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Vector dissipativity theory and stability of feedback interconnections for large-scale non-linear dynamical systems

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Recent technological demands have required the analysis and control design of increasingly complex, large-scale non-linear dynamical systems. In analysing these large-scale systems, it is often desirable to treat the overall system as a collection of interconnected subsystems. Solution properties of the large-scale system are then deduced from the solution properties of the individual subsystems and the nature of the system interconnections. In this paper we develop an analysis framework for large-scale dynamical systems based on *vector dissipativity* notions. Specifically, using vector storage functions and vector supply rates, dissipativity properties of the composite large-scale system are shown to be determined from the dissipativity properties of the subsystems and their interconnections. Furthermore, extended Kalman–Yakubovich–Popov conditions, in terms of the subsystem dynamics and interconnection constraints, characterizing vector dissipativeness via vector system storage functions are derived. Finally, these results are used to develop feedback interconnection stability results for large-scale non-linear dynamical systems using vector Lyapunov functions.

1. Introduction

Modern complex dynamical systems§ are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication network constraints. The sheer size (i.e. dimensionality) and complexity of these large-scale dynamical systems often necessitates a hierarchical decentralized architecture for analysing and controlling these systems. Specifically, in the analysis and control-system design of complex large-scale dynamical systems it is often desirable to treat the overall system as a collection of interconnected subsystems. The behaviour of the aggregate or composite (i.e. large-scale) system can then be predicted from the behaviours of the individual subsystems and their interconnections. The need for decentralized analysis and control design of large-scale systems is a direct consequence of the physical size and complexity of the dynamical model. In particular, computational complexity may be too large for model analysis while severe constraints on communication links between system sensors, actuators, and processors may render centralized control architectures impractical.

An approach to analysing large-scale dynamical systems was introduced by the pioneering work of Šiljak (1978) and involves the notion of *connective stability*. In particular, the large-scale dynamical system is decomposed into a collection of subsystems with local dynamics and uncertain interactions. Then, each subsystem is considered independently so that the stability of each subsystem is combined with the interconnection constraints to obtain a *vector Lyapunov function* for the composite large-scale dynamical system guaranteeing connective stability for the overall system. Vector Lyapunov functions were first introduced by Bellman (1962) and Matrosov (1972) and further developed by Martynyuk (1975), Michel and Miller (1977), Grujić *et al.* (1987), Lunze (1989) and Lakshmikantham *et al.* (1991), with Michel and Miller (1977), Šiljak (1978, 1983), Grujić *et al.* (1987), Lunze (1989), Martynyuk (1998, 2002) exploiting their utility for analysing large-scale systems. Extensions of vector Lyapunov function theory that include matrix-valued Lyapunov functions for stability analysis of large-scale dynamical systems appear in the monographs by Martynyuk (1998, 2002). The use of vector Lyapunov functions in large-scale system analysis offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function. Moreover, in large-scale systems several Lyapunov functions arise naturally from the stability properties of each subsystem. An alternative approach to vector Lyapunov functions for analysing large-scale dynamical systems is an input–output approach wherein stability criteria are derived by assuming that each subsystem is either finite gain, passive or conic (Araki 1976, Lasley and Michel 1976 a,b, Vidyasagar 1981).

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§ Here we have in mind large flexible space structures, aerospace systems, electric power systems, network systems, economic systems, and ecological systems, to cite but a few examples.

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In light of the fact that energy flow modelling arises naturally in large-scale dynamical systems and vector Lyapunov functions provide a powerful stability analysis framework for these systems, it seems natural that dissipativity theory (Willems 1972 a,b), on the subsystem level, should play a key role in unifying these analysis methods. Specifically, dissipativity theory provides a fundamental framework for the analysis and design of control systems using an input–output description based on system energy† related considerations (Willems 1972 a). The dissipation hypothesis on dynamical systems results in a fundamental constraint on their dynamic behaviour wherein a dissipative dynamical system can only deliver a fraction of its energy to its surroundings and can only store a fraction of the work done to it. Such conservation laws are prevalent in large-scale dynamical systems such as aerospace systems, power systems, network systems, structural systems and thermodynamic systems. Since these systems have numerous input–output properties related to conservation, dissipation, and transport of energy, extending dissipativity theory to capture conservation and dissipation notions on the subsystem level would provide a natural analysis framework for large-scale dynamical systems. Aggregating the dissipativity properties of each of the subsystems by appropriate storage functions and supply rates would allow us to study the dissipativity properties of the composite large-scale system using *vector storage functions* and *vector supply rates*. Furthermore, since vector Lyapunov functions can be viewed as generalizations of composite energy functions for all of the subsystems, a generalized notion of dissipativity; namely, *vector dissipativity*, with appropriate vector storage functions and vector supply rates, can be used to construct vector Lyapunov functions for non-linear feedback large-scale systems by appropriately combining vector storage functions for the forward and feedback large-scale systems. Finally, as in classical dynamical system theory, vector dissipativity theory can play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections and optimality for large-scale dynamical systems.

