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Energy dissipating hybrid control for impulsive dynamical systems☆

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Abstract

A novel class of fixed-order, energy-based hybrid controllers is proposed as a means for achieving enhanced energy dissipation in nonsmooth Euler–Lagrange, hybrid port-controlled Hamiltonian, and lossless impulsive dynamical systems. These dynamic controllers combine a logical switching architecture with hybrid dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework leads to hybrid closed-loop systems described by impulsive differential equations. Special cases of energy-based hybrid controllers involving state-dependent switching are described, and an illustrative numerical example is given to demonstrate the efficacy of the proposed approach.

Keywords: Hybrid control; Hybrid systems; Dynamic compensation; Impulsive dynamical systems; Lossless systems; Nonsmooth Euler–Lagrange systems

1. Introduction

In the recent paper [1], the authors develop a novel energy dissipating hybrid control framework for Lagrangian, port-controlled Hamiltonian, and lossless dynamical systems. Specifically, a fixed-order, energy-based hybrid controller is constructed to emulate a hybrid Hamiltonian dynamical system, and exploits the feature that the states of the dynamic controller may be reset to enhance the overall energy dissipation in the closed-loop system. An important feature of the hybrid controller is that its Hamiltonian structure is shown to be associated with a kinetic and potential energy function. In a mechanical Euler–Lagrange system, positions typically correspond to elastic deformations, which contribute to the potential energy of the system, whereas velocities typically correspond to momenta, which contribute to the kinetic energy of the system. On the other hand, while the energy-based hybrid controller developed in [1] has dynamical states that emulate the motion of a physical Hamiltonian system, these states only "exist" as numerical representations inside the processor. Consequently, while one can associate an *emulated energy* with these states, this energy is merely a mathematical construct and does not correspond to any physical form of energy. Since the states of the controller may be reset to enhance overall closed-loop energy dissipation, the dynamics of the hybrid dynamic controller are described by ordinary differential equations, except when a resetting event occurs. The resetting

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controller action is instantaneous and is described by a difference equation. Consequently, the closed-loop system is characterized by impulsive differential equations [2–8].

In this paper, we extend the results of [1] to lossless and dissipative impulsive dynamical systems [6]. In particular, we exploit the coupling between a physical nonsmooth dynamical system and an energy-based hybrid controller to efficiently remove energy from the physical impulsive system. Specifically, if a dissipative or lossless impulsive dynamical system is at high energy level, and a lossless feedback controller at a low energy level is attached to it, then energy will generally tend to flow from the plant into the controller, decreasing the plant energy and increasing the controller energy [7]. Of course, emulated energy, and not physical energy, is accumulated by the controller. Conversely, if the attached controller is at a high energy level and a plant is at a low energy level, then energy can flow from the controller to the plant, since a controller can generate real, physical energy to affect the required energy flow. Hence, if and when the controller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. In this case, the overall closed-loop system consisting of the impulsive plant and the hybrid controller possesses discontinuous flows since it combines logical switchings with hybrid dynamics.

This paper first establishes definitions, notation, and presents a review of some basic results on impulsive differential equations which provide the mathematical foundation for designing fixed-order, energy-based hybrid controllers. In Section 3, we present a general state-dependent hybrid control framework for lossless impulsive dynamical systems and impulsive port-controlled Hamiltonian systems. In Section 4, we introduce a nonsmooth Euler–Lagrange problem and present a state-dependent hybrid feedback control framework for Lagrangian systems with nonsmooth impact dynamics. The nonsmoothness in these mechanical systems can arise due to impulsive (impact or collision) dynamics as well as unilateral constraints on system positions [9]. Unlike standard energy-based controllers for continuous-time systems considered in the literature [10–20], the proposed approach does not achieve stabilization via passivation. Section 5 considers a numerical example that demonstrates the efficacy of the proposed approach. Finally, we draw conclusions in Section 6.

2. Hybrid control and impulsive dynamical systems

In this section, we establish definitions, notation, and review some basic results on impulsive dynamical systems [6]. Let $\mathbb{R}$ denote the set of real numbers, let $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, let $(\cdot)^{\mathrm{T}}$ denote transpose, and let I_n denote the $n \times n$ identity matrix. Furthermore, let $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, and $\overline{\mathcal{S}}$ denote the boundary, the interior, and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centered at α with radius ε, and $V'(x)$ for the Fréchet derivative of V at x. Finally, we write $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for each $\varepsilon > 0$ there exists $T > 0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$.

In this paper, we consider controlled impulsive dynamical systems of the form

$$\dot{x}_{\mathrm{p}}(t) = f_{\mathrm{cp}}(x_{\mathrm{p}}(t), u_{\mathrm{c}}(t)), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}0}, \quad (x_{\mathrm{p}}(t), u_{\mathrm{c}}(t)) \notin \mathcal{Z}_{\mathrm{p}}, \tag{1}$$

$$\Delta x_{\mathrm{p}}(t) = f_{\mathrm{dp}}(x_{\mathrm{p}}(t), u_{\mathrm{d}}(t)), \quad (x_{\mathrm{p}}(t), u_{\mathrm{c}}(t)) \in \mathcal{Z}_{\mathrm{p}}, \tag{2}$$

$$y(t) = h_{\mathrm{p}}(x_{\mathrm{p}}(t)), \tag{3}$$

where $t \geq 0$, $x_{\mathrm{p}}(t) \in \mathcal{D}_{\mathrm{p}} \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{p}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{p}}$, $\Delta x_{\mathrm{p}}(t) \triangleq x_{\mathrm{p}}(t^+) - x_{\mathrm{p}}(t)$, $u_{\mathrm{c}}(t) \in \mathbb{R}^{m_{\mathrm{c}}}$, $u_{\mathrm{d}}(t) \in \mathbb{R}^{m_{\mathrm{d}}}$, $f_{\mathrm{cp}} : \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^{m_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{p}}}$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}_{\mathrm{p}}$ and satisfies $f_{\mathrm{cp}}(0, 0) = 0$, $f_{\mathrm{dp}} : \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^{m_{\mathrm{d}}} \to \mathbb{R}^{n_{\mathrm{p}}}$ is continuous, $h_{\mathrm{p}} : \mathcal{D}_{\mathrm{p}} \to \mathbb{R}^l$ is continuous and satisfies $h_{\mathrm{p}}(0) = 0$, and $\mathcal{Z}_{\mathrm{p}} \triangleq \mathcal{Z}_{x_{\mathrm{p}}} \times \mathcal{Z}_{u_{\mathrm{c}}} \subset \mathcal{D}_{\mathrm{p}} \times \mathbb{R}^{m_{\mathrm{c}}}$ is the *resetting set*. Furthermore, we consider hybrid (resetting) dynamic controllers of the form

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t), y(t)) \notin \mathcal{Z}_{\mathrm{c}}, \tag{4}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \quad (x_{\mathrm{c}}(t), y(t)) \in \mathcal{Z}_{\mathrm{c}}, \tag{5}$$

$$u_{\mathrm{c}}(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t), y(t)), \tag{6}$$

$$u_{\mathrm{d}}(t) = h_{\mathrm{dc}}(x_{\mathrm{c}}(t), y(t)), \tag{7}$$

where $t \geq 0$, $x_{\mathrm{c}}(t) \in \mathcal{D}_{\mathrm{c}} \subseteq \mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0 \in \mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t) \triangleq x_{\mathrm{c}}(t^+) - x_{\mathrm{c}}(t)$, $f_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is smooth on $\mathcal{D}_{\mathrm{c}}$ and satisfies $f_{\mathrm{cc}}(0, 0) = 0$, $f_{\mathrm{dc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{n_{\mathrm{c}}}$ is continuous, $h_{\mathrm{cc}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \mathbb{R}^{m_{\mathrm{c}}}$ is continuous and satisfies $h_{cc}(0,0) = 0$, $h_{dc} : \mathcal{D}_c \times \mathbb{R}^l \to \mathbb{R}^{m_d}$ is continuous, and $\mathcal{Z}_c \subset \mathcal{D}_c \times \mathbb{R}^l$ is the resetting set. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_c which may be less than the plant order n_p.

The equations of motion for the closed-loop dynamical system (1)–(7) have the form

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}, \tag{8}$$

$$\Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{Z}, \tag{9}$$

where

$$x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix} \in \mathbb{R}^n, \qquad f_c(x) \triangleq \begin{bmatrix} f_{cp}(x_p, h_{cc}(x_c, h_p(x_p))) \\ f_{cc}(x_c, h_p(x_p)) \end{bmatrix}, \tag{10}$$

$$f_d(x) \triangleq \begin{bmatrix} f_{dp}(x_p, h_{dc}(x_c, h_p(x_p)))\chi_{\mathcal{Z}_1}(x) \\ f_{dc}(x_c, h_p(x_p))\chi_{\mathcal{Z}_2}(x) \end{bmatrix}, \qquad \chi_{\mathcal{Z}_i}(x) \triangleq \begin{cases} 1, & x \in \mathcal{Z}_i \\ 0, & x \notin \mathcal{Z}_i \end{cases}, \quad i = 1, 2, \tag{11}$$

and $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, $\mathcal{Z}_1 \triangleq \{x \in \mathcal{D} : (x_p, h_{cc}(x_c, h_p(x_p))) \in \mathcal{Z}_p\}$, $\mathcal{Z}_2 \triangleq \{x \in \mathcal{D} : (x_c, h_p(x_p)) \in \mathcal{Z}_c\}$, with $n \triangleq n_p + n_c$ and $\mathcal{D} \triangleq \mathcal{D}_p \times \mathcal{D}_c$. We refer to the differential equation (8) as the *continuous-time dynamics*, and we refer to the difference equation (9) as the *resetting law*. A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is a *solution* to the impulsive dynamical system (8) and (9) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$ if $x(\cdot)$ is left-continuous and $x(t)$ satisfies (8) and (9) for all $t \in \mathcal{I}_{x_0}$. For further discussion on solutions to impulsive differential equations, see [2–8,21,22,9]. For convenience, we use the notation $s(t, x_0)$ to denote the solution $x(t)$ of (8) and (9) at time $t \geq 0$ with initial condition $x(0) = x_0$.

For a particular closed-loop trajectory $x(t)$, we let $t_k \triangleq \tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (8) and (9) from the initial condition $x(0) = x_0$ is given by $\psi(t, x_0)$ for $0 < t \leq t_1$, where $\psi(t, x_0)$ denotes the solution to the continuous-time dynamics (8). If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $x_1 \in \mathcal{Z}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + f_d(x_1)$ according to the resetting law (9). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $\psi(t - t_1, x_1^+)$, and so on. Note that the solution $x(t)$ of (8) and (9) is left-continuous, that is, it is continuous everywhere except at the resetting times t_k, and

$$x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon), \tag{12}$$

$$x_k^+ \triangleq x(t_k) + f_d(x(t_k)) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon), \tag{13}$$

for $k = 1, 2, \ldots$.

To ensure the well-posedness of the resetting times, we make the following additional assumptions:

Assumption 1. If $x \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $\psi(\delta, x) \notin \mathcal{Z}$.

Assumption 2. If $x \in \mathcal{Z}$, then $x + f_d(x) \notin \mathcal{Z}$.

Assumption 1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$, that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, Assumption 2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) \notin \mathcal{Z}$, which serves as the initial condition for the continuous-time dynamics (8).

It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times $t_k = \tau_k(x_0)$ are distinct and well defined [6]. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, impulsive dynamical systems of the form (8) and (9) are time-invariant systems. These systems are called *state-dependent impulsive dynamical systems* [6]. Since the resetting times are well defined and distinct, and since the solution to (8) exists and is unique, it follows that the solution of the impulsive dynamical system (8) and (9) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit *Zenoness* and *beating*, as well as *confluence*, wherein solutions exhibit infinitely many resettings in a finite time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a certain point in time [6,8]. In this paper we allow for the possibility of confluence and Zeno solutions; however, Assumption 2 precludes the possibility of beating. Furthermore, since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details see [2–5].

For the statement of the next result the following key assumption is needed.

