptotically (resp., in finite time) to limit points that are Lyapunov stable. Semistability (resp., finite time semistability) for an equilibrium thus implies Lyapunov stability and is implied by asymptotic (resp., finite time) stability.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point [27]. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because the stability of sets is defined in terms of distance

(especially in case of noncompact sets), and it is possible to construct examples in which the dynamical system is semistable, but the domain of semistability contains no ε -neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

The theories of semistability and finite time semistability for deterministic dynamical systems developed in [25, 27–29] were extended in [58] to develop a rigorous framework for *stochastic semistability* and *stochastic finite time semistability*. In particular, [58] presents new Lyapunov theorems as well as the first converse Lyapunov theorem for stochastic semistability, which holds with a continuous Lyapunov function whose infinitesimal generator decreases along the stochastic dynamical system sample trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

Next, [58] establishes stochastic finite time semistability theory. In particular, we develop the notions of finite time convergence in probability and finite time semistability in probability for nonlinear stochastic dynamical systems driven by Markov diffusion processes. Furthermore, the continuity of a settling-time operator is established and a sufficient Lyapunov stability theorem for finite time semistability in probability is developed. Specifically, almost sure finite time convergence and stochastic Lyapunov stability properties are developed to address almost sure finite time semistability requiring that the sample trajectories of a nonlinear stochastic dynamical system converge almost surely in finite time to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable.

Even though convergence, semistability, finite time semistability, and optimality for *deterministic* multiagent network systems involving cooperative control tasks such as formation control, rendezvous, flocking, cyclic pursuit, and consensus have received considerable attention in the literature (see, for example, [1–5, 7–10, 10–15, 17, 18, 20–26, 60, 70–76]), stochastic multiagent networks have not been as plethorically developed; notable contributions include [77–81]. These contributions address asymptotic convergence [79], time-varying network topologies [78], communication delays [81], asynchronous switchings [80], and optimality [77]; however, with the exception of [58], none of the aforementioned references address the problems of stochastic semistability and stochastic finite time semistability.

As noted in the Introduction, a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics but depends on the initial system state as well. Thus, from a practical viewpoint, it is not sufficient for a nonlinear control protocol to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

To capture network system uncertainty and communication uncertainty between the agents in a network, wherein the evolution of each link of the network communication topology follows a Markov process for modeling unknown communication noise and attenuations, stochastic semistability and finite time semistability results can be used to develop almost sure consensus protocols for multiagent systems with nonlinear stochastic dynamics. Specifically, stochastic semistability and stochastic finite time semistability frameworks can be used to design distributed asymptotic and finite time consensus control protocols for nonlinear bidirectional dynamical networks with stochastic communication uncertainty. The proposed controller architectures can be predicated on the recently developed notion of stochastic dynamical thermodynamics [56, 69] resulting in controller architectures involving the exchange of generalized charge or energy state information between agents that guarantee that the closed-loop dynamical network is consistent with stochastic thermodynamic principles.

To elucidate the basis of this approach, we require some additional notation. Specifically, we denote by $\mathcal{G}$ a stochastic dynamical system generating a stochastic process $x : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ -algebra of subsets of Ω , and $\mathbb{P}$ defines a probability measure on $\mathcal{F}$. Moreover, all inequalities and equalities involving random process on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$ -almost surely (a.s.), and with $\mathbb{E}^{x_0}[\cdot]$ denoting expectation with respect to the classical Wiener measure $\mathbb{P}^{x_0}$ and $\mathcal{H}_n^{\mathcal{D}}$ denoting the Hilbert space of random variables $x \in \mathbb{R}^n$ with finite average power induced by $\mathcal{D}$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$dx(t) = f(x(t))dt + D(x(t))dw(t), \quad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq t_0, \quad (25.35)$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$ -measurable random state vector, $x(t_0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d -dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0))$, $t \geq t_0$, and $f : \mathcal{D} \rightarrow \mathbb{R}^n$ and $D : \mathcal{D} \rightarrow \mathbb{R}^{n \times d}$ are continuous functions and satisfy $f(x_e) = 0$ and $D(x_e) = 0$ for some $x_e \in \mathcal{D}$. An *equilibrium point* of (25.35) is a point $x_e \in \mathcal{D}$ such that $f(x_e) = 0$ and $D(x_e) = 0$. It is easy to see that x_e is an equilibrium point of (25.35) if and only if the constant stochastic process $x(\cdot) \stackrel{\text{a.s.}}{=} x_e$ is a solution of (25.35). We denote the set of equilibrium points of (25.35) by $\mathcal{E} \triangleq \{\omega \in \Omega : x(t, \omega) = x_e\} = \{x_e \in \mathcal{D} : f(x_e) = 0 \text{ and } D(x_e) = 0\}$. Here, we assume that all maximal pathwise solutions to (25.35) in $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume that (25.35) is *forward complete*.

The following definition introduces the notion of stochastic semistability.

Definition 25.3 An equilibrium solution $x(t) \stackrel{\text{a.s.}}{=} x_e \in \mathcal{E}$ of (25.35) is *stochastically semistable* if the following statements hold:

- (i) For every $\varepsilon > 0$, $\lim_{x_0 \rightarrow x_e} \mathbb{P}^{x_0} (\sup_{0 \leq t < \infty} \|x(t) - x_e\| > \varepsilon) = 0$. Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$,

$$\mathbb{P}^{x_0} \left(\sup_{0 \leq t < \infty} \|x(t) - x_e\| > \varepsilon \right) \leq \rho.$$

- (ii) $\lim_{\text{dist}(x_0, \mathcal{E}) \rightarrow 0} \mathbb{P}^{x_0} (\lim_{t \rightarrow \infty} \text{dist}(x(t), \mathcal{E}) = 0) = 1$. Equivalently, for every $\rho \in (0, 1)$, there exists $\delta = \delta(\rho) > 0$ such that if $\text{dist}(x_0, \mathcal{E}) \leq \delta$, then $\mathbb{P}^{x_0} (\lim_{t \rightarrow \infty} \text{dist}(x(t), \mathcal{E}) = 0) \geq 1 - \rho$.

The dynamical system (25.35) is *stochastically semistable* if every equilibrium solution of (25.35) is stochastically semistable. Finally, the dynamical system (25.35) is *globally stochastically semistable* if (i) holds and $\mathbb{P}^{x_0} (\lim_{t \rightarrow \infty} \text{dist}(x(t), \mathcal{E}) = 0) = 1$ for all $x_0 \in \mathbb{R}^n$.

Note that if $x(t) \stackrel{\text{a.s.}}{=} x_e \in \mathcal{E}$ only satisfies (i) in Definition 25.3, then the equilibrium solution $x(t) \stackrel{\text{a.s.}}{=} x_e \in \mathcal{E}$ of (25.35) is Lyapunov stable in probability.

The notion of stochastic finite time semistability involves finite time almost sure convergence along with stochastic semistability. For this definition, s^x represents a sample path trajectory of $\mathcal{G}$ with initial condition x .

Definition 25.4 An equilibrium solution $x(t) \stackrel{\text{a.s.}}{=} x_e \in \mathcal{E}$ of (25.35) is (globally) *stochastically finite time semistable* if there exists an operator $T : \mathcal{H}_n \rightarrow \mathcal{H}_1^{[0, \infty)}$, called the *stochastic settling-time operator*, such that the following statements hold:

- (i) *Finite time convergence in probability.* For every $x(0) \in \mathcal{H}_n \setminus \mathcal{E}$, $s^{x(0)}(t)$ is defined on $[0, T(x(0)))$, $s^{x(0)}(t) \in \mathcal{H}_n \setminus \mathcal{E}$ for all $t \in [0, T(x(0)))$, and

$$\mathbb{P}^{x_0} \left(\lim_{t \rightarrow T(x(0))} \text{dist}(s^{x(0)}(t), \mathcal{E}) = 0 \right) = 1.$$

- (ii) *Lyapunov stability in probability.* For every $\varepsilon > 0$,

$$\lim_{x_0 \rightarrow x_e} \mathbb{P}^{x_0} \left(\sup_{0 \leq t < \infty} \|s^{x(0)}(t) - x_e\| > \varepsilon \right) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$, $\mathbb{P}^{x_0} (\sup_{0 \leq t < \infty} \|s^{x(0)}(t) - x_e\| > \varepsilon) \leq \rho$.

- (iii) *Finiteness of the stochastic settling-time operator.* For every $x \in \mathcal{H}_n \setminus \mathcal{E}$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x [T(x)] < \infty$.

The dynamical system (25.35) is (globally) *stochastically finite time semistable* if every equilibrium solution of (25.35) is globally stochastically finite time semistable.

The following additional notion and definitions are needed. Specifically, let $\mathfrak{G}(C) = (\mathcal{V}, \mathcal{E})$ be a *directed graph* (or digraph) denoting the dynamical network

(or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \dots, q\}$ involving a finite nonempty set denoting the agents, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and a *connectivity matrix* $C \in \mathbb{R}^{q \times q}$ such that $C_{(i,j)} = 1, i, j = 1, \dots, q$, if $(j, i) \in \mathcal{E}$, while $C_{(i,j)} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent j can obtain information from agent i , but not necessarily vice versa. Moreover, we assume $C_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the connectivity matrix $C \in \mathbb{R}^{q \times q}$ is a directed graph for which the *arc set* is symmetric, that is, $C = C^T$. Weighted graphs can also be considered here; however, since this extension does not alter any of the conceptual results in the paper we do not consider this extension for simplicity of exposition.

To address the consensus problem with communication uncertainty, consider q continuous-time agents with dynamics

$$dx_i(t) = u_i(t)dt + \text{row}_i(D(x(t)))dw(t), \quad i = 1, \dots, q, \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \quad (25.36)$$

where $q \geq 2$ is the number of agents in the network with a communication graph topology $\mathfrak{G}(C)$, $D(x)dw$, where $D(x) = [\text{row}_1(D(x)), \dots, \text{row}_q(D(x))]^T : \mathbb{R}^q \rightarrow \mathbb{R}^q \times \mathbb{R}^d$, captures probabilistic variations in the information transfer rates between agents, and, for every $i \in \{1, \dots, q\}$, $x_i(t) \in \mathcal{H}_1$ denotes the information state of the i th agent and $u_i(t) \in \mathcal{H}_1$ denotes the control input of the i th agent. For a general distributed control architecture resulting in a network consensus action corresponding to an underlying conservation law, we assume $\mathbf{e}_q^T D(x) = 0, x \in \mathbb{R}^q$, where $\mathbf{e}_q \triangleq [1, \dots, 1]^T \in \mathbb{R}^q$, and where the agent state $x_i(t) \in \mathcal{H}_1$ denotes the generalized charge (i.e., Noether charge or simply charge) state and the control input $u_i(t) \in \mathcal{H}_1$ denotes the conserved current input for all $t \geq 0$.