In this paper we develop vector dissipativity notions for large-scale non-linear dynamical systems; a notion not previously considered in the literature. In particular, we introduce a generalized definition of dissipativity for large-scale non-linear dynamical systems in terms of a *vector inequality* involving a vector supply rate, a vector storage function, and a quasimonotone increasing function. On the subsystem level, the proposed approach provides an energy flow balance in terms of the stored subsystem energy, the supplied subsystem energy, the subsystem energy gained from all other subsystems independent of the subsystem coupling strengths, and the subsystem energy dissipated. Furthermore, for large-scale dynamical systems decomposed into interconnected subsystems, dissipativity of the composite system is shown to be determined from the dissipativity properties of the individual subsystems and the nature of the interconnections. In addition, we develop extended Kalman–Yakubovich–Popov conditions, in terms of the local subsystem dynamics and the interconnection constraints, for characterizing vector dissipativeness via vector storage functions for large-scale dynamical systems. Finally, using the concepts of vector dissipativity and vector storage functions as candidate vector Lyapunov functions, we develop feedback interconnection stability results for large-scale non-linear dynamical systems. General stability criteria are given for Lyapunov and asymptotic stability of feedback interconnections of large-scale dynamical systems. In the case of vector quadratic supply rates involving net subsystem powers and input–output subsystem energies, these results provide a positivity and small gain theorem for large-scale systems predicated on vector Lyapunov functions.

2. Mathematical preliminaries

In this section we introduce notation, several definitions, and some key results needed for analysing large-scale non-linear dynamical systems. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, $\mathbb{N}^n$ (respectively, $\mathbb{P}^n$) denote the the set of $n \times n$ non-negative (respectively, positive) definite matrices, $(\cdot)^{\mathrm{T}}$ denote transpose and let I_n or I denote the $n \times n$ identity matrix. For $v \in \mathbb{R}^q$ we write $v \geq\geq 0$ (respectively, $v \gg 0$) to indicate that every component of v is non-negative (respectively, positive). In this case we say that v is *non-negative* or *positive*, respectively. Let $\overline{\mathbb{R}}_+^q$ and $\mathbb{R}_+^q$ denote the non-negative and positive orthants of $\mathbb{R}^q$; that is, if $v \in \mathbb{R}^q$, then $v \in \overline{\mathbb{R}}_+^q$ and $v \in \mathbb{R}_+^q$ are equivalent, respectively, to $v \geq\geq 0$ and $v \gg 0$. Finally, we write $\| \cdot \|$ for the Euclidean vector norm, $V'(x)$ for the Fréchet derivative of V at x and $M \geq 0$ (respectively, $M > 0$) to denote the fact that the Hermitian matrix M is non-negative (respectively, positive) definite. The following definition introduces the notion of Z-, essentially non-negative, compartmental and non-negative matrices.

Definition 1 (Berman and Plemmons 1979, Bernstein and Hyland 1993)**:** Let $W \in \mathbb{R}^{q \times q}$. W is a *Z-matrix* if $W_{(i,j)} \leq 0$, $i, j = 1, \ldots, q$, $i \neq j$. W is *essentially non-negative* if $-W$ is a Z-matrix; that is, $W_{(i,j)} \geq 0$,

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† Here the notion of energy refers to abstract energy for which a physical system energy interpretation is not necessary.

$i, j = 1, \ldots, q$, $i \neq j$. W is *compartmental* if W is essentially non-negative and $\sum_{i=1}^{q} W_{(i,j)} \leq 0$, $j = 1, \ldots, q$. Finally, W is *non-negative*† (respectively, *positive*) if $W_{(i,j)} \geq 0$ (respectively, $W_{(i,j)} > 0$), $i, j = 1, \ldots, q$.

The following definition introduces the notion of class $\mathcal{W}$ functions involving quasimonotone increasing functions.

Definition 2 (Wazewski 1950, Šiljak 1978)**:** A function $w = [w_1, \ldots, w_q]^{\mathrm{T}}\colon \mathbb{R}^q \to \mathbb{R}^q$ is of *class* $\mathcal{W}$ if $w_i(r') \leq w_i(r'')$, $i = 1, \ldots, q$, for all $r', r'' \in \mathbb{R}^q$ such that $r'_j \leq r''_j$, $r'_i = r''_i$, $j = 1, \ldots, q$, $i \neq j$, where r_i denotes the ith component of r.