Assumption 3. Consider the impulsive dynamical system (8) and (9), and let $s(t, x_0)$, $t \geq 0$, denote the solution to (8) and (9) with initial condition x_0. Then for every $x_0 \notin \mathcal{Z}$ and every $\varepsilon > 0$ and $t \neq t_k$, there exists $\delta(\varepsilon, x_0, t) > 0$ such that if $\|x_0 - y\| < \delta(\varepsilon, x_0, t)$, $y \in \mathcal{D}$, then $\|s(t, x_0) - s(t, y)\| < \varepsilon$.

Assumption 3 is a weakened version of the quasi-continuous dependence assumption given in [8], and is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Since solutions of impulsive dynamical systems are not continuous in time and solutions are not continuous functions of the system initial conditions, Assumption 3 is needed to apply the hybrid invariance principle developed in [6,8] to hybrid closed-loop systems. Sufficient conditions that guarantee that the impulsive dynamical system (8) and (9) satisfies a stronger version of Assumption 3 are given in [8]. The following result provides a generalization of the results given in [8] for establishing sufficient conditions for guaranteeing that the impulsive dynamical system (8) and (9) satisfies Assumption 3.

Proposition 2.1. *Consider the impulsive dynamical system $\mathcal{G}$ given by (8) and (9). Assume that Assumptions 1 and 2 hold, $\tau_1(\cdot)$ is continuous at every $x \notin \overline{\mathcal{Z}}$ such that $0 < \tau_1(x) < \infty$, and if $x \in \mathcal{Z}$, then $x + f_{\mathrm{d}}(x) \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$. Furthermore, let $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ be such that $0 < \tau_1(x_0) < \infty$ and assume that the following statements hold:*

(i) *If a sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then either both $f_{\mathrm{d}}(x_0) = 0$ and $\lim_{i\to\infty} \tau_1(x_i) = 0$, or $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.*
(ii) *If a sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.*

Then $\mathcal{G}$ satisfies Assumption 3.

Proof. First, let $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ and let $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ be such that $f_{\mathrm{d}}(x_0) = 0$ and $\lim_{i\to\infty} \tau_1(x_i) = 0$ hold. Define $y_i \triangleq s(\tau_1(x_i), x_i) + f_{\mathrm{d}}(s(\tau_1(x_i), x_i)) = \psi(\tau_1(x_i), x_i) + f_{\mathrm{d}}(\psi(\tau_1(x_i), x_i))$, $i = 1, 2, \ldots$, where $\psi(t, x_0)$ denotes the solution to the continuous-time dynamics (8), and note that, since $f_{\mathrm{d}}(x_0) = 0$ and $\lim_{i\to\infty} \tau_1(x_i) = 0$, it follows that $\lim_{i\to\infty} y_i = x_0$. Hence, since by assumption $y_i \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, $i = 1, 2, \ldots$, it follows from (ii) that $\lim_{i\to\infty} \tau_1(y_i) = \tau_1(x_0)$, or, equivalently, $\lim_{i\to\infty} \tau_2(x_i) = \tau_1(x_0)$. Similarly, it can be shown that $\lim_{i\to\infty} \tau_{k+1}(x_i) = \tau_k(x_0)$, $k = 2, 3, \ldots$. Next, note that

$$\begin{aligned}\lim_{i\to\infty} s(\tau_2(x_i), x_i) &= \lim_{i\to\infty} \psi(\tau_2(x_i) - \tau_1(x_i), s(\tau_1(x_i), x_i) + f_{\mathrm{d}}(s(\tau_1(x_i), x_i)))\\ &= \psi(\tau_1(x_0), x_0)\\ &= s(\tau_1(x_0), x_0).\end{aligned}$$

Now, using mathematical induction it can be shown that $\lim_{i\to\infty} s(\tau_{k+1}(x_i), x_i) = s(\tau_k(x_0), x_0)$, $k = 2, 3, \ldots$.

Next, let $k \in \{1, 2, \ldots\}$ and let $t \in (\tau_k(x_0), \tau_{k+1}(x_0))$. Since $\lim_{i\to\infty} \tau_{k+1}(x_i) = \tau_k(x_0)$, it follows that there exists $I \in \{1, 2, \ldots\}$ such that $\tau_{k+1}(x_i) < t$ and $\tau_{k+2}(x_i) > t$ for all $i > I$. Hence, it follows that for every $t \in (\tau_k(x_0), \tau_{k+1}(x_0))$,

$$\begin{aligned}\lim_{i\to\infty} s(t, x_i) &= \lim_{i\to\infty} \psi(t - \tau_{k+1}(x_i), s(\tau_{k+1}(x_i), x_i) + f_{\mathrm{d}}(s(\tau_{k+1}(x_i), x_i)))\\ &= \psi(t - \tau_k(x_0), s(\tau_k(x_0), x_0) + f_{\mathrm{d}}(s(\tau_k(x_0), x_0)))\\ &= s(t, x_0).\end{aligned}$$

Alternatively, if $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ is such that $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$ holds, then using arguments identical to those above, it can be shown that $\lim_{i\to\infty} s(t, x_i) = s(t, x_0)$ for every $t \in (\tau_k(x_0), \tau_{k+1}(x_0))$, $k = 1, 2, \ldots$.

Finally, assume that for all $x_0 \notin \overline{\mathcal{Z}}$, $0 < \tau_1(x_0) < \infty$, and $\tau_1(\cdot)$ is continuous. In this case, it follows from the definition of $\tau_1(x_0)$ that for every $x_0 \notin \overline{\mathcal{Z}}$ and $t \in (\tau_1(x_0), \tau_2(x_0)]$,

$$s(t, x_0) = \psi(t - \tau_1(x_0), s(\tau_1(x_0), x_0) + f_{\mathrm{d}}(s(\tau_1(x_0), x_0))). \tag{14}$$

Since $\psi(\cdot,\cdot)$ is continuous in both its arguments, $\tau_1(\cdot)$ is continuous at x_0, and $f_{\mathrm{d}}(\cdot)$ is piecewise continuous, it follows that $s(t,\cdot)$ is continuous at x_0 for every $t \in (\tau_1(x_0), \tau_2(x_0))$. Next, for every sequence $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ such that $\lim_{i\to\infty} x_i = x_0$, it follows that $\lim_{i\to\infty} s(\tau_1(x_i), x_i) = \lim_{i\to\infty} \psi(\tau_1(x_i), x_i) = \psi(\tau_1(x_0), x_0) = s(\tau_1(x_0), x_0)$. Furthermore, note that by assumption $y_i \triangleq s(\tau_1(x_i), x_i) + f_{\mathrm{d}}(s(\tau_1(x_i), x_i)) \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, $i = 0, 1, \ldots$. Hence, it follows that for all $t \in (\tau_k(y_0), \tau_{k+1}(y_0))$, $k = 1, 2, \ldots$, $\lim_{i\to\infty} s(t, y_i) = s(t, y_0)$, or, equivalently, for all $t \in (\tau_k(x_0), \tau_{k+1}(x_0))$, $k = 2, 3, \ldots$, $\lim_{i\to\infty} s(t, x_i) = s(t, x_0)$, which proves the result. □

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, that is, $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$ for $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ and $\lim_{i\to\infty} x_i = x_0$. For this result, the following definition is needed. First, however, recall that the *Lie derivative* of a smooth function $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ along the vector field of the continuous-time dynamics $f_{\mathrm{c}}(x)$ is given by $L_{f_{\mathrm{c}}}\mathcal{X}(x) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{X}(\psi(t,x))|_{t=0} = \frac{\partial \mathcal{X}(x)}{\partial x} f_{\mathrm{c}}(x)$, and the *zeroth-* and *higher-order Lie derivatives* are, respectively, defined by $L_{f_{\mathrm{c}}}^0\mathcal{X}(x) \triangleq \mathcal{X}(x)$ and $L_{f_{\mathrm{c}}}^k\mathcal{X}(x) \triangleq L_{f_{\mathrm{c}}}(L_{f_{\mathrm{c}}}^{k-1}\mathcal{X}(x))$, where $k \geq 1$.

Definition 2.1. Let $\mathcal{M} \triangleq \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{p}}(x) = 0\} \cup \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{c}}(x) = 0\}$, where $\mathcal{X}_{\mathrm{p}} : \mathcal{D} \to \mathbb{R}$ and $\mathcal{X}_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}$ are infinitely differentiable functions. A point $x \in \mathcal{M}$ such that $f_{\mathrm{c}}(x) \neq 0$ is *transversal* to (8) if there exist $k_{\mathrm{p}} \in \{1, 2, \ldots\}$ and $k_{\mathrm{c}} \in \{1, 2, \ldots\}$ such that

$$L_{f_{\mathrm{c}}}^r\mathcal{X}_{\mathrm{p}}(x) = 0, \quad r = 0, \ldots, 2k_{\mathrm{p}} - 2, \quad L_{f_{\mathrm{c}}}^{2k_{\mathrm{p}}-1}\mathcal{X}_{\mathrm{p}}(x) \neq 0, \tag{15}$$

$$L_{f_{\mathrm{c}}}^r\mathcal{X}_{\mathrm{c}}(x) = 0, \quad r = 0, \ldots, 2k_{\mathrm{c}} - 2, \quad L_{f_{\mathrm{c}}}^{2k_{\mathrm{c}}-1}\mathcal{X}_{\mathrm{c}}(x) \neq 0. \tag{16}$$

Proposition 2.2. *Consider the impulsive dynamical system* (8) *and* (9). *Let* $\mathcal{X}_{\mathrm{p}} : \mathcal{D} \to \mathbb{R}$ *and* $\mathcal{X}_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}$ *be infinitely differentiable functions such that* $\overline{\mathcal{Z}} = \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{p}}(x) = 0\} \cup \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{c}}(x) = 0\}$, *and assume every* $x \in \overline{\mathcal{Z}}$ *is transversal to* (8). *Then at every* $x_0 \notin \overline{\mathcal{Z}}$ *such that* $0 < \tau_1(x_0) < \infty$, $\tau_1(\cdot)$ *is continuous. Furthermore, if* $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ *is such that* $\tau_1(x_0) \in (0, \infty)$ *and* $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ *or* $\lim_{i\to\infty} \tau_1(x_i) > 0$, *where* $\{x_i\}_{i=1}^{\infty} \notin \overline{\mathcal{Z}}$ *is such that* $\lim_{i\to\infty} x_i = x_0$ *and* $\lim_{i\to\infty} \tau_1(x_i)$ *exists, then* $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$.

Proof. Let $x_0 \notin \overline{\mathcal{Z}}$ be such that $0 < \tau_1(x_0) < \infty$. It follows from the definition of $\tau_1(\cdot)$ that $s(t, x_0) = \psi(t, x_0)$, $t \in [0, \tau_1(x_0)]$, $\mathcal{X}_{\mathrm{p}}(s(t, x_0))\mathcal{X}_{\mathrm{c}}(s(t, x_0)) \neq 0$, $t \in (0, \tau_1(x_0))$, and $\mathcal{X}_{\mathrm{p}}(s(\tau_1(x_0), x_0))\mathcal{X}_{\mathrm{c}}(s(\tau_1(x_0), x_0)) = 0$. Without loss of generality, let $\mathcal{X}_{\mathrm{p}}(s(t, x_0))\mathcal{X}_{\mathrm{c}}(s(t, x_0)) > 0$, $t \in (0, \tau_1(x_0))$. Since $\hat{x} \triangleq \psi(\tau_1(x_0), x_0) \in \overline{\mathcal{Z}}$ is transversal to (8), it follows that there exists $\theta > 0$ such that $\mathcal{X}_{\mathrm{p}}(\psi(t, \hat{x}))\mathcal{X}_{\mathrm{c}}(\psi(t, \hat{x})) > 0$, $t \in [-\theta, 0)$, and $\mathcal{X}_{\mathrm{p}}(\psi(t, \hat{x}))\mathcal{X}_{\mathrm{c}}(\psi(t, \hat{x})) < 0$, $t \in (0, \theta]$. (This fact can be easily shown by expanding $\mathcal{X}_{\mathrm{p}}(\psi(t, x))\mathcal{X}_{\mathrm{c}}(\psi(t, x))$ via a Taylor series expansion about $\hat{x}$ and using the fact that $\hat{x}$ is tranversal to (8).) Hence, $\mathcal{X}_{\mathrm{p}}(\psi(t, x_0))\mathcal{X}_{\mathrm{c}}(\psi(t, x_0)) > 0$, $t \in [\hat{t}_1, \tau_1(x_0))$, and $\mathcal{X}_{\mathrm{p}}(\psi(t, x_0))\mathcal{X}_{\mathrm{c}}(\psi(t, x_0)) < 0$, $t \in (\tau_1(x_0), \hat{t}_2]$, where $\hat{t}_1 \triangleq \tau_1(x_0) - \theta$ and $\hat{t}_2 \triangleq \tau_1(x_0) + \theta$.