The nonlinear consensus protocol is given by

$$u_i(t) = \sum_{j=1, j \neq i}^q C_{(i,j)} [\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))], \quad (25.37)$$

where $\sigma_{ij}(\cdot), i, j \in \{1, \dots, q\}, i \neq j$, are Lipschitz continuous. Here we assume that the control process $u_i(\cdot)$ in (25.37) is restricted to a class of admissible control protocols consisting of measurable functions adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that, for every $i \in \{1, \dots, q\}$, $u_i(\cdot) \in \mathcal{H}_1, t \geq 0$, and, for all $t \geq s$, $w_i(t) - w_i(s)$ is independent of $u_i(\tau), w_i(\tau), \tau \leq s$, and $x_i(0)$, and hence, $u_i(\cdot)$ is nonanticipative. Furthermore, we assume $u_i(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [82] that there exists a unique pathwise solution to (25.36) and (25.37) in $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}^{x_{i0}})$ for every $i \in \{1, \dots, q\}$. Finally, note that the closed-loop system (25.36) and (25.37) is given by

$$dx_i(t) = \sum_{j=1, j \neq i}^q C_{(i,j)} [\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))] dt + \text{row}_i(D(x(t))) dw(t),$$

$$i = 1, \dots, q, \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \quad (25.38)$$

Equation (25.38) represents the collective dynamics of q agents which interact by exchanging charge. The coefficients scaling the functions $\sigma_{ij}(\cdot)$, $i, j \in \{1, \dots, q\}$, $i \neq j$, appearing in (25.38) represent the topology of the charge exchange between the agents. More specifically, given $i, j \in \{1, \dots, q\}$, $i \neq j$, a coefficient of $C_{(i,j)} = 1$ denotes that subsystem j receives charge from subsystem i , and a coefficient of zero denotes that subsystem i and j are disconnected, and hence, cannot share any charge.

As in Sect. 25.2, our results can be directly extended to the case where (25.36) and (25.37) describe the dynamics of an aggregate multiagent system with an aggregate state vector $x(t) = [x_1^T(t), \dots, x_q^T(t)]^T \in \mathcal{H}_{Nq}$, where $x_i(t) \in \mathcal{H}_N$ and $u_i(t) \in \mathcal{H}_N$, $i = 1, \dots, q$, by using Kronecker algebra. Here, for simplicity of exposition, we focus on individual agent states evolving in $\mathcal{H}_1$ (i.e., $N = 1$).

Next, note that since

$$\mathbf{e}_q^T dx(t) = \mathbf{e}_q^T f(x(t)) dt + \mathbf{e}_q^T D(x(t)) dw(t) = 0, \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \quad (25.39)$$

it follows that $\sum_{i=1}^q dx_i(t) \stackrel{\text{a.s.}}{=} 0$, $t \geq 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law. Now, it follows from Nöether's theorem [83] that to every conservation law there corresponds a symmetry. To show this for our multiagent network, the following definition and assumptions are needed.

Definition 25.5 ([84]) A directed graph $\mathfrak{G}(C)$ is *strongly connected* if for every ordered pair of vertices (i, j) , $i \neq j$, there exists a *path* (i.e., a sequence of arcs) leading from i to j .

Recall that the connectivity matrix $C \in \mathbb{R}^{q \times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that C is cogredient to a lower block triangular matrix, if and only if $\mathfrak{G}(C)$ is strongly connected (see Theorem 2.7 of [84]).

Assumption 25.7 The connectivity matrix $C \in \mathbb{R}^{q \times q}$ associated with the multiagent stochastic dynamical system $\mathcal{G}$ is defined by

$$C_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0, \\ 1, & \text{otherwise,} \end{cases} \quad i \neq j, \quad i, j = 1, \dots, q, \quad (25.40)$$

and

$$C_{(i,i)} \triangleq - \sum_{k=1, k \neq i}^q C_{(k,i)}, \quad i = j, \quad i = 1, \dots, q, \quad (25.41)$$

with rank $C = q - 1$, and for $C_{(i,j)} = 1, i \neq j, \sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$.

Assumption 25.8 For $i, j = 1, \dots, q$,

$$\sum_{j=1, j \neq i}^q C_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -\text{row}_i(D(x))\text{row}_i^T(D(x)).$$

The information connectivity between the agents can be represented by the network communication graph topology $\mathfrak{G}(C)$ having q nodes such that $\mathfrak{G}(C)$ has an undirected edge from node i to node j if and only if agent j can receive charge from agent i . Since the coefficients scaling $\sigma_{ij}(\cdot), i, j \in \{1, \dots, q\}, i \neq j$, are constants, the communication graph topology of the network $\mathfrak{G}(C)$ is fixed. Furthermore, note that the graph $\mathfrak{G}$ is *weakly connected* since the underlying undirected graph is connected; that is, every agent receives charge from, or delivers charge to, at least one other agent.

The fact that $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j, i \neq j$, implies that agent i and j are *connected*, and hence, can share information; alternatively, $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0$ implies that agent i and j are *disconnected*, and hence, cannot share information. Assumption 25.7 thus implies that if the charge (or generalized energies) in the connected agents i and j are equal, then charge exchange between the agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $C = C^T$ and rank $C = q - 1$, then it follows that the connectivity matrix C is irreducible, which implies that for any pair of i and $j, i \neq j$, of $\mathcal{G}$ there exists a sequence information connectors (information arcs) of $\mathcal{G}$ that connect agents i and j .

Assumption 25.8 implies that charge flows from charge rich agents to charge poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (i.e., energy in transition) must flow in the direction of lower temperatures. It is important to note here that due to the stochastic term $D(x)d\mathbf{w}$ capturing probabilistic variations in the charge transfer (i.e., generalized current) between the agents, the second assumption requires that the scaled net charge flow $C_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]$ is bounded by the negative intensity of the diffusion coefficient given by $\frac{1}{2}\text{tr } D(x)D^T(x)$. For further details on Assumptions 25.7 and 25.8, see [56, 69].

The intensity $D(x)$ of the general probabilistic variations $D(x)d\mathbf{w}$ in the agent communication can take different forms to capture communication measurement noise or errors in the information transfer rates between agents. For example, we can consider $D(x) = M\hat{D}(x)$, where

$$\begin{aligned} M &\triangleq [m_{(1,2)}, \dots, m_{(1,q)}, m_{(2,3)}, \dots, m_{(2,q)}, \dots, m_{(q-1,q)}] \in \mathbb{R}^{q \times \frac{1}{2}q(q-1)} \\ \hat{D}(x) &\triangleq \text{diag}[d_{(1,2)}(x), \dots, d_{(1,q)}(x), d_{(2,3)}(x), \dots, d_{(2,q)}(x), \dots, d_{(q-1,q)}(x)] \\ &\in \mathbb{R}^{\frac{1}{2}q(q-1) \times \frac{1}{2}q(q-1)} \end{aligned}$$

and $m_{(i,j)}d_{(i,j)}(x_i, x_j)dw_i$ represents stochastic variations in the information flow between the i th and j th agent. Furthermore, considering

$$d_{(i,j)}(x_i, x_j) = C_{(i,j)}(x_j - x_i)^p, \quad (25.42)$$

where $p > 0$ and $m_{(i,j)} \in \mathbb{R}^q$ satisfies $m_{(i,j)_i} \geq 0$, $m_{(i,j)_j} \leq 0$, $m_{(i,j)_j} = -m_{(i,j)_i}$, $m_{(i,j)_k} = 0$, $k \neq i$, $k \neq j$, where $m_{(i,j)_i}$ denotes the i th component of $m_{(i,j)}$, it follows that $e_q^T m_{(i,j)} = 0$, and hence, it can be shown that (25.39) holds. Note that (25.42) captures nonlinear relative uncertainty between interagent communication. Of course, more general nonlinear uncertainties can also be considered.

For simplicity of exposition, in the reminder of this section we let $d = 1$ and $p = 1$, and consider q continuous-time agents with dynamics

$$\begin{aligned} dx_i(t) &= u_i(t)dt + \sum_{j=1, j \neq i}^q \gamma C_{(i,j)}[x_j(t) - x_i(t)]dw(t), \\ i &= 1, \dots, q, \quad x_i(0) \stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0, \end{aligned} \quad (25.43)$$

where $\gamma \in \mathbb{R}$, so that the closed-loop system (25.43) and (25.37) is given by

$$\begin{aligned} dx_i(t) &= \sum_{j=1, j \neq i}^q C_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]dt \\ &+ \sum_{j=1, j \neq i}^q \gamma C_{(i,j)}[x_j(t) - x_i(t)]dw(t), \quad i = 1, \dots, q, \\ x_i(0) &\stackrel{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \end{aligned} \quad (25.44)$$

In this case, (25.38) can be cast in the form of (25.35) with

$$f(x) = \begin{bmatrix} \sum_{j=1, j \neq 1}^q C_{(1,j)}[\sigma_{1j}(x_j) - \sigma_{j1}(x_1)] \\ \vdots \\ \sum_{j=1, j \neq q}^q C_{(q,j)}[\sigma_{qj}(x_j) - \sigma_{jq}(x_q)] \end{bmatrix}, \quad (25.45)$$

$$D(x) = \begin{bmatrix} \sum_{j=1, j \neq 1}^q \gamma C_{(1,j)}[x_j(t) - x_1(t)] \\ \vdots \\ \sum_{j=1, j \neq q}^q \gamma C_{(q,j)}[x_j(t) - x_q(t)] \end{bmatrix}, \quad (25.46)$$

where the stochastic term $D(x)dw$ represents probabilistic variations in the charge transfer rate (i.e., generalized currents) between the agents. Furthermore, Assumption 25.8 now takes the following form.

Assumption 25.9 For $i, j=1, \dots, q$, $C_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -(q-1)\gamma^2 C_{(i,j)}^2(x_i - x_j)^2$.

Theorem 25.4 ([58]) Consider the nonlinear stochastic multiagent system given by (25.44) and assume that Assumptions 25.7 and 25.9 hold. Then, for every $\alpha \in \mathbb{R}$, $\alpha \mathbf{e}_q$ is a stochastically semistable equilibrium state of (25.44). Furthermore, $x(t) \xrightarrow{a.s.} \frac{1}{q} \mathbf{e}_q \mathbf{e}_q^T x(0)$ as $t \rightarrow \infty$ and $\frac{1}{q} \mathbf{e}_q \mathbf{e}_q^T x(0)$ is a stochastically semistable equilibrium state.

Since in many consensus control protocol applications it is desirable for the closed-loop dynamical system that exhibits semistability to also possess the property that the system trajectories that almost surely converge to a Lyapunov stable in probability system state do so in finite time rather than merely asymptotically, next we develop a thermodynamically motivated finite time consensus framework for multiagent nonlinear stochastic systems that achieve finite time stochastic semistability and almost sure state equipartition.

Specifically, in place of (25.37) we can consider nonlinear consensus protocols of the form

$$\begin{aligned} u_i(t) = & \sum_{j=1, j \neq i}^q C_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))] \\ & + c \sum_{j=1, j \neq i}^q C_{(i,j)} \text{sign}(x_j(t) - x_i(t)) |x_j(t) - x_i(t)|^\theta, \end{aligned} \quad (25.47)$$

where $c > 0$ is a design constant, $0 < \theta < 1$, $\text{sign}(y) \triangleq y/|y|$, $y \neq 0$, with $\text{sign}(0) \triangleq 0$, and $\sigma_{ij}(\cdot)$, $i, j \in \{1, \dots, q\}$, $i \neq j$, are as in (25.37). Note that in this case the closed-loop system (25.36) and (25.47) is given by

$$\begin{aligned} dx_i(t) = & \sum_{j=1, j \neq i}^q C_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]dt \\ & + c \sum_{j=1, j \neq i}^q C_{(i,j)} \text{sign}(x_j(t) - x_i(t)) |x_j(t) - x_i(t)|^\theta \\ & + \sum_{j=1, j \neq i}^q \gamma C_{(i,j)}[x_j(t) - x_i(t)]dw_i(t), \quad i = 1, \dots, q, \\ & x_i(0) \stackrel{a.s.}{=} x_{i0}, \quad t \geq 0. \end{aligned} \quad (25.48)$$

Note that with $n = q$ and $d = 1$, (25.48) can be cast in the form of (25.35) with

$$f(x) = \begin{bmatrix} \sum_{j=1, j \neq 1}^q C_{(1,j)} [\sigma_{1j}(x_j) - \sigma_{1j}(x_1)] \\ + c \sum_{j=1, j \neq 1}^q C_{(1,j)} \text{sign}(x_j - x_1) |x_j - x_1|^\theta \\ \vdots \\ \sum_{j=1, j \neq q}^q C_{(q,j)} [\sigma_{qj}(x_j) - \sigma_{qj}(x_q)] \\ + c \sum_{j=1, j \neq q}^q C_{(q,j)} \text{sign}(x_j - x_q) |x_j - x_q|^\theta \end{bmatrix} \quad (25.49)$$

and $D(x)$ defined as in (25.46). Furthermore, note that since

$$\mathbf{e}_q^T dx(t) = \mathbf{e}_q^T f(x(t)) dt + \mathbf{e}_q^T D(x(t)) dw(t) = 0, \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0,$$

it follows that $\sum_{i=1}^q dx_i(t) \stackrel{\text{a.s.}}{=} 0, t \geq 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law.

Theorem 25.5 ([58]) *Consider the nonlinear stochastic multiagent system given by (25.48) with $c > 0$ and $\theta \in (0, 1)$, and assume that Assumptions 25.7 and 25.9 hold. Then, for every $\alpha \in \mathbb{R}$, $\alpha \mathbf{e}_q$ is a stochastically finite time semistable equilibrium state of (25.48). Moreover, $x(t) = \frac{1}{q} \mathbf{e}_q \mathbf{e}_q^T x(0)$ for all $t \geq T(x(0))$, where*

$$\mathbb{E}^{x_0}[T(x(0))] \leq \frac{4V(x_0)^{\frac{1-\theta}{2}}}{c(1-\theta)(4\lambda_2(L(C)))^{\frac{1+\theta}{2}}},$$

$\lambda_2(L(C))$ is the Fiedler eigenvalue of the graph Laplacian of $\mathfrak{G}(C)$, and

$$V(x_0) = \frac{1}{2} \left(x_0 - \frac{1}{q} \mathbf{e}_q \mathbf{e}_q^T x_0 \right)^T \left(x_0 - \frac{1}{q} \mathbf{e}_q \mathbf{e}_q^T x_0 \right).$$

Here, we once again use the set of aircrafts considered in Sect. 25.2 to demonstrate the proposed stochastic thermodynamic consensus framework for networks with communication uncertainty. However, instead of five aircrafts we consider three aircrafts with an uncertain triangular communication topology. For the higher level communication consensus controller design, we use (25.37) with $\sigma_{ij}(x_j) = x_j$ and $\sigma_{ji}(x_i) = x_i$ to generate $x_i(t), t \geq 0$, that has a direct effect on the lower level controller design to achieve pitch rate consensus. Figure 25.4 presents the sample trajectories along with the standard deviation of the states of each agent versus time for 10 sample paths for all initial conditions set to zero and $x_1(0) \stackrel{\text{a.s.}}{=} 8, x_2(0) \stackrel{\text{a.s.}}{=} 4$, and $x_3(0) \stackrel{\text{a.s.}}{=} 2$. The mean control profile is also plotted in Fig. 25.4.

More general probabilistic variations for interagent communication as in (25.42) as well as unidirectional communication topologies can also be explored.

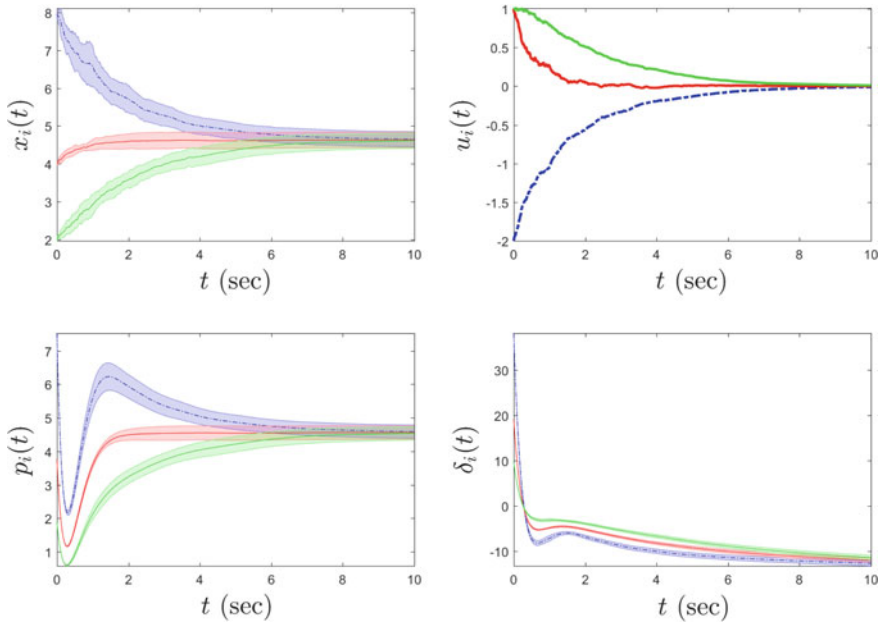


Fig. 25.4 Sample average along with the sample standard deviation versus time for agent guidance state ($x_i(t)$, $t \geq 0$), guidance input ($u_i(t)$, $t \geq 0$), pitch rate ($q_i(t)$, $t \geq 0$), and elevator control ($\delta_i(t)$, $t \geq 0$) responses for the standard consensus protocol given by (25.37) with $k_1 = 1$. The control profile is plotted as the mean of the 10 sample runs

25.4 Systems Biology, Neurophysiology, Thermodynamics, and Dynamic Switching Communication Topologies for Large-Scale Multilayered Networks

To develop consensus control design protocols for nonlinear networks with dynamic switching topologies, we can look to the field of neuroscience [85–88] for inspiration. And in particular the study of the central nervous system that exhibits nearly discontinuous transitions between macroscopic states. One such example exhibiting this phenomenon is the induction of general anesthesia [89–92] resulting in a transition from consciousness to unconsciousness (i.e., a consensus state). In current clinical practice of general anesthesia, potent drugs are administered which profoundly influence levels of consciousness and vital respiratory (ventilation and oxygenation) and cardiovascular (heart rate, blood pressure, and cardiac output) functions. These variation patterns of the physiologic parameters (i.e., ventilation, oxygenation, heart rate variability, blood pressure, and cardiac output) and their alteration with levels of consciousness can provide scale-invariant fractal temporal structures to characterize the degree of consciousness in sedated patients [89].

The degree of consciousness reflects the adaptability of the central nervous system and is proportional to the maximum work output under a fully conscious state divided by the work output of a given anesthetized state in analogy to Carnot's theorem [56]. A reduction in maximum work output (and cerebral oxygen consumption) or elevation in the anesthetized work output (or cerebral oxygen consumption) will thus reduce the degree of consciousness. Hence, the fractal nature (i.e., complexity) of conscious variability is a self-organizing emergent property of the large-scale interconnected biological neuronal network since it enables the central nervous system to maximize entropy production and dissipate energy gradients.

Complex dynamical systems involving self-organizing components forming spatiotemporal evolving structures that exhibit a hierarchy of emergent system properties form the underpinning of the central nervous system. These complex dynamical systems are ubiquitous in nature and engineering systems and are not limited to the central nervous system. Such systems include, for example, biological systems, immune systems, ecological systems, quantum particle systems, chemical reaction systems, economic systems, cellular systems, modern network systems, and battle space management systems involving distributed command and control between heterogeneous air and ground assets, to cite but a few examples. The connection between the local subsystem interactions and the globally complex system behavior is often elusive. These systems are known as dissipative systems [56, 93, 94] and consume energy and matter while maintaining their stable structure by dissipating entropy to the environment.

In the central nervous system billions of neurons interact to form self-organizing dissipative nonequilibrium structures [56, 61, 93, 94]. As noted in introduction, the fundamental common phenomenon among these systems is that they evolve in accordance with the laws of (nonequilibrium) thermodynamics, which are among the most firmly established laws of nature. Dynamical thermodynamics, in the sense of [56], involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics. Self-organization can spontaneously occur in such systems by invoking the two fundamental axioms of the science of heat.

Namely, (i) if the energies in the connected subsystems of an interconnected system are equal, then energy exchange between these subsystems is not possible, and (ii) energy flows from more energetic subsystems to less energetic subsystems. These axioms establish the existence of a system entropy function as well as lead to equipartition of energy [56, 61, 95, 96] in system thermodynamics, synchronization [97] in biological neuronal networks, and information consensus [60] in cooperative networks; an emergent behavior in thermodynamic systems as well as neuroscience and swarm dynamics. Hence, in complex interconnected dynamical systems, self-organization is not a property of the systems' parts but rather emerges as a result of the nonlinear subsystem interactions.