Next, let $\varepsilon \triangleq \min\{|\mathcal{X}_{\mathrm{p}}(\psi(\hat{t}_1, x_0))\mathcal{X}_{\mathrm{c}}(\psi(\hat{t}_1, x_0))|, |\mathcal{X}_{\mathrm{p}}(\psi(\hat{t}_2, x_0))\mathcal{X}_{\mathrm{c}}(\psi(\hat{t}_2, x_0))|\}$. Now, it follows from the continuity of $\mathcal{X}_{\mathrm{p}}(\cdot)\mathcal{X}_{\mathrm{c}}(\cdot)$ and the continuous dependence of $\psi(\cdot,\cdot)$ on the system initial conditions that there exists $\delta > 0$ such that

$$\sup_{0 \leq t \leq \hat{t}_2} |\mathcal{X}_{\mathrm{p}}(\psi(t, x))\mathcal{X}_{\mathrm{c}}(\psi(t, x)) - \mathcal{X}_{\mathrm{p}}(\psi(t, x_0))\mathcal{X}_{\mathrm{c}}(\psi(t, x_0))| < \varepsilon, \quad x \in \mathcal{B}_\delta(x_0), \tag{17}$$

which implies that $\mathcal{X}_{\mathrm{p}}(\psi(\hat{t}_1, x))\mathcal{X}_{\mathrm{c}}(\psi(\hat{t}_1, x)) > 0$ and $\mathcal{X}_{\mathrm{p}}(\psi(\hat{t}_2, x))\mathcal{X}_{\mathrm{c}}(\psi(\hat{t}_2, x)) < 0$, $x \in \mathcal{B}_\delta(x_0)$. Hence, it follows that $\hat{t}_1 < \tau_1(x) < \hat{t}_2$, $x \in \mathcal{B}_\delta(x_0)$. The continuity of $\tau_1(\cdot)$ at x_0 now follows immediately on noting that θ can be chosen arbitrarily small. Finally, let $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ be such that $\lim_{i\to\infty} x_i = x_0$ for some sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$. Then using arguments similar to those above it can be shown that $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$. Alternatively, if $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ is such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i) > 0$ for some sequence $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$, then it follows that there exist sufficiently small $\hat{t} > 0$ and $I \in \mathbb{Z}_+$ such that $s(\hat{t}, x_i) = \psi(\hat{t}, x_i)$, $i = I, I+1, \ldots$, which implies that $\lim_{i\to\infty} s(\hat{t}, x_i) = s(\hat{t}, x_0)$. Next, define $y_i \triangleq \psi(\hat{t}, x_i)$, $i = 0, 1, \ldots$, so that $\lim_{i\to\infty} y_i = y_0$ and note that it follows from the transversality assumption that $y_0 \notin \overline{\mathcal{Z}}$, which implies that $\tau_1(\cdot)$ is continuous at y_0. Hence, $\lim_{i\to\infty} \tau_1(y_i) = \tau_1(y_0)$. The result now follows on noting that $\tau_1(x_i) = \hat{t} + \tau_1(y_i)$, $i = 1, 2, \ldots$. □

Remark 2.1. Let $x_0 \notin \mathcal{Z}$ be such that $\lim_{i\to\infty} \tau_1(x_i) \neq \tau_1(x_0)$ for some sequence $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$. Then it follows from Proposition 2.2 that $\lim_{i\to\infty} \tau_1(x_i) = 0$.

The next result characterizes impulsive dynamical system limit sets in terms of continuously differentiable functions. In particular, we show that the system trajectories of a state-dependent impulsive dynamical system converge to an invariant set contained in a union of level surfaces characterized by the continuous-time system dynamics and the resetting system dynamics. For the next result assume that $f_c(\cdot)$, $f_d(\cdot)$, and $\mathcal{Z}$ are such that the dynamical system $\mathcal{G}$ given by (8) and (9) satisfies Assumptions 1–3.

Theorem 2.1. *Consider the impulsive dynamical system (8) and (9), assume $\mathcal{D}_{ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to (8) and (9), assume that if $x_0 \in \mathcal{Z}$ then $x_0 + f_d(x_0) \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, and assume that there exists a continuously differentiable function $V : \mathcal{D}_{ci} \to \mathbb{R}$ such that*

$$V'(x) f_c(x) \leq 0, \quad x \in \mathcal{D}_{ci}, \ x \notin \mathcal{Z}, \tag{18}$$

$$V(x + f_d(x)) \leq V(x), \quad x \in \mathcal{D}_{ci}, \ x \in \mathcal{Z}. \tag{19}$$

Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_{ci} : x \notin \mathcal{Z}, V'(x) f_c(x) = 0\} \cup \{x \in \mathcal{D}_{ci} : x \in \mathcal{Z}, V(x + f_d(x)) = V(x)\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $x_0 \in \mathcal{D}_{ci}$, then $x(t) \to \mathcal{M}$ as $t \to \infty$. Furthermore, if $0 \in \overset{\circ}{\mathcal{D}}_{ci}$, $V(0) = 0$, $V(x) > 0$, $x \neq 0$, and the set $\mathcal{R}$ contains no invariant set other than the set $\{0\}$, then the zero solution $x(t) \equiv 0$ to (8) and (9) is asymptotically stable and $\mathcal{D}_{ci}$ is a subset of the domain of attraction of (8) and (9).

Proof. The proof is similar to the proof of Corollary 5.1 given in [8] and, hence, is omitted. □

Remark 2.2. Setting $\mathcal{D} = \mathbb{R}^n$ and requiring $V(x) \to \infty$ as $\|x\| \to \infty$ in Theorem 2.1, it follows that the zero solution $x(t) \equiv 0$ to (8) and (9) is globally asymptotically stable. A similar remark holds for Theorem 2.2 below.

Theorem 2.2. *Consider the impulsive dynamical system (8) and (9), assume $\mathcal{D}_{ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to (8) and (9) such that $0 \in \overset{\circ}{\mathcal{D}}_{ci}$, assume that if $x_0 \in \mathcal{Z}$ then $x_0 + f_d(x_0) \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, and assume that for all $x_0 \in \mathcal{D}_{ci}$, $x_0 \neq 0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, where $x(t)$, $t \geq 0$, denotes the solution to (8) and (9) with the initial condition x_0. Furthermore, assume there exists a continuously differentiable function $V : \mathcal{D}_{ci} \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$, $x \neq 0$,*

$$V'(x) f_c(x) \leq 0, \quad x \in \mathcal{D}_{ci}, \ x \notin \mathcal{Z}, \tag{20}$$

$$V(x + f_d(x)) < V(x), \quad x \in \mathcal{D}_{ci}, \ x \in \mathcal{Z}. \tag{21}$$

Then the zero solution $x(t) \equiv 0$ to (8) and (9) is asymptotically stable and $\mathcal{D}_{ci}$ is a subset of the domain of attraction of (8) and (9).

Proof. It follows from (21) that $\mathcal{R} = \{x \in \mathcal{D}_{ci} : x \notin \mathcal{Z}, V'(x) f_c(x) = 0\}$. Since for all $x_0 \in \mathcal{D}_{ci}$, $x_0 \neq 0$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$, it follows that the largest invariant set contained in $\mathcal{R}$ is $\{0\}$. Now, the result is a direct consequence of Theorem 2.1. □

3. Hybrid control design for lossless impulsive dynamical systems

In this section, we present a hybrid controller design framework for lossless impulsive dynamical systems [6]. Specifically, we consider impulsive dynamical systems $\mathcal{G}_p$ of the form given by (1)–(3) where $u(\cdot)$ satisfies sufficient regularity conditions such that (1) has a unique solution between the resetting times. Furthermore, we consider hybrid resetting dynamic controllers $\mathcal{G}_c$ of the form

$$\dot{x}_c(t) = f_{cc}(x_c(t), y(t)), \quad x_c(0) = x_{c0}, \ (x_c(t), y(t)) \notin \mathcal{Z}_c, \tag{22}$$

$$\Delta x_c(t) = \eta(y(t)) - x_c(t), \quad (x_c(t), y(t)) \in \mathcal{Z}_c, \tag{23}$$

$$y_{cc}(t) = h_{cc}(x_c(t), u_{cc}(t)), \tag{24}$$

$$y_{dc}(t) = h_{dc}(x_c(t), y(t)), \tag{25}$$

where $x_{\rm c}(t) \in \mathcal{D}_{\rm c} \subseteq \mathbb{R}^{n_{\rm c}}$, $\mathcal{D}_{\rm c}$ is an open set with $0 \in \mathcal{D}_{\rm c}$, $y(t) \in \mathbb{R}^l$, $y_{\rm cc}(t) \in \mathbb{R}^{m_{\rm c}}$, $y_{\rm dc}(t) \in \mathbb{R}^{m_{\rm d}}$, $f_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{n_{\rm c}}$ is smooth on $\mathcal{D}_{\rm c}$ and satisfies $f_{\rm cc}(0, 0) = 0$, $\eta : \mathbb{R}^l \to \mathcal{D}_{\rm c}$ is continuous and satisfies $\eta(0) = 0$, $h_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{m_{\rm c}}$ is continuous and satisfies $h_{\rm cc}(0, 0) = 0$, and $h_{\rm dc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{m_{\rm d}}$ is continuous.

Recall that for the impulsive dynamical system $\mathcal{G}_{\rm p}$ given by (1)–(3), a function $(s_{\rm c}(u_{\rm c}, y), s_{\rm d}(u_{\rm d}, y))$, where $s_{\rm c} : \mathbb{R}^{m_{\rm c}} \times \mathbb{R}^l \to \mathbb{R}$ and $s_{\rm d} : \mathbb{R}^{m_{\rm d}} \times \mathbb{R}^l \to \mathbb{R}$ are such that $s_{\rm c}(0, 0) = 0$ and $s_{\rm d}(0, 0) = 0$, is called a *hybrid supply rate* [6] if it is locally integrable for all input–output pairs satisfying (1)–(3), that is, for all input–output pairs $u_{\rm c} \in \mathcal{U}_{\rm c}$ and $y \in \mathcal{Y}$ satisfying (1) and (3), $s_{\rm c}(\cdot, \cdot)$ satisfies $\int_t^{\hat{t}} |s_{\rm c}(u_{\rm c}(\sigma), y(\sigma))| {\rm d}\sigma < \infty$, $t, \hat{t} \geq 0$. Here, $\mathcal{U}_{\rm c}$ and $\mathcal{Y}$ are input and output spaces, respectively, that are assumed to be closed under the shift operator. Note that since all input–output pairs $u_{\rm d}(t_k) \in \mathcal{U}_{\rm d}$ and $y(t_k) \in \mathcal{Y}$ satisfying (2) and (3) are defined for discrete instants, $s_{\rm d}(\cdot, \cdot)$ satisfies $\sum_{k\in\mathbb{Z}_{[t,\hat{t})}} |s_{\rm d}(u_{\rm d}(t_k), y(t_k))| < \infty$, where $\mathcal{U}_{\rm d}$ is an input space and $\mathbb{Z}_{[t,\hat{t})} \triangleq \{k : t \leq t_k < \hat{t}\}$. Furthermore, we assume that $\mathcal{G}_{\rm p}$ is *lossless with respect to the hybrid supply rate* $(s_{\rm c}(u_{\rm c}, y), s_{\rm d}(u_{\rm d}, y))$, and hence, there exists a continuous, nonnegative-definite *storage function* $V_{\rm s} : \mathcal{D}_{\rm p} \to \overline{\mathbb{R}}_+$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(x_{\rm p}(t)) = V_{\rm s}(x_{\rm p}(t_0)) + \int_{t_0}^{t} s_{\rm c}(u_{\rm c}(\sigma), y(\sigma)){\rm d}\sigma + \sum_{k\in\mathbb{Z}_{[t,t_0)}} s_{\rm d}(u_{\rm d}(t_k), y(t_k)), \quad t \geq t_0, \tag{26}$$

for all $t_0, t \geq 0$, where $x_{\rm p}(t)$, $t \geq t_0$, is the solution to (1) and (2) with $(u_{\rm c}, u_{\rm d}) \in \mathcal{U}_{\rm c} \times \mathcal{U}_{\rm d}$. Equivalently, over the interval $t \in (t_k, t_{k+1}]$, (26) can be written as [6]

$$V_{\rm s}(x_{\rm p}(\hat{t})) - V_{\rm s}(x_{\rm p}(t)) = \int_{t}^{\hat{t}} s_{\rm c}(u_{\rm c}(\sigma), y(\sigma)){\rm d}\sigma, \quad t_k < t \leq \hat{t} \leq t_{k+1},\ k \in \overline{\mathbb{Z}}_+, \tag{27}$$

$$V_{\rm s}(x_{\rm p}(t_k) + f_{\rm dp}(x_{\rm p}(t_k), u_{\rm d}(t_k))) - V_{\rm s}(x_{\rm p}(t_k)) = s_{\rm d}(u_{\rm d}(t_k), y(t_k)). \tag{28}$$