In recent research [56, 61, 95, 96, 98–100], we combined the two universalisms of thermodynamics and dynamical systems theory under a single umbrella to develop a dynamical system formalism for classical thermodynamics so as to harmonize it

with classical mechanics. In particular, our dynamical system formalism captures all of the key aspects of thermodynamics, including its fundamental laws, while providing a mathematically rigorous formulation for thermodynamical systems out of equilibrium by unifying the theory of heat transfer with that of classical thermodynamics. In addition, the concept of entropy for a nonequilibrium state of a dynamical process is defined, and its global existence and uniqueness is established. This state space formalism of thermodynamics shows that the behavior of heat, as described by the conservation equations of thermal transport and as described by classical thermodynamics, can be derived from the same basic principles and is part of the same scientific discipline. Connections between irreversibility, the second law of thermodynamics, and the entropic arrow of time are also established in [56, 61, 96, 98].

Building on the results in [56, 61, 95, 96, 98, 99], dynamical thermodynamics can be merged with neuroscience to develop a neuroinspired control design framework for network consensus using control architectures predicated on the network control properties of the brain by rigorously emulating the large-scale interconnected biological neuronal network control model given in [89–92]. Specifically, as in thermodynamics, neuroscience is a theory of large-scale systems wherein graph theory [101] can be used in capturing the (possibly dynamic) connectivity properties of network interconnections, with neurons represented by nodes, synapses represented by edges or arcs, and synaptic efficacy captured by edge weighting giving rise to a weighted adjacency matrix governing the underlying directed dynamic graph network topology. However, unlike thermodynamics, wherein heat (i.e., energy in transition) spontaneously flows from a state of higher temperature to a state of lower temperature, neuron membrane potential variations occur due to ion species exchanges, which evolve from regions of higher chemical potentials to regions of lower chemical potentials (i.e., Gibbs' chemical potential [56, 99]). And this evolution does not occur spontaneously but rather requires a hierarchical continuous-discrete (i.e., hybrid) control architecture for the opening and closing of specific gates within specific ion channels.

Merging the dynamical neuroscience framework developed in [89–92] with dynamical thermodynamics [56, 61, 96, 99] by embedding thermodynamic state notions (i.e., entropy, energy, free energy, chemical potential, etc.) within our dynamical systems framework can allow us to directly exploit the otherwise mathematically complex and computationally prohibitive large-scale dynamical system model given in [89–92] for designing consensus and synchronization control protocols for nonlinear multilayered network systems. In particular, a thermodynamically consistent neuroinspired controller architecture would emulate the clinically observed self-organizing, spatiotemporal fractal structures that optimally dissipate energy and optimize entropy production in thalamocortical circuits of fully conscious patients. This thermodynamically consistent neuroinspired control framework can provide the necessary tools involving multistability, synaptic drive equipartitioning (i.e., synchronization or consensus across time scales), energy dispersal, and entropy production for connecting network consensus for large-scale network dynamical systems to synchronization (i.e., agreement over time instants) in thalamocortical circuits in the necortex. This is a subject of current research.

25.5 Nonlinear Stochastic Optimal Control and Learning

Since multiagent systems can involve information laws governed by nodal dynamics and rerouting strategies that can be modified to minimize waiting times and optimize system throughput, optimality considerations in network systems is of paramount importance. In view of the advantages of nonlinear optimal controllers over linear controllers [29, 102], it is not surprising that a significant effort has been devoted to developing a theory of optimal nonlinear regulation [103–137]. For problems involving nonquadratic cost functionals on the infinite interval, asymptotic stability is guaranteed by means of a Lyapunov function for the closed-loop system. This Lyapunov function is given as the solution to a steady-state form of the Hamilton–Jacobi–Bellman equation [29, 102].

We propose to reverse the situation somewhat by fixing the structure of the Lyapunov function, cost functional, and feedback law prior to optimization. In this case, the structure of the Lyapunov function can be viewed as providing the *framework* for controller synthesis by guaranteeing local or global semistability or asymptotic stability for a class of feedback controllers. The *actual* controller chosen for implementation can thus be the member of this candidate class that minimizes the given performance functional. In LQG theory, for example, the Lyapunov function is the familiar quadratic functional $V(x) = x^T P x$, while the gains for the linear feedback control are chosen to minimize a quadratic performance functional. Thus, Lyapunov function theory provides the framework, whereas optimization fixes the gains.

To elucidate the basis for such an approach for nonlinear stochastic systems, consider the problem of evaluating a nonlinear-nonquadratic cost functional depending upon a controlled nonlinear stochastic differential equation. It can be shown that the cost functional can be evaluated in closed form so long as the cost functional is related in a specific way to an underlying Lyapunov function [138]. In accordance with practical motivations, we restrict our attention to time-invariant systems on the infinite horizon. Furthermore, for simplicity of exposition, we shall define all functions globally and assume that existence and uniqueness properties of the given stochastic differential equations are satisfied in forward time. For details, see [138].

Let $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $D : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times d}$, assume that $F(0, 0) = 0$, $D(0, 0) = 0$, and consider the nonlinear controlled stochastic dynamical system

$$dx(t) = F(x(t), u(t))dt + D(x(t), u(t))dw(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \quad (25.50)$$

where $w(t)$ is a d -dimensional independent standard Wiener process. The control $u(\cdot)$ in (25.50) is nonanticipative and is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in \mathcal{H}_m^U$, $t \geq 0$, where $U \subseteq \mathbb{R}^m$ is given. Next, let $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, let $V'(x)$ and $V''(x)$ denote the Fréchet derivative and the Hessian of V at x , respectively, and, for $p \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$, define

$$H(x, p, Q, u) \triangleq L(x, u) + p^T F(x, u) + \frac{1}{2} \text{tr } D^T(x, u) Q D(x, u). \quad (25.51)$$

Furthermore, define the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (49)} \\ \text{satisfies } x(t) \xrightarrow{\text{a.s.}} 0 \text{ as } t \rightarrow \infty\}.$$

Theorem 25.6 ([138]) *Consider the nonlinear controlled stochastic dynamical system (25.50) with performance functional*

$$J(x_0, u(\cdot)) \triangleq \mathbb{E}^{x_0} \left[\int_0^\infty L(x(t), u(t)) dt \right], \quad (25.52)$$

where $u(\cdot)$ is an admissible control. Assume that there exists a two-times continuously differentiable radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ and a control law $\phi : \mathbb{R}^n \rightarrow U$ such that

$$V(0) = 0, \quad V(x) > 0, \quad \phi(0) = 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (25.53)$$

$$V'(x)F(x, \phi(x)) + \frac{1}{2} \text{tr} D^T(x, \phi(x))V''(x)D(x, \phi(x)) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (25.54)$$

$$H(x, V^T(x), V''(x), \phi(x)) = 0, \quad x \in \mathbb{R}^n, \quad (25.55)$$

$$H(x, V^T(x), V''(x), u) \geq 0, \quad x \in \mathbb{R}^n, \quad u \in U. \quad (25.56)$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system (25.50) is globally asymptotically stable in probability and $J(x_0, \phi(x(\cdot))) = V(x_0)$, $x_0 \in \mathbb{R}^n$. Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)). \quad (25.57)$$

In spite of the appealing nature of the stochastic Hamilton–Jacobi–Bellman theory, its current state of development entails several limitations in addressing real-world problems. In particular, these include (i) the ability to design static and dynamic output-feedback compensators and (ii) the development of robust controllers for uncertain plants. Hence, a principal goal is to extend the current state of the theory by removing these limitations.

For the class of network control problems that we are addressing there are typically only a small number of measurements available. An indirect solution to this problem is to implement an estimator to reconstruct the state from the available measurements. However, besides being a difficult problem itself, there is no reason to

expect that certainty equivalence (separation) will hold in the presence of nonlinearities. Consequently, we must consider the problem of designing output-feedback laws that are pre-constrained to operate solely upon the available measurements. Such controllers may be either static (proportional) or dynamic.

To achieve this goal it appears natural to develop a *fixed-structure* Hamilton–Jacobi–Bellman theory in which one can prespecify the structure of the feedback law with respect to, for example, the order of nonlinearities appearing in the dynamic compensator. The actual gain maps can then be determined by solving algebraic relations in much the same way full-state feedback controllers can be obtained.

By assuming polynomial forms for the controlled drift and diffusion terms $F(x, u)$ and $D(x, u)$, and considering polynomial or multilinear parameterizations of the Lyapunov function $V(x)$ and the control policy $\phi(x)$, the optimal stochastic stability criterion can be casted as in (25.54). In addition to the development of the fixed-structure synthesis for optimal stabilizing control under full-state feedback, this framework can be used to address the problem of static and dynamic output-feedback control [139].

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval.

In order to achieve convergence in finite time for deterministic dynamical systems, the closed-loop system dynamics need to be non-Lipschitzian giving rise to non-uniqueness of solutions in backward time. Uniqueness of solutions in forward time, however, can be preserved in the case of finite time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity for deterministic dynamical systems are given in [140–143], whereas [144, 145] give sufficient conditions that ensure uniqueness of solutions for stochastic dynamical systems in forward time in the absence of a uniform Lipschitz continuity and a growth restriction condition on the system drift and diffusion functions.

For deterministic dynamical system, finite time convergence to a Lyapunov stable equilibrium, that is, *finite time stability*, was rigorously studied in [146, 147] using Hölder continuous Lyapunov functions. Finite time stabilization of second-order systems was considered in [148, 149]. More recently, researchers have considered finite time stabilization of higher order systems [150] as well as finite time stabilization using output feedback [151]. Design of globally strongly stabilizing continuous controllers for nonlinear systems using the theory of homogeneous systems was studied in [152]. Alternatively, discontinuous finite time stabilizing feedback controllers have also been developed in the literature [153–155]. However, for practical implementations, discontinuous feedback controllers can lead to chattering due to system

uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

The problem of finite time *optimal* control has received very little attention in the literature. When the performance functional (25.52) involves terms of order x^p , where $p < 2$, then we call the cost criterion *subquadratic*. Subquadratic cost criteria pay close attention to the behavior of the state near the origin since, for example $x^{\frac{1}{2}} \gg x^2$ for x in a neighborhood of zero. Our interest in subquadratic cost criteria stems from the fact that optimal controllers for such criteria are sublinear, and thus, exhibit finite settling-time behavior. This phenomenon was studied in [128, 148] and applied to spacecraft control in [156].

The aforementioned results yield finite interval controllers even though the original cost criterion is defined on the infinite horizon. Hence, one advantage of this approach for certain applications is to obtain finite interval controllers without the computational complexities of two-point boundary value problems. We also note that if the order of the subquadratic state terms appearing in the cost functional is sufficiently small, then the controllers actually optimize a minimum-time cost criterion. Currently, such results are only obtainable using the maximum principle, which generally does not yield feedback controllers.

The nonlinear control framework presented in this section can be readily extendable to address *optimal stochastic finite time stabilization* using a feedback control architecture by requiring (25.54) to satisfy a fractional Lyapunov differential inequality [157, 158]. To state this extension the following definition is required.

Definition 25.6 ([158]) The zero solution $x(t) \stackrel{\text{a.s.}}{=} 0$ to (25.35) is (globally) *stochastically finite time stable* if there exists an operator $T : \mathcal{H}_n \rightarrow \mathcal{H}_1^{(0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold:

- (i) *Finite time convergence in probability.* For every $x(0) \in \mathcal{H}_n$, $s^{x(0)}(t)$ is defined on $[0, T(x(0)))$, $s^{x(0)}(t) \in \mathcal{H}_n$ for all $t \in [0, T(x(0)))$, and

$$\mathbb{P}^{x_0} \left(\lim_{t \rightarrow T(x(0))} \|s^{x(0)}(t)\| = 0 \right) = 1.$$

- (ii) *Lyapunov stability in probability.* For every $\varepsilon > 0$,

$$\mathbb{P}^{x_0} \left(\sup_{t \in [0, T(x(0))]} \|s^{x(0)}(t)\| > \varepsilon \right) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$, $\mathbb{P}^{x_0} \left(\sup_{t \in [0, T(x(0))]} \|s^{x(0)}(t)\| > \varepsilon \right) \leq \rho$.

- (iii) *Finiteness of the stochastic settling-time operator.* For every $x \in \mathcal{H}_n$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x [T(x)] < \infty$.

Theorem 25.7 ([158]) Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (25.35) with $\mathcal{D} = \mathbb{R}^n$. If there exist a radially unbounded positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and a function $\eta : \overline{\mathbb{R}}_+ \rightarrow \overline{\mathbb{R}}_+$ such that $V(0) = 0$, $V(x)$ is two-times continuously differentiable for all $x \in \mathbb{R}^n$, $\eta(\cdot)$ is continuously differentiable, and, for all $x \in \mathbb{R}^n$,

$$V'(x)f(x) + \frac{1}{2}\text{tr } D^T(x)V''(x)D(x) \leq -\eta(V(x)), \quad (25.58)$$

$$\int_0^\varepsilon \frac{dv}{\eta(v)} < \infty, \quad \varepsilon \in [0, \infty), \quad (25.59)$$

$$\eta'(v) > 0, \quad v \geq 0, \quad (25.60)$$

then $\mathcal{G}$ is globally stochastically finite time stable. Moreover, there exists a settling-time operator $T : \mathcal{H}_n \rightarrow \mathcal{H}_1^{(0, \infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_0)] \leq \int_0^{V(x_0)} \frac{dv}{\eta(v)}, \quad x_0 \in \mathbb{R}^n. \quad (25.61)$$

Note that if $\eta(V) = cV^\theta$, where $c > 0$ and $\theta \in (0, 1)$, then $\eta(\cdot)$ satisfies (25.59) and (25.60). In this case, (25.61) becomes

$$\mathbb{E}^{x_0}[T(x(0))] \leq \frac{V(x_0)^{1-\theta}}{c(1-\theta)}.$$

For deterministic dynamical systems, this specialization recovers the finite time stability result given in [146].

Using the control synthesis version (rather than the analysis version) of Theorem 25.7, a unified framework for finite time stochastic stabilization was developed in [158]. Furthermore, building on the results of [159–161] as well as on the recent stochastic finite time stabilization framework of [158] we developed a constructive universal feedback control law for stochastic finite time stabilization of stochastic dynamical systems [162]. In addition, we presented necessary and sufficient conditions for continuity of such controllers. Furthermore, we showed that for every nonlinear stochastic dynamical system for which a stochastic control Lyapunov function can be constructed there exists an inverse optimal feedback control law in the sense of [138] with guaranteed sector and gain margins of $(\frac{1}{2}, \infty)$.

Moreover, using the newly developed notion of stochastic dissipativity [59], in [163] we derive a return difference inequality to provide connections between stochastic dissipativity and optimality of nonlinear controllers for stochastic dynamical systems. In particular, analogous to deterministic controlled dynamical systems,

using extended Kalman–Yakubovich–Popov conditions characterizing stochastic dissipativity [59] we show that our optimal feedback control law satisfies a return difference inequality predicated on the infinitesimal generator of a controlled Markov diffusion process if and only if the controller is stochastically dissipative with respect to a specific quadratic supply rate. This allows us to develop connections between stochastic dissipativity and stochastic optimal control to address robust stability and robust stabilization problems involving both stochastic and deterministic uncertainty as well as both averaged and worst-case performance criteria.

A key extension to optimal stochastic Hamilton–Jacobi–Bellman theory is the design of *semistabilizing* optimal controllers involving controlled stochastic dynamical systems with a continuum of equilibria. In [164], we developed $\mathcal{H}_2$ optimal semistable control theory for *linear* dynamical systems necessary for addressing *optimal consensus control* problems for multiagent systems. Unlike the standard $\mathcal{H}_2$ optimal control problem, it is shown in [164, 165] that a complicating factor of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. In addition, we showed that the $\mathcal{H}_2$ optimal solution is given by a least squares solution to the closed-loop Lyapunov equation over all possible semistabilizing solutions. Moreover, it is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem. Building on the results of this section we can address the problem of finding a state feedback *nonlinear* control law $u = \phi(x)$ that minimizes the performance measure (25.52) and guarantees stochastic semistability of *nonlinear* dynamical systems of the form (25.50).

A key limitation of the solution procedures for most optimal control problems is that they require full knowledge of the system dynamics. To this end, adaptive control along with robust control theory has been developed to address the problem of system uncertainty in control system design. In contrast to fixed-gain robust controllers, which are predicated on a mathematical model of the system uncertainty, and which maintain specified constants within the feedback control law to *sustain* robust stability and performance over the range of system uncertainty, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop system stability and *improve* system performance in the face of system uncertainties. However, unlike most robust controllers, adaptive controllers are not usually optimal with respect to a prespecified performance criterion.

Online adaptive learning control algorithms have recently been developed for optimal control problems using measurements along the dynamical system trajectories. These control algorithms build on adaptive dynamic programming and neurodynamic programming methods [166, 167] and are known as *reinforcement learning* [168]. Reinforcement learning controllers are based on adaptive critic or actor–critic structures that involve simultaneous tuning of the actor–critic parameters that can learn optimal control solutions by (approximately) solving the Hamilton–Jacobi–Bellman equation in real time without requiring the full knowledge of the system dynamics. Thus, reinforcement learning-based approaches allow for the development of control algorithms that can be used in real time to affect optimal and adaptive control in the presence of system uncertainty [169, 170].

Reinforcement learning methods have been developed by observing learning mechanisms of living organisms in nature [171]. Most living things in nature are chronognostic and adopt their behavior to a dynamic (i.e., changing) environment. Thus, the behavior has a temporal component, wherein all living organisms perceive their future relative to the present. Since behavioral anticipation is vital for survival, every decision-making organism interacts with its environment and adapts its actions based on the learned responses that it receives from its actions.

In controlled dynamical systems, reinforcement learning refers to a method in which the controller can learn the unknown and (possibly) time-varying dynamics of the system by interacting with the system and its environment [169, 171]. The actor–critic structure of the algorithm evaluates the current control policy and iteratively updates it to meet a given performance measure. The control policy update is carried out by observing the response of the system predicated on the current control policy. Therefore, the reinforcement learning-based controller can learn optimal actions in the presence of system parameter uncertainty and in the absence of the complete knowledge of the system dynamics.

A direct extension of the optimal control framework outlined in this section would involve a stochastic optimal control framework based on the development of algorithms for general models of stochasticity and multi-trajectory optimization and reinforcement learning. In particular, many large-scale multilayered network systems with communication uncertainty, as well as biological, financial, and engineered processes, can be represented stochastically involving distributions such as Levy distributions and jump diffusion processes. The optimal control method presented in this section can be readily extended to address these stochastic models by generalizing work on risk sensitivity and logarithmic transformations for the case of Markov-jump diffusion processes. Whereas standard reinforcement learning methods utilize trajectory optimizers based on single trajectory optimization, the proposed extension would allow for the development of algorithms for multi-trajectory optimization and control of the full probability density of the dynamical system's states by exploring generalizations of the Hamilton–Jacobi–Bellman equation to optimal control of infinite-dimensional systems. In addition, using the framework outlined in Sects. 25.2 and 25.3, system thermodynamic notions can be used to derive symmetric Fourier type distributed control law protocols predicated on online policy iteration algorithms that achieve information consensus in the face of uncertain network communication topologies.

25.6 Complexity, Thermodynamics, Information Theory, and Swarm Dynamics

To enable the autonomous operation for large-scale multiagent network systems, the development of functional algorithms for agent coordination, control, and learning is needed. In particular, control algorithms need to address agent interactions, coop-

erative and non-cooperative control, task assignments, and resource allocations. To realize these tasks, appropriate sensory and cognitive capabilities such as adaptation, learning, decision-making, and agreement (or consensus) on the agent and multiagent levels are required. The common approach for addressing the autonomous operation of multiagent systems is using distributed control algorithms involving neighbor-to-neighbor interaction between agents wherein agents update their information state based on the information states of the neighboring agents.

Since most multiagent network systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks, these systems are characterized by high-dimensional, large-scale interconnected dynamical systems. To develop distributed methods for control and coordination of autonomous multiagent systems, many researchers have looked to autonomous *swarm* systems appearing in nature for inspiration [1, 2, 6, 52–54].

Biology has shown that many species of animals such as insect swarms, ungulate flocks, fish schools, ant colonies, and bacterial colonies *self-organize* in nature [172–175]. These biological aggregations give rise to remarkably complex global behaviors from simple local interactions between large numbers of relatively unintelligent agents without the need for centralized control. The spontaneous development (i.e., self-organization) of these autonomous biological systems and their spatiotemporal evolution to more complex states often appears without any external system interaction. In other words, structure morphing into coherent groups is internal to the system and results from local interactions among subsystem components that are independent of the physical nature of the individual components. These local interactions often comprise a simple set of rules that lead to remarkably complex global behaviors. *Complexity* here refers to the quality of a system wherein interacting subsystems self-organize to form hierarchical evolving structures exhibiting *emergent* system properties. In addition, the spatially distributed sensing and actuation control architecture prevalent in such systems is inherently *robust* to individual subsystem (or agent) failures and unplanned behavior at the individual subsystem (or agent) level.