In addition, we assume that the nonlinear impulsive dynamical system $\mathcal{G}_{\rm p}$ is *completely reachable* [6] and *zero-state observable* [6], and there exist functions $\kappa_{\rm c} : \mathbb{R}^l \to \mathbb{R}^{m_{\rm c}}$ and $\kappa_{\rm d} : \mathbb{R}^l \to \mathbb{R}^{m_{\rm d}}$ such that $\kappa_{\rm c}(0) = 0$, $\kappa_{\rm d}(0) = 0$, $s_{\rm c}(\kappa_{\rm c}(y), y) < 0$, $y \neq 0$, and $s_{\rm d}(\kappa_{\rm d}(y), y) < 0$, $y \neq 0$, so that all storage functions $V_{\rm s}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, of $\mathcal{G}_{\rm p}$ are positive definite [6]. Finally, we assume that $V_{\rm s}(\cdot)$ is continuously differentiable.

Next, consider the negative feedback interconnection of $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ given by $y = u_{\rm cc}$ and $(u_{\rm c}, u_{\rm d}) = (-y_{\rm cc}, -y_{\rm dc})$. In this case, the closed-loop system $\mathcal{G}$ is given by

$$\dot{x}(t) = f_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z},\ t \geq 0, \tag{29}$$

$$\Delta x(t) = f_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}, \tag{30}$$

where $t \geq 0$, $x(t) \triangleq [x_{\rm p}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$, $\mathcal{Z} \triangleq \mathcal{Z}_1 \cup \mathcal{Z}_2$, $\mathcal{Z}_1 \triangleq \{x \in \mathcal{D} : (x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \in \mathcal{Z}_{\rm p}\}$, $\mathcal{Z}_2 \triangleq \{x \in \mathcal{D} : (x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{Z}_{\rm c}\}$,

$$f_{\rm c}(x) \triangleq \begin{bmatrix} f_{\rm cp}(x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \\ f_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \end{bmatrix}, \qquad f_{\rm d}(x) \triangleq \begin{bmatrix} f_{\rm dp}(x_{\rm p}, -h_{\rm dc}(x_{\rm c}, h_{\rm p}(x_{\rm p})))\chi_{\mathcal{Z}_1}(x) \\ (\eta(h_{\rm p}(x_{\rm p})) - x_{\rm c})\chi_{\mathcal{Z}_2}(x) \end{bmatrix}. \tag{31}$$

Assume that there exists an infinitely differentiable function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$, and

$$\dot{V}_{\rm c}(x_{\rm c}(t), y(t)) = s_{\rm cc}(u_{\rm cc}(t), y_{\rm cc}(t)), \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c},\ t \geq 0, \tag{32}$$

where $s_{\rm cc} : \mathbb{R}^l \times \mathbb{R}^{m_{\rm c}} \to \mathbb{R}$ is such that $s_{\rm cc}(0, 0) = 0$.

We associate with the plant a positive-definite, continuously differentiable function $V_{\rm p}(x_{\rm p}) \triangleq V_{\rm s}(x_{\rm p})$, which we will refer to as the *plant energy*. Furthermore, we associate with the controller a nonnegative-definite, infinitely differentiable function $V_{\rm c}(x_{\rm c}, y)$ called the controller *emulated energy*. Finally, we associate with the closed-loop system the function

$$V(x) \triangleq V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})), \tag{33}$$

called the *total energy*.

Next, we construct the resetting set for $\mathcal{G}_{\mathrm{c}}$ in the following form:

$$\begin{aligned}\mathcal{Z}_2 &= \left\{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = 0 \text{ and } V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) > 0\right\} \\ &= \left\{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : s_{\mathrm{cc}}(h_{\mathrm{p}}(x_{\mathrm{p}}), h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))) = 0 \text{ and } V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) > 0\right\}. \end{aligned} \tag{34}$$

The resetting set $\mathcal{Z}_2$ is thus defined to be the set of all points in the closed-loop state space that correspond to decreasing controller emulated energy. By resetting the controller states, the plant energy can never increase after the first resetting event. Furthermore, if the closed-loop system total energy is conserved between resetting events, then a decrease in plant energy is accompanied by a corresponding increase in emulated energy. Hence, this approach allows the plant energy to flow to the controller, where it increases the emulated energy but does not allow the emulated energy to flow back to the plant after the first resetting event. This energy dissipating hybrid controller effectively enforces a one-way energy transfer between the plant and the controller after the first resetting event. The next theorem gives sufficient conditions for asymptotic stability of the closed-loop system $\mathcal{G}$ using state-dependent hybrid controllers. For practical implementation, knowledge of x_{c} and y is sufficient for determining whether or not the closed-loop state vector is in the set $\mathcal{Z}_2$.

Theorem 3.1. *Consider the closed-loop impulsive dynamical system $\mathcal{G}$ given by (29) and (30) with the resetting set $\mathcal{Z}_2$ given by (34). Assume that $\mathcal{D}_{\mathrm{ci}} \subset \mathcal{D}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{\mathrm{ci}}$, assume that if $x_0 \in \mathcal{Z}_1$ then $x_0 + f_{\mathrm{d}}(x_0) \in \overline{\mathcal{Z}}_1 \setminus \mathcal{Z}_1$, and if $x_0 \in \overline{\mathcal{Z}}_1 \setminus \mathcal{Z}_1$, then $f_{\mathrm{dp}}(x_{\mathrm{p}0}, -h_{\mathrm{dc}}(x_{\mathrm{c}0}, h_{\mathrm{p}}(x_{\mathrm{p}0}))) = 0$, where $\overline{\mathcal{Z}}_1 = \{x \in \mathcal{D} : \mathcal{X}_{\mathrm{p}}(x) = 0\}$ with an infinitely differentiable function $\mathcal{X}_{\mathrm{p}}(\cdot)$, and assume that $\mathcal{G}_{\mathrm{p}}$ is lossless with respect to the hybrid supply rate $(s_{\mathrm{c}}(u_{\mathrm{c}}, y), s_{\mathrm{d}}(u_{\mathrm{d}}, y))$ and with a positive-definite, continuously differentiable storage function $V_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}} \in \mathcal{D}_{\mathrm{p}}$. In addition, assume there exists a smooth (i.e., infinitely differentiable) function $V_{\mathrm{c}} : \mathcal{D}_{\mathrm{c}} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\mathrm{c}}(x_{\mathrm{c}}, y) \geq 0$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $y \in \mathbb{R}^l$, $V_{\mathrm{c}}(x_{\mathrm{c}}, y) = 0$ if and only if $x_{\mathrm{c}} = \eta(y)$, and (32) holds. Furthermore, assume that every $x_0 \in \overline{\mathcal{Z}}$ is transversal to (29) with $\mathcal{X}_{\mathrm{c}}(x) = \frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$, and*

$$s_{\mathrm{c}}(u_{\mathrm{c}}, y) + s_{\mathrm{cc}}(u_{\mathrm{cc}}, y_{\mathrm{cc}}) = 0, \quad x \notin \mathcal{Z}, \tag{35}$$

$$s_{\mathrm{d}}(u_{\mathrm{d}}, y) < 0, \quad x \in \mathcal{Z}_1, \tag{36}$$

where $y = u_{\mathrm{cc}} = h_{\mathrm{p}}(x_{\mathrm{p}})$, $u_{\mathrm{c}} = -y_{\mathrm{cc}} = -h_{\mathrm{cc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$ and $u_{\mathrm{d}} = -y_{\mathrm{dc}} = -h_{\mathrm{dc}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$. Then the zero solution $x(t) \equiv 0$ to the closed-loop system $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ given by (33) is strictly decreasing across resetting events. Finally, if $\mathcal{D}_{\mathrm{p}} = \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{c}} = \mathbb{R}^{n_{\mathrm{c}}}$, and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

Proof. First, note that since $V_{\mathrm{c}}(x_{\mathrm{c}}, y) \geq 0$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $y \in \mathbb{R}^l$, it follows that

$$\begin{aligned}\overline{\mathcal{Z}} &= \overline{\mathcal{Z}}_1 \cup \{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = 0 \text{ and } V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) \geq 0\} \\ &= \overline{\mathcal{Z}}_1 \cup \{(x_{\mathrm{p}}, x_{\mathrm{c}}) \in \mathcal{D}_{\mathrm{p}} \times \mathcal{D}_{\mathrm{c}} : \mathcal{X}_{\mathrm{c}}(x) = 0\},\end{aligned} \tag{37}$$

where $\mathcal{X}_{\mathrm{c}}(x) = L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}}))$. Next, we show that if the transversality condition (15) holds, then Assumptions 1–3 hold and, for every $x_0 \in \mathcal{D}_{\mathrm{ci}}$, there exists $\tau \geq 0$ such that $x(\tau) \in \mathcal{Z}$. Note that if $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, that is, $\mathcal{X}_{\mathrm{p}}(x(0)) = 0$, or $V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x(0))) = 0$ and $L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x_{\mathrm{p}}(0))) = 0$, it follows from the transversality condition that there exists $\delta > 0$ such that for all $t \in (0, \delta]$, $\mathcal{X}_{\mathrm{p}}(x(t)) \neq 0$ and $L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) \neq 0$. Hence, since $V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = V_{\mathrm{c}}(x_{\mathrm{c}}(0), h_{\mathrm{p}}(x_{\mathrm{p}}(0))) + t L_{f_{\mathrm{c}}} V_{\mathrm{c}}(x_{\mathrm{c}}(\tau), h_{\mathrm{p}}(x_{\mathrm{p}}(\tau)))$ for some $\tau \in (0, t]$ and $V_{\mathrm{c}}(x_{\mathrm{c}}, y) \geq 0$, $x_{\mathrm{c}} \in \mathcal{D}_{\mathrm{c}}$, $y \in \mathbb{R}^l$, it follows that $V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) > 0$, $t \in (0, \delta]$, which implies that Assumption 1 is satisfied. Furthermore, if $x \in \mathcal{Z}$ then, since $V_{\mathrm{c}}(x_{\mathrm{c}}, y) = 0$ if and only if $x_{\mathrm{c}} = \eta(y)$, it follows from (30) that $x + f_{\mathrm{d}}(x) \in \overline{\mathcal{Z}}_2 \setminus \mathcal{Z}_2$, and hence, $x + f_{\mathrm{d}}(x) \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$. Hence, Assumption 2 holds.