The connection between the local subsystem interactions and the globally complex system behavior is often elusive. Complex dynamical systems involving self-organizing components forming spatiotemporally evolving structures that exhibit a hierarchy of emergent system properties are not limited to biological aggregation systems. Such systems include, for example, nervous systems, immune systems, ecological systems, quantum particle systems, chemical reaction systems, economic systems, cellular systems, and galaxies, to cite but a few examples. As discussed in Sect. 25.4, these systems are known as *dissipative systems* [61, 176] and consume energy and matter while maintaining their stable structure by dissipating entropy to the environment. For example, as in neurophysiology involving billions of interacting neurons, in the physical universe billions of stars and galaxies interact to form self-organizing dissipative nonequilibrium structures [176]. The fundamental common phenomenon among these systems are that they evolve in accordance with the laws of (nonequilibrium) thermodynamics which are among the most firmly established laws of nature.

As discussed in Sect. 25.3, dynamical thermodynamics, in the sense of [56], involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics. Self-organization can spontaneously occur in such systems by invoking the two fundamental axioms of the science of heat. Namely, (i) if the energies in the connected subsystems of an interconnected system are equal, then energy exchange between these subsystems is not possible, and (ii) energy flows from more energetic subsystems to less energetic subsystems. These axioms establish the existence of a system entropy function as well as *equipartition of energy* [56] in system thermodynamics and *information consensus* [61] in cooperative networks; an *emergent* behavior in thermodynamic systems as well as swarm dynamics. Hence, in complex interconnected dynamical systems, self-organization is not a property of the system's parts but rather emerges as a result of the nonlinear subsystem interactions.

In light of the above discussion, engineering swarm systems necessitates the development of relatively simple autonomous agents that are inherently distributed, self-organized, and truly scalable. Scalability follows from the fact that such systems do not involve centralized control and communication architectures. In addition, engineered swarming systems should be inherently robust to individual agent failures, unplanned task assignment changes, and environmental changes. Mathematical models for large-scale swarms can involve Lagrangian and Eulerian models. In a Lagrangian model, each agent is modeled as a particle governed by a difference or differential equation, whereas an Eulerian model describes the local energy or information flux for a distribution of swarms with an advection–diffusion (conservation) equation. The two formulations can be connected by a Fokker–Plank approximation relating jump distance distributions of individual agents to terms in the advection–diffusion equation [56, 175].

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein, as discussed in Sect. 25.2, a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or *consensus*. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, it was argued in Sects. 25.2 and 25.3 that semistability [27, 177], and not asymptotic stability, is the relevant notion of stability.

Dynamical thermodynamics [56] can be used to develop distributed boundary control algorithms for addressing the consensus problem for an Eulerian swarm model [95]. The distributed boundary controller architectures are predicated on the recently developed notion of continuum dynamical thermodynamics [56] resulting in controller architectures involving the exchange of information between uniformly distributed swarms over an n -dimensional (not necessarily Euclidian) space that guarantees that the closed-loop system is consistent with basic thermodynamic principles. For such a thermodynamically consistent model the existence of a unique continu-

ously differentiable entropy functional for all equilibrium and nonequilibrium states of our system can also be established. Information consensus and semistability follow using the well-known Sobolev embedding theorems and the notion of generalized (or weak) solutions. Since the closed-loop system can be designed in such a way to satisfy basic thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior is automatically guaranteed. For details, see [56, 95].

25.7 Thermodynamic Entropy, Shannon Entropy, Bode Integrals, and Performance Limitations in Nonlinear Systems

Thermodynamics grew out of steam tables and the desire to design and build efficient heat engines, with its central problem involving hard limits on the efficiency of heat engines. Using the laws of thermodynamics, which are among the most firmly established laws of nature, Carnot's principle states that it is impossible to perform a repeatable cycle in which the only result is the performance of positive work [56]. In particular, Carnot showed that the *efficiency* of a reversible cycle—that is, the ratio of the total work produced during the cycle and the amount of heat transferred from a boiler to a cooler—is bounded by a universal maximum, and this maximum is only a function of the temperatures of the boiler and the cooler. In other words, Carnot's principle shows that for any cyclic process that is shielded from heat exchange with its environment, it is impossible to extract work from heat without at the same time discarding some heat, giving rise to an increasing quantity known as (thermodynamic) *entropy*. From a system-theoretic point of view, entropy production places hard limits on system (heat engine) performance.

Whereas thermodynamic entropy captures a measure of the amount of wasted energy in a dynamical (energy) transformation from one state (form) to another, *Shannon entropy* captures a measure of information contained in a random signal with prescribed distribution. In particular, Shannon's source-coding theorem [178] states that for a randomly generated data signal there exists a simple coding procedure, wherein the average code length for the data of length l is approximately l times the entropy of the signal source. In other words, Shannon entropy places hard limits of information performance that can be reliably retrieved from a compressed memoryless signal source.

Fundamental limits of achievable performance in linear feedback control systems were first investigated by Bode [179]. Specifically, Bode's integral theorem states that for a single-input, single-output stable system transfer function with a stable loop-gain and relative degree greater than or equal to two, the integral over all frequencies of the natural logarithm of the magnitude of the sensitivity transfer function $S(s)$ vanishes, that is,

$$\int_0^\infty \log_e |S(j\omega)| d\omega = 0. \quad (25.62)$$

This result shows that it is not possible to decrease $|S(j\omega)|$ below the value of 1 over all frequencies imposing fundamental limitations on achievable tracking, disturbance rejection, and robust performance for the closed-loop system. Bode's integral limitation theorem has been extended to multi-input, multi-output unstable systems [180]. In particular, the authors in [180] show that the integral over all frequencies of the natural logarithm of the magnitude of the determinant of the sensitivity transfer function is proportional to the sum of the unstable loop-gain poles, that is,

$$\int_0^\infty \log_e |\det S(j\omega)| d\omega = \pi \sum_{i=1}^{n_u} \operatorname{Re} p_i > 0, \quad (25.63)$$

where $p_i, i = 1, \dots, n_u$, denotes the i th unstable loop-gain pole. The unstable poles in the right-hand side of (25.63) worsens the achievable tracking, disturbance rejection, and robust performance for the closed-loop system. Nonlinear extensions of Bode's integral theorem based on an information-theoretic interpretation, singular control, and Markov chains appear in [181–183].

Merging dynamical thermodynamics [56], information theory [184], and nonlinear dynamical systems theory, a unified nonlinear stabilization framework with a priori achievable system performance guarantees can be developed. The fact that classical thermodynamics is a physical theory concerning systems in equilibrium, information theory resorts to statistical (subjective or informational) probabilities, and control theory is based on a dynamical systems theory made it all but impossible to unify these theories, leaving these disciplines to stand in sharp contrast to one another in the seven decades of their coexistence. Yet all three theories involve fundamental limitations of performance giving rise to system entropy notions.

Using the dynamical systems framework for nonequilibrium thermodynamics recently developed in [56], we can harmoniously amalgamate thermodynamics, information theory, and control theory under a single umbrella for quantifying limits of performance for nonlinear system stabilization. The starting point of such a unified framework is to place information theory on a state space footing using graph-theoretic notions [56]. As in the case of thermodynamic entropy for systems out of equilibrium [56], this will allow us to develop an analytical description of an objective property of information entropy that can potentially offer a conceptual advantage over the subjective or informational expressions for information entropy proposed in the literature (e.g., Shannon entropy, von Neumann entropy, Kolmogorov–Sinai entropy). This can potentially allow us to quantify fundamental limitations for robustness and disturbance rejection of nonlinear feedback systems with finite capacity input–output signal communication rates.

To highlight some of the features of the aforementioned ideas and elucidate how thermodynamic entropy concepts can be merged with dynamical systems theory,

consider the nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (25.64)$$

$$y(t) = h(x(t)) + J(x(t))u(t), \quad (25.65)$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ denotes the state vector, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the control input, $y(t) \in Y \subseteq \mathbb{R}^l$ denotes the system output, and $f : \mathcal{D} \rightarrow \mathbb{R}^n$, $G : \mathcal{D} \rightarrow \mathbb{R}^{n \times m}$, $h : \mathcal{D} \rightarrow \mathbb{R}^l$, and $J : \mathcal{D} \rightarrow \mathbb{R}^{l \times m}$. For the dynamical system $\mathcal{G}$ given by (25.64) and (25.65) defined on the state space $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{U}$ and $\mathcal{Y}$ define input and output spaces, respectively, consisting of continuous bounded U -valued and Y -valued functions on the semi-infinite interval $[0, \infty)$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator. Here, for simplicity of exposition, the mappings $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are assumed to be continuously differentiable and $f(\cdot)$ has at least one equilibrium point $x_e \in \mathcal{D}$ so that $f(x_e) + G(x_e)u_e = 0$ and $y_e = h(x_e) + J(x_e)u_e$ for some $u_e \in U$. Finally, we assume that $\mathcal{G}$ is completely reachable [29].

A thermodynamic feedback control framework can be developed based on the notion of the thermodynamic entropy developed in [56]. To elucidate such a framework, the following definition of *feedback dissipativity* is needed. Feedback dissipative systems define a class of dynamical systems for which a continuously differentiable feedback transformation exists that renders the system $\mathcal{G}$ dissipative and is a *generalization* of the *feedback passivation* notion introduced in [185].

Definition 25.7 The nonlinear dynamical system $\mathcal{G}$ is *state feedback dissipative* if there exists a state feedback transformation $u = \phi(x) + \beta(x)v$, where $\phi : \mathcal{D} \rightarrow \mathbb{R}^m$ and $\beta : \mathcal{D} \rightarrow \mathbb{R}^{m \times m}$ are continuously differentiable, with $\det \beta(x) \neq 0$, $x \in \mathcal{D}$, such that the nonlinear dynamical system $\mathcal{G}_s$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)) + G(x(t))\beta(x(t))v(t), \quad x(0) = x_0, \quad t \geq 0, \quad (25.66)$$

$$y(t) = h(x(t)) + J(x(t))\phi(x(t)) + J(x(t))\beta(x(t))v(t), \quad (25.67)$$

is dissipative with respect to the supply rate $r(v, y)$, where $r : U \times Y \rightarrow \mathbb{R}$ is locally integrable for all input–output pairs satisfying (25.66) and (25.67), and $r(0, 0) = 0$. If $r(v, y) = v^T y$, then $\mathcal{G}$ is *state feedback passive*.