Next, consider the set $\mathcal{M}_\gamma \triangleq \left\{x \in \mathcal{D}_{\mathrm{ci}} : V_{\mathrm{c}}(x_{\mathrm{c}}, h_{\mathrm{p}}(x_{\mathrm{p}})) = \gamma\right\}$, where $\gamma \geq 0$. It follows from the transversality condition that for every $\gamma \geq 0$, $\mathcal{M}_\gamma$ does not contain any nontrivial trajectory of $\mathcal{G}$. To see this, suppose, *ad absurdum*, there exists a nontrivial trajectory $x(t) \in \mathcal{M}_\gamma$, $t \geq 0$, for some $\gamma \geq 0$. In this case, it follows that $\frac{\mathrm{d}^k}{\mathrm{d}t^k} V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) = L_{f_{\mathrm{c}}}^k V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) \equiv 0$, $k = 1, 2, \ldots$, which contradicts the transversality condition.

Next, we show that for every $x_0 \notin \mathcal{Z}$, $x_0 \neq 0$, there exists $\tau > 0$ such that $x(\tau) \in \mathcal{Z}$. To see this, suppose, *ad absurdum*, $x(t) \notin \mathcal{Z}$, $t \geq 0$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) \neq 0, \quad t \geq 0, \tag{38}$$

or

$$V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) = 0, \quad t \geq 0. \tag{39}$$

If (38) holds, then it follows that $V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t)))$ is a (decreasing or increasing) monotonic function of time. Hence, $V_{\mathrm{c}}(x_{\mathrm{c}}(t), h_{\mathrm{p}}(x_{\mathrm{p}}(t))) \to \gamma$ as $t \to \infty$, where $\gamma \geq 0$ is a constant, which implies that the positive limit set of the closed-loop system is contained in $\mathcal{M}_\gamma$ for some $\gamma \geq 0$, and hence, is a contradiction. Similarly, if (39) holds then $\mathcal{M}_0$ contains a nontrivial trajectory of $\mathcal{G}$ also leading to a contradiction. Hence, for every $x_0 \notin \mathcal{Z}$, there exists $\tau > 0$ such that $x(\tau) \in \mathcal{Z}$. Thus, it follows that for every $x_0 \notin \mathcal{Z}$, $0 < \tau_1(x_0) < \infty$. Now, it follows from Proposition 2.2 that $\tau_1(\cdot)$ is continuous at $x_0 \notin \overline{\mathcal{Z}}$. Furthermore, for all $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ and for every sequence $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ converging to $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$, it follows from the transversality condition and Proposition 2.2 that $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$. Next, let $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ and let $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}_{\mathrm{ci}}$ be such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists. In this case, it follows from Proposition 2.2 that either $\lim_{i\to\infty} \tau_1(x_i) = 0$ or $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$. Furthermore, since $x_0 \in \overline{\mathcal{Z}} \setminus \mathcal{Z}$ corresponds to the case where $f_{\mathrm{dp}}(x_{\mathrm{p0}}, -h_{\mathrm{dc}}(x_{\mathrm{c0}}, h_{\mathrm{p}}(x_{\mathrm{p0}}))) = 0$ or $V_{\mathrm{c}}(x_{\mathrm{c0}}, h_{\mathrm{p}}(x_{\mathrm{p0}})) = 0$, if $V_{\mathrm{c}}(x_{\mathrm{c0}}, h_{\mathrm{p}}(x_{\mathrm{p0}})) = 0$, then it follows that $x_{\mathrm{c0}} = \eta(h_{\mathrm{p}}(x_{\mathrm{p0}}))$, and hence, $f_{\mathrm{d}}(x_0) = 0$. Now, it follows from Proposition 2.1 that Assumption 3 holds.

To show that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable, consider the Lyapunov function candidate corresponding to the total energy function $V(x)$ given by (33). Since $\mathcal{G}_{\mathrm{p}}$ is lossless with respect to the hybrid supply rate $(s_{\mathrm{c}}(u_{\mathrm{c}}, y), s_{\mathrm{d}}(u_{\mathrm{d}}, y))$ and (32) and (35) hold, it follows that

$$\dot{V}(x(t)) = s_{\mathrm{c}}(u_{\mathrm{c}}(t), y(t)) + s_{\mathrm{cc}}(u_{\mathrm{cc}}(t), y_{\mathrm{cc}}(t)) = 0, \quad x(t) \notin \mathcal{Z}. \tag{40}$$

Furthermore, it follows from (28), (31) and (34) that

$$\begin{aligned}
\Delta V(x(t_k)) &= V_{\mathrm{p}}(x_{\mathrm{p}}(t_k^+)) - V_{\mathrm{p}}(x_{\mathrm{p}}(t_k)) + V_{\mathrm{c}}(x_{\mathrm{c}}(t_k^+), h_{\mathrm{p}}(x_{\mathrm{p}}(t_k^+))) - V_{\mathrm{c}}(x_{\mathrm{c}}(t_k), h_{\mathrm{p}}(x_{\mathrm{p}}(t_k)))\\
&= s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y(t_k))\chi_{\mathcal{Z}_1}(x(t_k))\\
&\quad + [V_{\mathrm{c}}(\eta(h_{\mathrm{p}}(x_{\mathrm{p}}(t_k))), h_{\mathrm{p}}(x_{\mathrm{p}}(t_k))) - V_{\mathrm{c}}(x_{\mathrm{c}}(t_k), h_{\mathrm{p}}(x_{\mathrm{p}}(t_k)))]\chi_{\mathcal{Z}_2}(x(t_k))\\
&= s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y(t_k))\chi_{\mathcal{Z}_1}(x(t_k)) - V_{\mathrm{c}}(x_{\mathrm{c}}(t_k), h_{\mathrm{p}}(x_{\mathrm{p}}(t_k)))\chi_{\mathcal{Z}_2}(x(t_k))\\
&< 0, \quad x(t_k) \in \mathcal{Z}, \; k \in \overline{\mathbb{Z}}_+.
\end{aligned} \tag{41}$$

Thus, it follows from Theorem 2.2 that the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable. Finally, if $\mathcal{D}_{\mathrm{p}} = \mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{c}} = \mathbb{R}^{n_{\mathrm{c}}}$, and $V(\cdot)$ is radially unbounded, then global asymptotic stability is immediate. □

Remark 3.1. If $V_{\mathrm{c}} = V_{\mathrm{c}}(x_{\mathrm{c}}, y)$ is only a function of x_{c} and $V_{\mathrm{c}}(x_{\mathrm{c}})$ is a positive-definite function, then we can choose $\eta(y) \equiv 0$. In this case, $V_{\mathrm{c}}(x_{\mathrm{c}}) = 0$ if and only if $x_{\mathrm{c}} = 0$, and hence, Theorem 3.1 specializes to the case of a negative feedback interconnection of two hybrid lossless dynamical systems $\mathcal{G}_{\mathrm{p}}$ and $\mathcal{G}_{\mathrm{c}}$ [7].

Remark 3.2. In the proof of Theorem 3.1, we assume that $x_0 \notin \mathcal{Z}$ for $x_0 \neq 0$. This proviso is necessary since it may be possible to reset the states of the closed-loop system to the origin, in which case $x(s) = 0$ for a finite value of s. In this case, for $t > s$, we have $V(x(t)) = V(x(s)) = V(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.

Remark 3.3. Theorem 3.1 can be trivially generalized to the case where $\mathcal{G}_{\mathrm{p}}$ is *dissipative* with respect to the hybrid supply rate $(s_{\mathrm{c}}(u_{\mathrm{c}}, y), s_{\mathrm{d}}(u_{\mathrm{d}}, y))$ in the sense of [6],

$$V_{\mathrm{s}}(x_{\mathrm{p}}(\hat{t})) = V_{\mathrm{s}}(x_{\mathrm{p}}(t)) + \int_t^{\hat{t}} s_{\mathrm{c}}(u_{\mathrm{c}}(\sigma), y(\sigma))\mathrm{d}\sigma, \quad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{42}$$

$$V_{\mathrm{s}}(x_{\mathrm{p}}(t_k) + f_{\mathrm{dp}}(x_{\mathrm{p}}(t_k), u_{\mathrm{d}}(t_k))) \leq V_{\mathrm{s}}(x_{\mathrm{p}}(t_k)) + s_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y(t_k)), \quad k \in \overline{\mathbb{Z}}_+. \tag{43}$$

In this case, the dissipation rate function inherent in (43) does not add any additional complexity to the hybrid stabilization process. Similar remarks hold for impulsive port-controlled Hamiltonian systems considered below.

Finally, we specialize the hybrid controller design framework just presented to *impulsive port-controlled Hamiltonian systems* [23]. Specifically, consider the state-dependent impulsive port-controlled Hamiltonian system given by

$$\dot{x}_{\mathrm{p}}(t)=\mathcal{J}_{\mathrm{cp}}(x_{\mathrm{p}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}(t))\right)^{\mathrm{T}}+G_{\mathrm{p}}(x_{\mathrm{p}}(t))u_{\mathrm{c}}(t),\quad x_{\mathrm{p}}(0)=x_{\mathrm{p}0},\ (x_{\mathrm{p}}(t),u_{\mathrm{c}}(t))\notin\mathcal{Z}_{\mathrm{p}},\tag{44}$$

$$\Delta x_{\mathrm{p}}(t)=\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}(t))\right)^{\mathrm{T}}+G_{\mathrm{p}}(x_{\mathrm{p}}(t))u_{\mathrm{d}}(t),\quad (x_{\mathrm{p}}(t),u_{\mathrm{c}}(t))\in\mathcal{Z}_{\mathrm{p}},\tag{45}$$

$$y(t)=G_{\mathrm{p}}^{\mathrm{T}}(x_{\mathrm{p}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}(t))\right)^{\mathrm{T}},\tag{46}$$

where $t\geq 0$, $x_{\mathrm{p}}(t)\in\mathcal{D}_{\mathrm{p}}\subseteq\mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{p}}$ is an open set with $0\in\mathcal{D}_{\mathrm{p}}$, $u_{\mathrm{c}}(t)\in\mathbb{R}^m$, $u_{\mathrm{d}}(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^m$, $\mathcal{H}_{\mathrm{p}}:\mathcal{D}_{\mathrm{p}}\to\mathbb{R}$ is an infinitely differentiable Hamiltonian function for the system (44)–(46), $\mathcal{J}_{\mathrm{cp}}:\mathcal{D}_{\mathrm{p}}\to\mathbb{R}^{n_{\mathrm{p}}\times n_{\mathrm{p}}}$ is such that $\mathcal{J}_{\mathrm{cp}}(x_{\mathrm{p}})=-\mathcal{J}_{\mathrm{cp}}^{\mathrm{T}}(x_{\mathrm{p}})$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, $\mathcal{J}_{\mathrm{cp}}(x_{\mathrm{p}})(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}))^{\mathrm{T}}$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, is smooth on $\mathcal{D}_{\mathrm{p}}$, $G_{\mathrm{p}}:\mathcal{D}_{\mathrm{p}}\to\mathbb{R}^{n_{\mathrm{p}}\times m}$, $\mathcal{J}_{\mathrm{dp}}:\mathcal{D}_{\mathrm{p}}\to\mathbb{R}^{n_{\mathrm{p}}\times n_{\mathrm{p}}}$ is such that $\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})=-\mathcal{J}_{\mathrm{dp}}^{\mathrm{T}}(x_{\mathrm{p}})$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, $\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}))^{\mathrm{T}}$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, is smooth on $\mathcal{D}_{\mathrm{p}}$, and $\mathcal{Z}_{\mathrm{p}}\triangleq\mathcal{Z}_{x_{\mathrm{p}}}\times\mathcal{Z}_{u_{\mathrm{c}}}\subset\mathcal{D}_{\mathrm{p}}\times\mathbb{R}^m$ is the resetting set. The skew-symmetric matrix functions $\mathcal{J}_{\mathrm{cp}}(x_{\mathrm{p}})$ and $\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, capture the internal hybrid system interconnection structure and the input matrix function $G_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, captures interconnections with the environment. Furthermore, we assume $\mathcal{H}_{\mathrm{p}}(\cdot)$ is such that

$$\mathcal{H}_{\mathrm{p}}\left(x_{\mathrm{p}}+\mathcal{J}_{\mathrm{dp}}(x_{\mathrm{p}})\left(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}})\right)^{\mathrm{T}}+G_{\mathrm{p}}(x_{\mathrm{p}})u_{\mathrm{d}}\right)=\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})+\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}})G_{\mathrm{p}}(x_{\mathrm{p}})u_{\mathrm{d}},\quad x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}},\ u_{\mathrm{d}}\in\mathbb{R}^m.\tag{47}$$

Finally, we assume that $\mathcal{H}_{\mathrm{p}}(0)=0$ and $\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})>0$ for all $x_{\mathrm{p}}\neq 0$ and $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$.

Next, consider the fixed-order, energy-based hybrid controller

$$\dot{x}_{\mathrm{c}}(t)=\mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}}+G_{\mathrm{cc}}(x_{\mathrm{c}}(t))y(t),\quad x_{\mathrm{c}}(0)=x_{\mathrm{c}0},\ (x_{\mathrm{c}}(t),y(t))\notin\mathcal{Z}_{\mathrm{c}},\tag{48}$$

$$\Delta x_{\mathrm{c}}(t)=-x_{\mathrm{c}}(t),\quad (x_{\mathrm{c}}(t),y(t))\in\mathcal{Z}_{\mathrm{c}},\tag{49}$$

$$u_{\mathrm{c}}(t)=-G_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}(t))\right)^{\mathrm{T}},\tag{50}$$

$$u_{\mathrm{d}}(t)=-G_{\mathrm{p}}^{\mathrm{T}}(x_{\mathrm{p}}(t))\left(\frac{\partial\mathcal{H}_{\mathrm{p}}}{\partial x_{\mathrm{p}}}(x_{\mathrm{p}}(t))\right)^{\mathrm{T}},\tag{51}$$

where $t\geq 0$, $x_{\mathrm{c}}(t)\in\mathcal{D}_{\mathrm{c}}\subseteq\mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0\in\mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t)\triangleq x_{\mathrm{c}}(t^+)-x_{\mathrm{c}}(t)$, $\mathcal{H}_{\mathrm{c}}:\mathcal{D}_{\mathrm{c}}\to\mathbb{R}$ is an infinitely differentiable Hamiltonian function for (48), $\mathcal{J}_{\mathrm{cc}}:\mathcal{D}_{\mathrm{c}}\to\mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{c}}}$ is such that $\mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}})=-\mathcal{J}_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathcal{D}_{\mathrm{c}}$, $\mathcal{J}_{\mathrm{cc}}(x_{\mathrm{c}})(\frac{\partial\mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}}))^{\mathrm{T}}$, $x_{\mathrm{c}}\in\mathcal{D}_{\mathrm{c}}$, is smooth on $\mathcal{D}_{\mathrm{c}}$, $G_{\mathrm{cc}}:\mathcal{D}_{\mathrm{c}}\to\mathbb{R}^{n_{\mathrm{c}}\times m}$, and the resetting set $\mathcal{Z}_{\mathrm{c}}\subset\mathcal{D}_{\mathrm{p}}\times\mathcal{D}_{\mathrm{c}}$ given by

$$\mathcal{Z}_{\mathrm{c}}\triangleq\left\{(x_{\mathrm{p}},x_{\mathrm{c}})\in\mathcal{D}_{\mathrm{p}}\times\mathcal{D}_{\mathrm{c}}:\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})=0\text{ and }\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})>0\right\},\tag{52}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t))\triangleq\lim_{\tau\to t^-}\frac{1}{t-\tau}[\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(t))-\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}}(\tau))]$ whenever the limit on the right-hand side exists. Here, we assume that $\mathcal{H}_{\mathrm{c}}(0)=0$ and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})>0$ for all $x_{\mathrm{c}}\neq 0$ and $x_{\mathrm{c}}\in\mathcal{D}_{\mathrm{c}}$.

Note that $\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})$, $x_{\mathrm{p}}\in\mathcal{D}_{\mathrm{p}}$, is the plant energy and $\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$, $x_{\mathrm{c}}\in\mathcal{D}_{\mathrm{c}}$, is the controller emulated energy. Furthermore, the closed-loop system energy is given by $\mathcal{H}(x_{\mathrm{p}},x_{\mathrm{c}})\triangleq\mathcal{H}_{\mathrm{p}}(x_{\mathrm{p}})+\mathcal{H}_{\mathrm{c}}(x_{\mathrm{c}})$. The resetting set $\mathcal{Z}$ is given by $\mathcal{Z}\triangleq\mathcal{Z}_1\cup\mathcal{Z}_2$, where

$$\mathcal{Z}_1\triangleq\left\{(x_{\mathrm{p}},x_{\mathrm{c}})\in\mathcal{D}_{\mathrm{p}}\times\mathcal{D}_{\mathrm{c}}:\left(x_{\mathrm{p}},-G_{\mathrm{cc}}^{\mathrm{T}}(x_{\mathrm{c}})\left(\frac{\partial\mathcal{H}_{\mathrm{c}}}{\partial x_{\mathrm{c}}}(x_{\mathrm{c}})\right)^{\mathrm{T}}\right)\in\mathcal{Z}_{\mathrm{p}}\right\},\tag{53}$$

$$\mathcal{Z}_2 \triangleq \left\{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \left(x_{\rm c}, G_{\rm p}^{\rm T}(x_{\rm p})\left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p})\right)^{\rm T}\right) \in \mathcal{Z}_{\rm c}\right\}. \tag{54}$$

Here, we assume that $\overline{\mathcal{Z}}_1 = \{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \mathcal{X}_1(x_{\rm p}, x_{\rm c}) = 0\}$. Furthermore, if $(x_{\rm p}, x_{\rm c}) \in \mathcal{Z}_1$ then $x_{\rm p} + \mathcal{J}_{\rm dp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} - G_{\rm p}(x_{\rm p})G_{\rm p}^{\rm T}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} \in \overline{\mathcal{Z}}_1 \setminus \mathcal{Z}_1$, and if $(x_{\rm p}, x_{\rm c}) \in \overline{\mathcal{Z}}_1 \setminus \mathcal{Z}_1$ then $\mathcal{J}_{\rm dp}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} - G_{\rm p}(x_{\rm p})G_{\rm p}^{\rm T}(x_{\rm p})(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}))^{\rm T} = 0$. Finally, we assume that

$$\mathcal{Z}_1 \cap \left\{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : G_{\rm p}^{\rm T}(x_{\rm p})\left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p})\right)^{\rm T} = 0\right\} = \emptyset. \tag{55}$$

Next, note that total energy function $\mathcal{H}(x_{\rm p}, x_{\rm c})$ along the trajectories of the closed-loop dynamics (44)–(54) satisfies

$$\frac{\rm d}{{\rm d}t}\mathcal{H}(x_{\rm p}(t), x_{\rm c}(t)) = 0, \quad (x_{\rm p}(t), x_{\rm c}(t)) \notin \mathcal{Z}, \tag{56}$$

$$\begin{aligned}\Delta\mathcal{H}(x_{\rm p}(t_k), x_{\rm c}(t_k)) = & -\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t_k))G_{\rm p}(x_{\rm p}(t_k))G_{\rm p}^{\rm T}(x_{\rm p}(t_k))\left(\frac{\partial \mathcal{H}_{\rm p}}{\partial x_{\rm p}}(x_{\rm p}(t_k))\right)^{\rm T} \\ & \cdot \chi_{\mathcal{Z}_1}(x_{\rm p}(t_k), x_{\rm c}(t_k)) - \mathcal{H}_{\rm c}(x_{\rm c}(t_k))\chi_{\mathcal{Z}_2}(x_{\rm p}(t_k), x_{\rm c}(t_k)), \\ & (x_{\rm p}(t_k), x_{\rm c}(t_k)) \in \mathcal{Z},\ k \in \overline{\mathbb{Z}}_+.\end{aligned} \tag{57}$$

Here, we assume that every $(x_{\rm p0}, x_{\rm c0}) \in \overline{\mathcal{Z}}$ is transversal to the closed-loop dynamical system given by (44)–(54) with $\mathcal{X}_{\rm p}(x_{\rm p}, x_{\rm c}) = \mathcal{X}_1(x_{\rm p}, x_{\rm c})$ and $\mathcal{X}_{\rm c}(x_{\rm p}, x_{\rm c}) = \frac{\rm d}{{\rm d}t}\mathcal{H}_{\rm c}(x_{\rm c})$. Furthermore, we assume $\mathcal{D}_{\rm ci} \subset \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c}$ is a compact positively invariant set with respect to the closed-loop dynamical system (44)–(54), such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$. In this case, it follows from Theorem 3.1, with $V_{\rm s}(x_{\rm p}) = \mathcal{H}_{\rm p}(x_{\rm p})$, $V_{\rm c}(x_{\rm c}, y) = \mathcal{H}_{\rm c}(x_{\rm c})$, $s_{\rm c}(u_{\rm c}, y) = u_{\rm c}^{\rm T}y$, $s_{\rm d}(u_{\rm d}, y) = u_{\rm d}^{\rm T}y$, and $s_{\rm cc}(u_{\rm cc}, y_{\rm cc}) = u_{\rm cc}^{\rm T}y_{\rm cc}$, that the zero solution $(x_{\rm p}(t), x_{\rm c}(t)) \equiv (0, 0)$ to the closed-loop system (44)–(54) is asymptotically stable.

4. Hybrid control design for nonsmooth Euler–Lagrange systems

In this section, we present a hybrid feedback control framework for nonsmooth Euler–Lagrange dynamical systems. Consider the governing equations of motion of an $\hat{n}_{\rm p}$-degree-of-freedom dynamical system given by the *hybrid Euler–Lagrange* equation

$$\frac{\rm d}{{\rm d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}(q(t), \dot{q}(t))\right]^{\rm T} - \left[\frac{\partial \mathcal{L}}{\partial q}(q(t), \dot{q}(t))\right]^{\rm T} = u_{\rm c}(t), \quad q(0) = q_0,\ \dot{q}(0) = \dot{q}_0, \quad (q(t), \dot{q}(t)) \notin \mathcal{Z}_{\rm p}, \tag{58}$$

$$\begin{bmatrix}\Delta q(t)\\ \Delta\dot{q}(t)\end{bmatrix} = \begin{bmatrix}P(q(t)) - q(t)\\ Q(\dot{q}(t)) - \dot{q}(t)\end{bmatrix}, \quad (q(t), \dot{q}(t)) \in \mathcal{Z}_{\rm p}, \tag{59}$$

with outputs

$$y = \begin{bmatrix}h_1(q)\\ h_2(\dot{q})\end{bmatrix}, \tag{60}$$

where $t \geq 0$, $q \in \mathbb{R}^{\hat{n}_{\rm p}}$ represents the generalized system positions, $\dot{q} \in \mathbb{R}^{\hat{n}_{\rm p}}$ represents the generalized system velocities, $\mathcal{L} : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ denotes the system Lagrangian given by $\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - U(q)$, where $T : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is the system kinetic energy and $U : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is the system potential energy, $u_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm p}}$ is the vector of generalized control forces acting on the system, $\mathcal{Z}_{\rm p} \subset \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}}$ is the resetting set such that the closure of $\mathcal{Z}_{\rm p}$ is given by

$$\overline{\mathcal{Z}}_{\rm p} \triangleq \{(q, \dot{q}) : H(q, \dot{q}) = 0\}, \tag{61}$$

where $H : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is an infinitely differentiable function, $\Delta q(t) \triangleq q(t^+) - q(t)$, $\Delta\dot{q}(t) \triangleq \dot{q}(t^+) - \dot{q}(t)$, $P : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{\hat{n}_{\rm p}}$ and $Q : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{\hat{n}_{\rm p}}$ are smooth functions such that if $(q, \dot{q}) \in \mathcal{Z}_{\rm p}$, then $(P(q), Q(\dot{q})) \in \overline{\mathcal{Z}}_{\rm p} \setminus \mathcal{Z}_{\rm p}$, and if $(q,\dot q)\in\overline{\mathcal{Z}}_{\rm p}\setminus\mathcal{Z}_{\rm p}$, then $(P(q),Q(\dot q))=(q,\dot q)$, $T(P(q),Q(\dot q))+U(P(q))<T(q,\dot q)+U(q)$, $(q,\dot q)\in\mathcal{Z}_{\rm p}$, $h_1:\mathbb{R}^{\hat n_{\rm p}}\to\mathbb{R}^{l_1}$ and $h_2:\mathbb{R}^{\hat n_{\rm p}}\to\mathbb{R}^{l-l_1}$ are smooth functions, $h_1(0)=0$, $h_2(0)=0$, and $h_1(q)\not\equiv 0$. We assume that the system kinetic energy is such that $T(q,\dot q)=\frac{1}{2}\dot q^{\rm T}[\frac{\partial T}{\partial\dot q}(q,\dot q)]^{\rm T}$, $T(q,0)=0$, and $T(q,\dot q)>0$, $\dot q\neq 0$, $\dot q\in\mathbb{R}^{\hat n_{\rm p}}$.