For simplicity of exposition, here we assume that $\beta(x) = I_m$.

The nonlinear dynamical system $\mathcal{G}$ given by (25.64) and (25.65) is feedback equivalent to a passive system with a two-times continuously differentiable storage function if and only if $\mathcal{G}$ has (vector) relative degree $\{1, \dots, 1\}$ at $x = 0$ and is weakly minimum phase [29]. Alternatively, the Kalman–Yakubovich–Popov lemma [29] can be used to construct smooth state feedback controllers that guarantee feedback passivation as well as feedback dissipativity [185].

The following result is a direct consequence of dissipativity theory [29]. Here we assume that all storage functions $V_s(\cdot)$ of the nonlinear dynamical system $\mathcal{G}_s$ are continuously differentiable.

Proposition 25.1 ([29]) *Consider the nonlinear dynamical system $\mathcal{G}$ given by (25.64) and (25.65), and assume that $\mathcal{G}$ is state feedback dissipative. Then there exist functions $V_s : \mathbb{R}^n \rightarrow \mathbb{R}$, $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times m}$ such that $V_s(\cdot)$ is continuously differentiable and non-negative definite, $V_s(x_e) = V_{se}$, and $\dot{V}_s(x) = r(v, y) - [\ell(x) + \mathcal{W}(x)v]^T[\ell(x) + \mathcal{W}(x)v]$.*

Defining $d(x, v) \triangleq [\ell(x) + \mathcal{W}(x)v]^T[\ell(x) + \mathcal{W}(x)v]$, where $d : \mathcal{D} \times U \rightarrow \overline{\mathbb{R}}_+$ is a continuous, non-negative-definite dissipation rate function, and $dQ(t) \triangleq [r(v(t), y(t)) - d(x(t), v(t))]dt$, where $dQ(t)$ is the amount of energy (heat) received or dissipated by the state feedback dissipative system over the infinitesimal time interval dt , we arrive at a *Clausius-type equality* for $\mathcal{G}_s$. To see this, let $\oint$ denote a cyclic integral evaluated along an arbitrary closed path of $\mathcal{G}_s$, that is, $\oint \triangleq \int_{t_0}^{t_f}$ with $t_f \geq t_0$ and $v(\cdot) \in \mathcal{U}$ such that $x(t_f) = x(t_0) = x_0 \in \mathcal{D}$.

Proposition 25.2 *Consider the nonlinear dynamical system $\mathcal{G}$ given by (25.64) and (25.65), and assume that $\mathcal{G}$ is state feedback dissipative. Then, for all $t_f \geq t_0 \geq 0$ and $v(\cdot) \in \mathcal{U}$ such that $V_s(x(t_f)) = V_s(x(t_0))$,*

$$\int_{t_0}^{t_f} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_s(x(t))} dt = \oint \frac{dQ(t)}{c + V_s(x(t))} = 0, \quad (25.68)$$

where $c > 0$.

Proof It follows from Proposition 25.1 that

$$\begin{aligned} \oint \frac{dQ(t)}{c + V_s(x(t))} &= \int_{t_0}^{t_f} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_s(x(t))} dt \\ &= \log_e \frac{c + V_s(x(t_f))}{c + V_s(x(t_0))} \\ &= \log_e 1 \\ &= 0, \end{aligned} \quad (25.69)$$

which proves the assertion. $\square$

In light of Proposition 25.2, we can give a definition of entropy for a feedback dissipative system.

Definition 25.8 For the nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67) a function $S : \mathcal{D} \rightarrow \mathbb{R}$ satisfying

$$S(x(t_2)) \geq S(x(t_1)) + \int_{t_1}^{t_2} \frac{dQ(t)}{c + V_s(x(t))} \quad (25.70)$$

for every $t_2 \geq t_1 \geq 0$ and $v(\cdot) \in \mathcal{U}$ is called the *entropy* function of $\mathcal{G}_s$.

Recalling that $dQ(t) = [r(v(t), y(t)) - d(x(t), v(t))]dt$ is the infinitesimal amount of the net energy received or dissipated by $\mathcal{G}_s$ over the infinitesimal time interval dt , it follows from (25.70) that

$$dS(x(t)) \geq \frac{dQ(t)}{c + V_s(x(t))}, \quad t \geq t_0. \quad (25.71)$$

Inequality (25.71) is analogous to the classical thermodynamic inequality for the variation of entropy during an infinitesimal irreversible transformation with the shifted system energy $c + V_s(x)$ playing the role of the thermodynamic temperature. Specifically, note that since $\frac{dS}{dQ} = \frac{1}{c + V_s}$, it follows that $\frac{dS}{dQ}$ defines the reciprocal of the system thermodynamic temperature T_e . That is,

$$\frac{1}{T_e} \triangleq \frac{dS}{dQ}$$

and $T_e > 0$.

Next, we show that if $\mathcal{G}_s$ given by (25.66) and (25.67) is *locally controllable* at $\hat{x}$, that is, the set of points that can be reached from and to $\hat{x}$ in finite time T using admissible inputs $v : [0, T] \rightarrow U$, satisfying $\|v(t) - \hat{v}\| < \varepsilon$, contains a neighborhood of $\hat{x}$ [29, p. 333], then all entropy functions for $\mathcal{G}_s$ are continuous on $\mathcal{D}$.

Theorem 25.8 *Consider the dissipative nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67). Assume that $\mathcal{G}_s$ is completely reachable and assume that for every $x_e \in \mathcal{D}$, there exists $v_e \in \mathbb{R}^m$ such that $x(t) \equiv x_e$ and $v(t) \equiv v_e$, $t \geq 0$, satisfy (25.66), and $\mathcal{G}_s$ is locally controllable at every $x_e \in \mathcal{D}$. Then every entropy function $S(x)$, $x \in \mathcal{D}$, of $\mathcal{G}_s$ is continuous on $\mathcal{D}$.*

Proof Let $x_e \in \mathcal{D}$ and $v_e \in \mathbb{R}^m$ be such that $f(x_e) + G(x_e)\phi(x_e) + G(x_e)v_e = 0$, that is, x_e is an equilibrium point of $\mathcal{G}_s$ with $v(t) \equiv v_e$. Now, let $\delta > 0$ and note that it follows from the continuity of $f(\cdot)$, $G(\cdot)$, and $\phi(\cdot)$ that there exist $T > 0$ and $\varepsilon > 0$ such that for every $v : [0, T] \rightarrow \mathbb{R}^n$ and $\|v(t) - v_e\| < \varepsilon$, $\|x(t) - x_e\| < \delta$, $t \in [0, T]$, where $v(\cdot) \in \mathcal{U}$ and $x(t)$, $t \in [0, T]$, denotes the solution to (25.64) with the initial condition x_e . Furthermore, it follows from the local controllability of $\mathcal{G}_s$ that for every $\hat{T} \in (0, T]$, there exists a strictly increasing, continuous function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ such that $\gamma(0) = 0$, and for every $x_0 \in \mathcal{D}$ such that $\|x_0 - x_e\| \leq \gamma(\hat{T})$, there exist $\hat{t} \in [0, \hat{T}]$ and an input $v : [0, \hat{T}] \rightarrow \mathbb{R}^m$ such that $\|v(t) - v_e\| < \varepsilon$, $t \in [0, \hat{t}]$, and $x(\hat{t}) = x_0$. Hence, there exists $\rho > 0$ such that for every $x_0 \in \mathcal{D}$ such that $\|x_0 - x_e\| \leq \rho$, there exists $\hat{t} \in [0, \gamma^{-1}(\|x_0 - x_e\|)]$ and an input $v : [0, \hat{t}] \rightarrow \mathbb{R}^n$ such that $\|v(t) - v_e\| < \varepsilon$, $t \in [0, \hat{t}]$, and $x(\hat{t}) = x_0$.

Next, since $r(\cdot, \cdot)$ is locally integrable for all input–output pairs satisfying (25.66) and (25.67), it follows that there exists $M \in (0, \infty)$ such that

$$\begin{aligned}
\left| \int_0^{\hat{t}} \frac{r(v(\sigma), y(\sigma)) - d(x(\sigma), v(\sigma))}{c + V_s(x(\sigma))} d\sigma \right| &\leq \int_0^{\hat{t}} \left| \frac{dQ(\sigma)}{c + V_s(x(\sigma))} \right| \\
&\leq M\hat{t} \\
&\leq M\gamma^{-1}(\|x_0 - x_e\|).
\end{aligned} \tag{25.72}$$

Now, if $S(\cdot)$ is an entropy function of $\mathcal{G}_s$, then

$$S(x(\hat{t})) \geq S(x_e) + \int_0^{\hat{t}} \frac{dQ(\sigma)}{c + V_s(x(\sigma))}, \tag{25.73}$$

or, equivalently,

$$- \int_0^{\hat{t}} \frac{dQ(\sigma)}{c + V_s(x(\sigma))} \geq S(x_e) - S(x(\hat{t})). \tag{25.74}$$

If $S(x_e) \geq S(x(\hat{t}))$, then combining (25.72) and (25.74) yields

$$|S(x_e) - S(x(\hat{t}))| \leq M\gamma^{-1}(\|x_0 - x_e\|). \tag{25.75}$$

Alternatively, if $S(x(\hat{t})) \geq S(x_e)$, then (25.75) can be derived by reversing the roles of x_e and $x(\hat{t})$. In particular, using the assumption that $\mathcal{G}_s$ is locally controllable from and to x_e , similar arguments can be used to show that the set of points that can be steered in small time to x_e contains a neighborhood of $x(\hat{t})$. Hence, since $\gamma(\cdot)$ is continuous and $x(\hat{t})$ is arbitrary, it follows that $S(\cdot)$ is continuous on $\mathcal{D}$. $\square$

Next, we characterize a continuously differentiable entropy function for state feedback dissipative systems.

Proposition 25.3 *Consider the dissipative nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67). Then the continuously differentiable function $S : \mathcal{D} \rightarrow \mathbb{R}$ given by*

$$S(x) \triangleq \log_e [c + V_s(x)] - \log_e c, \tag{25.76}$$

where $c > 0$, is an entropy function of $\mathcal{G}_s$.

Proof Using Proposition 25.1 it follows that

$$\begin{aligned}
\dot{S}(x(t)) &= \frac{\dot{V}_s(x(t))}{c + V_s(x(t))} \\
&= \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_s(x(t))} \\
&= \frac{\dot{Q}(t)}{c + V_s(x(t))}, \quad t \geq 0.
\end{aligned} \tag{25.77}$$

Now, integrating (25.77) over $[t_1, t_2]$ yields (25.70). $\square$

In [56], we showed that the entropy function for an energy balance equation involving a large-scale, compartmental thermodynamic model is unique. However, whether or not there exists a unique continuously differentiable entropy function for the general nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67) is an open problem.