Furthermore, let $\mathcal{H}:\mathbb{R}^{\hat n_{\rm p}}\times\mathbb{R}^{\hat n_{\rm p}}\to\mathbb{R}$ denote the *Legendre transformation* of the Lagrangian function $\mathcal{L}(q,\dot q)$ with respect to the generalized velocity $\dot q$ defined by $\mathcal{H}(q,p)\triangleq\dot q^{\rm T}p-\mathcal{L}(q,\dot q)$, where p denotes the vector of generalized momenta given by

$$p(q,\dot q)=\left[\frac{\partial\mathcal{L}}{\partial\dot q}(q,\dot q)\right]^{\rm T},\tag{62}$$

where the map from the generalized velocities $\dot q$ to the generalized momenta p is assumed to be *bijective* (i.e., one-to-one and onto). Now, if $\mathcal{H}(q,p)$ is lower bounded, then we can always shift $\mathcal{H}(q,p)$ so that, with a minor abuse of notation, $\mathcal{H}(q,p)\geq 0$, $(q,p)\in\mathbb{R}^{\hat n_{\rm p}}\times\mathbb{R}^{\hat n_{\rm p}}$. In this case, using (58) and the fact that

$$\frac{\rm d}{{\rm d}t}[\mathcal{L}(q,\dot q)]=\frac{\partial\mathcal{L}}{\partial q}(q,\dot q)\dot q+\frac{\partial\mathcal{L}}{\partial\dot q}(q,\dot q)\ddot q,\quad (q,\dot q)\notin\mathcal{Z}_{\rm p},\tag{63}$$

it follows that $\frac{\rm d}{{\rm d}t}\mathcal{H}(q,p)=u_{\rm c}^{\rm T}\dot q$, $(q,\dot q)\notin\mathcal{Z}_{\rm p}$. We also assume that the system potential energy $U(\cdot)$ is such that $U(0)=0$ and $U(q)>0$, $q\neq 0$, $q\in\mathcal{D}_{\rm q}\subseteq\mathbb{R}^{\hat n_{\rm p}}$, which implies that $\mathcal{H}(q,p)=T(q,\dot q)+U(q)>0$, $(q,\dot q)\neq 0$, $(q,\dot q)\in\mathcal{D}_{\rm q}\times\mathbb{R}^{\hat n_{\rm p}}$.

Next, consider the energy-based hybrid controller

$$\frac{\rm d}{{\rm d}t}\left[\frac{\partial\mathcal{L}_{\rm c}}{\partial\dot q_{\rm c}}(q_{\rm c}(t),\dot q_{\rm c}(t),y_q(t))\right]^{\rm T}-\left[\frac{\partial\mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t),\dot q_{\rm c}(t),y_q(t))\right]^{\rm T}=0,\quad q_{\rm c}(0)=q_{\rm c0},\ \dot q_{\rm c}(0)=\dot q_{\rm c0},$$
$$(q_{\rm c}(t),\dot q_{\rm c}(t),y(t))\notin\mathcal{Z}_{\rm c},\tag{64}$$

$$\begin{bmatrix}\Delta q_{\rm c}(t)\\ \Delta\dot q_{\rm c}(t)\end{bmatrix}=\begin{bmatrix}\eta(y_q(t))-q_{\rm c}(t)\\ -\dot q_{\rm c}(t)\end{bmatrix},\quad (q_{\rm c}(t),\dot q_{\rm c}(t),y(t))\in\mathcal{Z}_{\rm c},\tag{65}$$

$$u_{\rm c}(t)=\left[\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot q_{\rm c}(t),y_q(t))\right]^{\rm T},\tag{66}$$

where $t\geq 0$, $q_{\rm c}\in\mathbb{R}^{\hat n_{\rm c}}$ represents virtual controller positions, $\dot q_{\rm c}\in\mathbb{R}^{\hat n_{\rm c}}$ represents virtual controller velocities, $y_q\triangleq h_1(q)$, $\mathcal{L}_{\rm c}:\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{l_1}\to\mathbb{R}$ denotes the controller Lagrangian given by $\mathcal{L}_{\rm c}(q_{\rm c},\dot q_{\rm c},y_q)\triangleq T_{\rm c}(q_{\rm c},\dot q_{\rm c})-U_{\rm c}(q_{\rm c},y_q)$, where $T_{\rm c}:\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{\hat n_{\rm c}}\to\mathbb{R}$ is the controller kinetic energy, $U_{\rm c}:\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{l_1}\to\mathbb{R}$ is the controller potential energy, $\eta(\cdot)$ is a continuously differentiable function such that $\eta(0)=0$, $\mathcal{Z}_{\rm c}\subset\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{\hat n_{\rm c}}\times\mathbb{R}^{l}$ is the resetting set, $\Delta q_{\rm c}(t)\triangleq q_{\rm c}(t^+)-q_{\rm c}(t)$, and $\Delta\dot q_{\rm c}(t)\triangleq\dot q_{\rm c}(t^+)-\dot q_{\rm c}(t)$. We assume that the controller kinetic energy $T_{\rm c}(q_{\rm c},\dot q_{\rm c})$ is such that $T_{\rm c}(q_{\rm c},\dot q_{\rm c})=\frac{1}{2}\dot q_{\rm c}^{\rm T}[\frac{\partial T_{\rm c}}{\partial\dot q_{\rm c}}(q_{\rm c},\dot q_{\rm c})]^{\rm T}$, with $T_{\rm c}(q_{\rm c},0)=0$ and $T_{\rm c}(q_{\rm c},\dot q_{\rm c})>0$, $\dot q_{\rm c}\neq 0$, $\dot q_{\rm c}\in\mathbb{R}^{\hat n_{\rm c}}$. Furthermore, we assume that $U_{\rm c}(\eta(y_q),y_q)=0$ and $U_{\rm c}(q_{\rm c},y_q)>0$ for $q_{\rm c}\neq\eta(y_q)$, $q_{\rm c}\in\mathcal{D}_{q_{\rm c}}\subseteq\mathbb{R}^{\hat n_{\rm c}}$.

As in Section 3, note that $V_{\rm p}(q,\dot q)\triangleq T(q,\dot q)+U(q)$ is the plant energy and $V_{\rm c}(q_{\rm c},\dot q_{\rm c},y_q)\triangleq T_{\rm c}(q_{\rm c},\dot q_{\rm c})+U_{\rm c}(q_{\rm c},y_q)$ is the controller emulated energy. Furthermore, $V(q,\dot q,q_{\rm c},\dot q_{\rm c})\triangleq V_{\rm p}(q,\dot q)+V_{\rm c}(q_{\rm c},\dot q_{\rm c},y_q)$ is the total energy of the closed-loop system. It is important to note that the Lagrangian dynamical system (58) is *not* lossless with outputs y_q or y. Next, we study the behavior of the total energy function $V(q,\dot q,q_{\rm c},\dot q_{\rm c})$ along the trajectories of the closed-loop system dynamics. For the closed-loop system, we define our resetting set as $\mathcal{Z}\triangleq\mathcal{Z}_1\cup\mathcal{Z}_2$, where $\mathcal{Z}_1\triangleq\{(q,\dot q,q_{\rm c},\dot q_{\rm c}):(q,\dot q)\in\mathcal{Z}_{\rm p}\}$ and $\mathcal{Z}_2\triangleq\{(q,\dot q,q_{\rm c},\dot q_{\rm c}):(q_{\rm c},\dot q_{\rm c},y)\in\mathcal{Z}_{\rm c}\}$. Note that

$$\frac{\rm d}{{\rm d}t}V_{\rm p}(q,\dot q)=\frac{\rm d}{{\rm d}t}\mathcal{H}(q,p)=u_{\rm c}^{\rm T}\dot q,\quad (q,\dot q,q_{\rm c},\dot q_{\rm c})\notin\mathcal{Z}.\tag{67}$$

To obtain an expression for $\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot q_{\rm c},y_q)$ when $(q,\dot q,q_{\rm c},\dot q_{\rm c})\notin\mathcal{Z}$, define the controller Hamiltonian by

$$\mathcal{H}_{\rm c}(q_{\rm c},\dot q_{\rm c},p_{\rm c},y_q)\triangleq\dot q_{\rm c}^{\rm T}p_{\rm c}-\mathcal{L}_{\rm c}(q_{\rm c},\dot q_{\rm c},y_q),\tag{68}$$

where the virtual controller momentum $p_{\rm c}$ is given by $p_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)=\left[\frac{\partial\mathcal{L}_{\rm c}}{\partial\dot{q}_{\rm c}}(q_{\rm c},\dot{q}_{\rm c},y_q)\right]^{\rm T}$. Then $\mathcal{H}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},p_{\rm c},y_q)=T_{\rm c}(q_{\rm c},\dot{q}_{\rm c})+U_{\rm c}(q_{\rm c},y_q)$. Now, it follows from (64) and the structure of $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c})$ that, for $t\in(t_k,t_{k+1}]$,

$$\begin{aligned}
0 &= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]^{\rm T}\dot{q}_{\rm c}(t)-\frac{\partial\mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t)\\
&= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t)\right]-p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\ddot{q}_{\rm c}(t)+\frac{\partial\mathcal{L}_{\rm c}}{\partial\dot{q}_{\rm c}}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\ddot{q}_{\rm c}(t)\\
&\quad+\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)-\frac{\rm d}{{\rm d}t}\mathcal{L}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\\
&= \frac{\rm d}{{\rm d}t}\left[p_{\rm c}^{\rm T}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}_{\rm c}(t)-\mathcal{L}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\right]+\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= \frac{\rm d}{{\rm d}t}\mathcal{H}_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),p_{\rm c}(t),y_q(t))+\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))+\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t),\quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t))\notin\mathcal{Z}.
\end{aligned}\tag{69}$$

Hence,

$$\begin{aligned}
\frac{\rm d}{{\rm d}t}V(q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t)) &= u_{\rm c}^{\rm T}(t)\dot{q}(t)-\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\
&= 0,\quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t))\notin\mathcal{Z},\ t_k<t\le t_{k+1},
\end{aligned}\tag{70}$$

which implies that the total energy of the closed-loop system between resetting events is conserved.

The total energy difference across resetting events is given by

$$\begin{aligned}
\Delta V(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k)) &= V_{\rm p}(q(t_k^+),\dot{q}(t_k^+))-V_{\rm p}(q(t_k),\dot{q}(t_k))\\
&\quad+T_{\rm c}(q_{\rm c}(t_k^+),\dot{q}_{\rm c}(t_k^+))+U_{\rm c}(q_{\rm c}(t_k^+),y_q(t_k))-V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k))\\
&= [V_{\rm p}(P(q(t_k)),Q(\dot{q}(t_k)))-V_{\rm p}(q(t_k),\dot{q}(t_k))]\\
&\quad\cdot\chi_{\mathcal{Z}_1}(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))-V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k))\\
&\quad\cdot\chi_{\mathcal{Z}_2}(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))\\
&< 0,\quad (q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))\in\mathcal{Z},\ k\in\overline{\mathbb{Z}}_+,
\end{aligned}\tag{71}$$

which implies that the resetting law (65) ensures the total energy decrease across resetting events.

Here, we concentrate on an energy dissipating state-dependent resetting controller that affects a one-way energy transfer between the plant and the controller. Specifically, consider the closed-loop system (58)–(66), where $\mathcal{Z}_{\rm c}$ is defined by

$$\mathcal{Z}_{\rm c}\triangleq\left\{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}):\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)=0\text{ and }V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)>0\right\}.\tag{72}$$

Since $y_q=h_1(q)$ and

$$\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)=-\left[\frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c},\dot{q}_{\rm c},y_q)\right]\dot{q}=\left[\frac{\partial U_{\rm c}}{\partial q}(q_{\rm c},y_q)\right]\dot{q},\quad (q_{\rm c},\dot{q}_{\rm c},y)\notin\mathcal{Z}_{\rm c},\tag{73}$$

it follows that (72) can be equivalently rewritten as

$$\mathcal{Z}_{\rm c}=\left\{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}):\left[\frac{\partial U_{\rm c}}{\partial q}(q_{\rm c},h_1(q))\right]\dot{q}=0\text{ and }V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},h_1(q))>0\right\}.\tag{74}$$

Once again, for practical implementation, knowledge of $q_{\rm c}$, $\dot{q}_{\rm c}$, and y is often sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}_{\rm c}$.

The next theorem gives sufficient conditions for stabilization of nonsmooth Euler–Lagrange dynamical systems using state-dependent hybrid controllers. For this result define the closed-loop system states $x\triangleq[q^{\rm T},\dot{q}^{\rm T},q_{\rm c}^{\rm T},\dot{q}_{\rm c}^{\rm T}]^{\rm T}$.

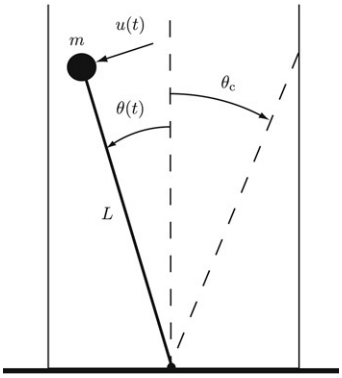

Fig. 5.1. Constrained inverted pendulum.

Theorem 4.1. *Consider the closed-loop dynamical system $\mathcal{G}$ given by (58)–(66), with the resetting set $\mathcal{Z}_c$ given by (72). Assume that $\mathcal{D}_{ci} \subset \mathcal{D}_q \times \mathbb{R}^{\hat{n}_p} \times \mathcal{D}_{q_c} \times \mathbb{R}^{\hat{n}_c}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{ci}$. Furthermore, assume that the transversality condition (15) and (16) holds with $\mathcal{X}_p(x) = H(q, \dot{q})$ and $\mathcal{X}_c(x) = \frac{d}{dt} V_c(q_c, \dot{q}_c, y_q)$. Then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ is strictly decreasing across resetting events. Finally, if $\mathcal{D}_q = \mathbb{R}^{\hat{n}_p}$, $\mathcal{D}_{q_c} = \mathbb{R}^{\hat{n}_c}$, and the total energy function $V(x)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.*

Proof. The proof is similar to the proof of Theorem 3.1 with $V_p(x_p) = V_p(q, \dot{q})$, $V_c(x_c, y) = V_c(q_c, \dot{q}_c, y_q)$, $y = u_{cc} = x_p$, $u_c = -y_{cc} = \frac{\partial \mathcal{L}_c}{\partial q}$, $s_c(u_c, y) = u_c^T \rho(y)$, $s_d(u_d, y) = 0$, $V_p(P(q), Q(\dot{q})) - V_p(q, \dot{q}) < 0$, $(q, \dot{q}) \in \mathcal{Z}_p$, $s_{cc}(u_{cc}, y_{cc}) = y_{cc}^T \rho(u_c)$, where $\rho(y) = \rho\left(\begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right) = \dot{q}$, $\eta(y)$ replaced by $\begin{bmatrix} \eta(y_q) \\ 0 \end{bmatrix}$, and noting that (70) and (71) hold. □

5. Hybrid control design for impact mechanics

In this section, we apply the energy dissipating hybrid controller synthesis framework to the constrained inverted pendulum shown in Fig. 5.1, where $m = 1$ kg and $L = 1$ m. In the case where $|\theta(t)| < \theta_c \le \frac{\pi}{2}$, the system is governed by the dynamic equation of motion

$$\ddot{\theta}(t) - g \sin\theta(t) = u_c(t), \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0, \ t \ge 0, \tag{75}$$

where g denotes the gravitational acceleration and $u_c(\cdot)$ is a (thruster) control force. At the instant of collision with the vertical constraint $|\theta(t)| = \theta_c$, the system resets according to the resetting law

$$\theta(t_k^+) = \theta(t_k), \quad \dot{\theta}(t_k^+) = -e\dot{\theta}(t_k), \tag{76}$$

where $e \in [0, 1)$ is the coefficient of restitution. Defining $q = \theta$ and $\dot{q} = \dot{\theta}$, we can rewrite the continuous-time dynamics (75) and resetting dynamics (76) in Lagrangian form (58) and (59) with $\mathcal{L}(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - g\cos q$, $P(q) = q$, $Q(\dot{q}) = -e\dot{q}$, and $\mathcal{Z}_p = \{(q, \dot{q}) \in \mathbb{R}^2 : q = \theta_c, \dot{q} > 0\} \cup \{(q, \dot{q}) \in \mathbb{R}^2 : q = -\theta_c, \dot{q} < 0\}$.

Next, to stabilize the equilibrium point $(q_e, \dot{q}_e) = (0, 0)$, consider the hybrid dynamic compensator

$$\ddot{q}_c(t) + k_c q_c(t) = k_c q(t), \quad q_c(0) = q_{c0}, \ (q(t), \dot{q}(t), q_c(t), \dot{q}_c(t)) \notin \mathcal{Z}_c, \ t \ge 0, \tag{77}$$

$$\begin{bmatrix} \Delta q_c(t) \\ \Delta \dot{q}_c(t) \end{bmatrix} = \begin{bmatrix} q(t) - q_c(t) \\ -\dot{q}_c(t) \end{bmatrix}, \quad (q(t), \dot{q}(t), q_c(t), \dot{q}_c(t)) \in \mathcal{Z}_c, \tag{78}$$

$$u_c(t) = -k_p q + k_c(q_c(t) - q(t)), \tag{79}$$

where $k_p > g$ and $k_c > 0$, with the resetting set (72) taking the form

$$\mathcal{Z}_c = \left\{ (q, \dot{q}, q_c, \dot{q}_c) : k_c(q_c - q)\dot{q} = 0 \text{ and } \begin{bmatrix} q - q_c \\ -\dot{q}_c \end{bmatrix} \ne 0 \right\}. \tag{80}$$

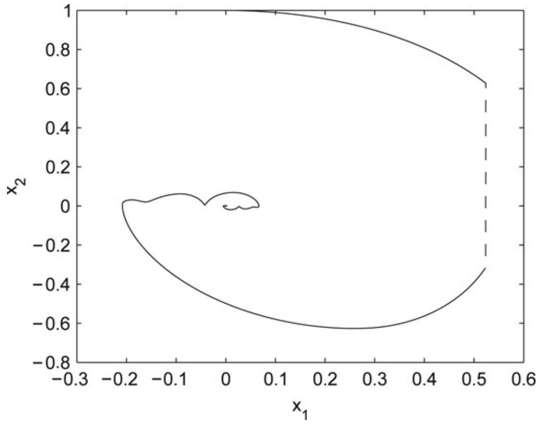

Fig. 5.2. Phase portrait of the constrained inverted pendulum.

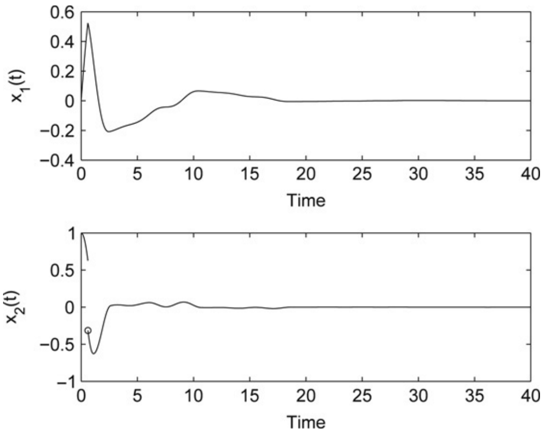

Fig. 5.3. Plant position and velocity versus time.

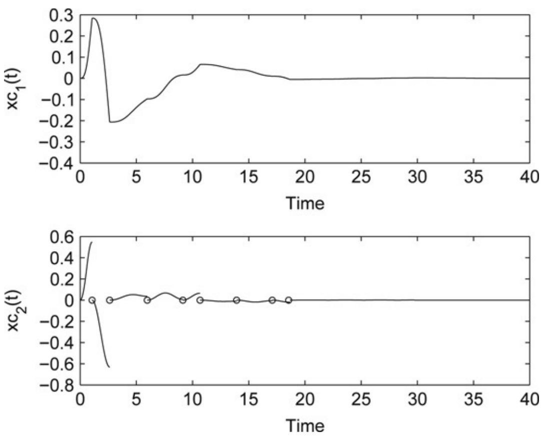

Fig. 5.4. Controller position and velocity versus time.

To illustrate the behavior of the closed-loop impulsive dynamical system, let $\theta_{\mathrm{c}} = \frac{\pi}{6}$, $g = 9.8$, $e = 0.5$, $k_{\mathrm{p}} = 9.9$, and $k_{\mathrm{c}} = 2$ with initial conditions $q(0) = 0$, $\dot{q}(0) = 1$, $q_{\mathrm{c}}(0) = 0$, and $\dot{q}_{\mathrm{c}}(0) = 0$. For this system a straightforward, but lengthy, calculation shows that Assumptions 1 and 2 hold. However, the transversality condition is sufficiently complex that we have been unable to show it analytically. This condition was verified numerically, and hence,

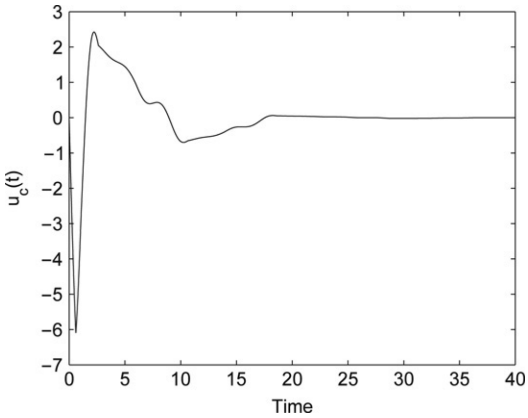

Fig. 5.5. Control signal versus time.

Assumption 3 holds. Fig. 5.2 shows the phase portrait of the closed-loop impulsive dynamical system with $x_1 = q$ and $x_2 = \dot{q}$. Fig. 5.3 shows the controlled plant position and velocity states versus time, while Fig. 5.4 shows the controller position and velocity versus time. Fig. 5.5 shows the control force versus time. Note that for this example the plant velocity and the controller velocity are the only states that reset. Furthermore, in this case, the control force is continuous since the plant position and the controller position are continuous functions of time.

6. Conclusion

In this paper, we developed a general energy-based hybrid control framework for nonsmooth Lagrangian, hybrid port-controlled Hamiltonian, and hybrid lossless dynamical systems. Specifically, an energy dissipating state-dependent hybrid controller is developed and analyzed, and an example is given to illustrate the enhanced ability of those controllers to remove energy from the open-loop system dynamics. In addition, unlike standard energy-based controllers for continuous-time systems, the proposed approach does not achieve stabilization via passivation.

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