Finally, the following result presenting an upper and lower bound of the entropy function for a state feedback dissipative system is needed.

Proposition 25.4 *Consider the nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67), and let $S : \mathcal{D} \rightarrow \mathbb{R}$ given by (25.76) be an entropy function of $\mathcal{G}_s$. Then,*

$$\frac{V_s(x)}{c + V_s(x)} \leq S(x) \leq \frac{1}{c} V_s(x), \quad x \in \mathcal{D}. \quad (25.78)$$

Proof Note that (25.76) can be rewritten as $S(x) = \log_e[1 + V_s(x)/c]$. The assertion now follows as a direct consequence of the inequality $z/(1+z) \leq \log_e(1+z) \leq z$, $z > -1$. $\square$

Using the notion of feedback dissipation, we can develop a framework for optimal *semistabilization* of nonlinear systems; a notion briefly discussed at the end of Sect. 25.4. To address the state feedback, thermodynamic-based semistabilization problem, consider the nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67) with performance criterion

$$J(x_0, \phi(\cdot)) = \lim_{t \rightarrow \infty} \left[\frac{1}{t} \int_0^t S(x(\sigma)) d\sigma \right]. \quad (25.79)$$

The performance criterion $J(x_0, \phi(\cdot))$ can be interpreted as the time-average of the entropy function for the dissipative nonlinear dynamical system $\mathcal{G}_s$. The key feature of this optimal control problem is that it addresses semistability instead of asymptotic stability. In the absence of energy exchange with the environment, a thermodynamically consistent nonlinear dynamical system model possesses a continuum of equilibria, and hence, is semistable; that is, the system states converge to Lyapunov stable energy equilibria determined by the system initial conditions [56]. A key question that arises is whether or not this optimal control problem is well defined; that is, whether $J(x_0, \phi(\cdot))$ is finite, and if there exists a state feedback controller such that $J(x_0, \phi(\cdot))$ is minimized.

The first question is addressed by the following proposition.

Proposition 25.5 *Consider the nonlinear dissipative dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67). If there exists $\phi : \mathcal{D} \rightarrow \mathbb{R}^m$ such that (25.66), with $v(t) \equiv 0$, is semistable, then $|J(x_0, \phi(\cdot))| < \infty$.*

Proof Since (25.66) with $v(t) \equiv 0$ is semistable, it follows that $x(t)$ is bounded for all $t \geq 0$. Furthermore, it follows from Theorem 25.8 that $S(\cdot)$ is a continuous entropy function on $\mathcal{D}$ for $\mathcal{G}_s$. Hence, $S(x(t))$ is bounded for all $t \geq 0$. Now, let

$|S(x(t))| \leq c$ for all $t \geq 0$. Then, it follows that $-c \leq (1/t) \int_0^t S(x(\sigma))d\sigma \leq c$ for all $t \geq 0$, which proves the result. $\square$

To address the question of existence of a semistabilizing controller such that $J(x_0, \phi(\cdot))$ given by (25.79) is minimized, we consider an *auxiliary minimization problem* involving the performance criterion

$$\mathcal{J}(x_0, \phi(\cdot)) = \lim_{t \rightarrow \infty} \left[\frac{1}{t} \int_0^t V_s(x(\sigma))d\sigma \right]. \quad (25.80)$$

It follows from the auxiliary minimization problem that we seek feedback controllers that minimize the stored energy in the system in order to attain a stable minimum energy level determined by the system initial conditions and the control system effort.

The following lemma is necessary for proving the main result of this section.

Lemma 25.1 *Consider the dissipative nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67) with continuously differentiable storage function $V_s : \mathcal{D} \rightarrow \mathbb{R}_+$. Suppose there exists $\phi^* : \mathcal{D} \rightarrow \mathbb{R}^m$ such that (25.66) with $v(t) \equiv 0$ is semistable, $\dot{V}_s(x(t)) \leq 0$, $t \geq 0$, and $\mathcal{J}(x_0, \phi(\cdot))$ is minimized. If $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then $\arg \min_{\phi(\cdot) \in \mathbb{R}^m} \mathcal{J}(x_0, \phi(\cdot)) = \arg \min_{\phi(\cdot) \in \mathbb{R}^m} J(x_0, \phi(\cdot))$ and $J(x_0, \phi^*(\cdot)) = 0$. Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then $J(x_0, \phi^*(\cdot)) = S(x_e)$, where $x_e = \lim_{t \rightarrow \infty} x(t)$.*

Proof It follows from Proposition 25.4 and $\dot{V}_s(x(t)) \leq 0$, $t \geq 0$, that

$$\frac{V_s(x(t))}{c + V_s(x(0))} \leq S(x(t)) \leq \frac{V_s(x(t))}{c}, \quad t \geq 0. \quad (25.81)$$

Hence,

$$\frac{\mathcal{J}(x_0, \phi(\cdot))}{(c + V_s(x(0)))} \leq J(x_0, \phi(\cdot)) \leq \frac{\mathcal{J}(x_0, \phi(\cdot))}{c}.$$

Now, if $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then $J(x_0, \phi(\cdot))$ is minimized and $J(x_0, \phi^*(t)) = 0$, $t \geq 0$.

Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then it follows from the definition of $\mathcal{J}(x_0, \phi^*(\cdot))$ that there exists $c^* > 0$ such that $\mathcal{J}(x_0, \phi^*(t)) \geq c^*$ for all $t \geq 0$, and hence,

$$\lim_{t \rightarrow \infty} \mathcal{J}(x_0, \phi^*(t))t / (c + V_s(x(0))) = \infty.$$

Thus,

$$\lim_{t \rightarrow \infty} \int_0^t S(x(\sigma))d\sigma = \infty.$$

Now, using l'Hôpital's rule, it follows that $J(x_0, \phi^*(t)) = S(x_e)$, where $x_e = \lim_{t \rightarrow \infty} x(t)$. $\square$

Theorem 25.9 Consider the dissipative nonlinear dynamical system $\mathcal{G}_s$ given by (25.66) and (25.67) with continuously differentiable storage function $V_s : \mathcal{D} \rightarrow \mathbb{R}_+$. Assume that there exists $\phi^* : \mathcal{D} \rightarrow \mathbb{R}^m$ such that (25.80) is minimized, (25.66), with $v(t) \equiv 0$, is semistable, and $\dot{V}_s(x(t)) \leq 0$, $t \geq 0$. Then, for $S : \mathcal{D} \rightarrow \mathbb{R}$ given by (25.76), $\arg \min_{\phi(\cdot) \in \mathbb{R}^m} \mathcal{J}(x_0, \phi(\cdot)) = \arg \min_{\phi(\cdot) \in \mathbb{R}^m} J(x_0, \phi(\cdot))$.

Proof If $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then it follows from Lemma 25.1 that $\phi^*(\cdot) = \arg \min_{\phi(\cdot) \in \mathbb{R}^m} J(x_0, \phi(\cdot))$. Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then, using similar arguments as in the proof of Lemma 25.1, $\lim_{t \rightarrow \infty} t \mathcal{J}(x_0, \phi^*(t)) = \infty$, and hence, $\int_0^t V_s(x(\sigma)) d\sigma = \infty$ as $t \rightarrow \infty$. Hence, using l'Hôpital's rule, it follows that $\mathcal{J}(x_0, \phi^*(\cdot)) = V_s(x_e)$.

Next, since for all $\phi : \mathcal{D} \rightarrow \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite,

$$\lim_{t \rightarrow \infty} \mathcal{J}(x_0, \phi(t))t / (c + V_s(x(0))) = \infty,$$

it follows from (25.81) that

$$\lim_{t \rightarrow \infty} \int_0^t S(x(\sigma)) d\sigma = \infty.$$

Consequently, for all $\phi : \mathcal{D} \rightarrow \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite and $\mathcal{G}_s$ is semistable, it follows from l'Hôpital's rule that $J(x_0, \phi(\cdot)) = S(x_e) = \log_e(1 + V_s(x_e))$, where $x_e = \lim_{t \rightarrow \infty} x(t)$. Next, assume that $\phi_J^* : \mathcal{D} \rightarrow \mathbb{R}^m$ is such that $\mathcal{G}_s$ is semistable, $\dot{V}_s(x(t)) \leq 0$, $t \geq 0$, and $J(x_0, \phi(\cdot))$ is minimized. Then, it follows that $J(x_0, \phi_J^*(\cdot)) = S(x_e) = \log_e(1 + V_s(x_e))$.

Finally, by uniqueness of solutions of $x(\cdot)$ it follows that $\phi^*(\cdot)$ uniquely determines x_e and $\dot{V}_s(x(t))$, $t \geq 0$. Choosing $V_s(x_e) = V_{se}$, where $V_{se} \in \mathbb{R}$, it follows that $\phi^*(\cdot)$ uniquely determines $V_s(x_e)$, and hence, $J(x_0, \phi^*(\cdot)) = \log_e(1 + V_s(x_e))$, which proves the result. $\square$

It follows from Theorem 25.9 that an optimal semistable controller minimizing $\mathcal{J}(x_0, v(\cdot))$ given by (25.80) also minimizes the entropy functional $J(x_0, v(\cdot))$ given by (25.79). Since quadratic cost functions arise naturally in dissipativity theory [29, 186, 187], addressing the auxiliary cost (25.80) can be simpler than addressing the entropy (logarithmic) cost functional (25.79). The afore highlighted framework can be explored to quantify fundamental limits of performance for nonlinear dynamical systems using thermodynamic entropy notions. This is a subject of ongoing research.

25.8 Conclusion

In this paper, we highlighted the salient features for developing a general framework for the analysis and control design of complex, large-scale network systems. This framework is predicated on fundamental principles from systems biology, neurophysiology, and dynamical thermodynamics to address issues of robustness, semistability, hybrid stabilization, stochasticity, optimality, learning, system mode switching,

controller complexity, system complexity, scalability, and limits of performance in nonlinear system stabilization and spatially invariant multiagent systems. Such a framework would significantly advance the state of the art in embedded hybrid control system design as well as cooperative control and coordination of multiagent systems. In addition to network systems, the framework can be applied to queuing systems, ecological systems, economic systems, telecommunications systems, transportation systems, power systems, mean field neural systems, and large-scale aerospace systems, to cite but a few examples. Payoffs would arise from improvements in system performance, robustness, reliability, controller complexity, and maintainability, as well as reduced development cost, design time, and hardware/software implementation requirements for hierarchical and hybrid control systems.

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