If $w(\cdot) \in \mathcal{W}$ we say that w satisfies the *Kamke condition*. Note that if $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then the function $w(\cdot)$ is of class $\mathcal{W}$ if and only if W is essentially non-negative. Furthermore, note that it follows from Definition 2 that any scalar ($q = 1$) function $w(r)$ is of class $\mathcal{W}$. The following definition introduces the notion of essentially non-negative functions (Bernstein and Bhat 1999, Haddad *et al.* 2001).

Definition 3: Let $w = [w_1, \ldots, w_q]^{\mathrm{T}}\colon \mathcal{V} \to \mathbb{R}^q$, where $\mathcal{V}$ is an open subset of $\mathbb{R}^q$ that contains $\overline{\mathbb{R}}_+^q$. Then w is *essentially non-negative* if $w_i(r) \geq 0$ for all $i = 1, \ldots, q$ and $r \in \overline{\mathbb{R}}_+^q$ such that $r_i = 0$.

Note that if $w\colon \mathbb{R}^q \to \mathbb{R}^q$ is such that $w(\cdot) \in \mathcal{W}$ and $w(0) \geq\geq 0$, then w is essentially non-negative; the converse however is not generally true. However, if $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$ is essentially non-negative, then $w(\cdot)$ is essentially non-negative and $w(\cdot) \in \mathcal{W}$.

Proposition 1 (Bernstein and Bhat 1999, Haddad *et al.* 2001)**:** *Suppose* $\overline{\mathbb{R}}_+^q \subset \mathcal{V}$. *Then* $\overline{\mathbb{R}}_+^q$ *is an invariant set with respect to*

$$\dot{r}(t) = w(r(t)), \quad r(t_0) = r_0, \quad t \geq t_0, \tag{1}$$

where $r_0 \in \overline{\mathbb{R}}_+^q$, *if and only if* $w\colon \mathcal{V} \to \mathbb{R}^q$ *is essentially non-negative.*

It follows from Proposition 1 that if $r_0 \geq\geq 0$, then $r(t) \geq\geq 0$, $t \geq t_0$, if and only if $w(\cdot)$ is essentially non-negative. In this case, the usual stability definitions for the equilibrium solution $r(t) \equiv r_{\mathrm{e}}$ to (1) are not valid. In particular, stability notions need to be defined with respect to relatively open subsets of $\overline{\mathbb{R}}_+^q$ containing r_{e} (Haddad *et al.* 2001).

Next, we present a stability result for large-scale non-linear dynamical systems using vector Lyapunov functions. In particular, we consider non-linear dynamical systems of the form

$$\dot{x}(t) = F(x(t)), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{2}$$

where $F\colon \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$ and $F(0) = 0$. Here, we assume that (2) characterizes a large-scale non-linear dynamical system composed of q interconnected subsystems such that, for all $i = 1, \ldots, q$, each element of $F(x)$ is given by $F_i(x) = f_i(x_i) + \mathcal{I}_i(x)$, where $f_i\colon \mathcal{D}_i \subseteq \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ defines the vector field of each isolated subsystem of (2), $\mathcal{I}_i\colon \mathcal{D} \to \mathbb{R}^{n_i}$ defines the structure of the interconnection dynamics of the ith subsystem with all other subsystems, $x_i \in \mathcal{D}_i \subseteq \mathbb{R}^{n_i}$, $f_i(0) = 0$, $\mathcal{I}_i(0) = 0$, and $\sum_{i=1}^{q} n_i = n$. For the large-scale non-linear dynamical system (2) we note that the subsystem states $x_i(t)$, $t \geq t_0$, for all $i = 1, \ldots, q$ belong to $\mathcal{D}_i \subseteq \mathbb{R}^{n_i}$ as long as $x(t) \triangleq [x_1^{\mathrm{T}}(t), \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{D}$, $t \geq t_0$. The next theorem presents a stability result for (2) via vector Lyapunov functions by relating the stability properties of a *comparison system* to the stability properties of the large-scale non-linear dynamical system.

Theorem 1 (Šiljak 1978)**:** *Consider the large-scale non-linear dynamical system given by* (2). *Suppose there exist a continuously differentiable vector function* $V\colon \mathcal{D} \to \overline{\mathbb{R}}_+^q$ *and a positive vector* $p \in \mathbb{R}_+^q$ *such that* $V(0) = 0$, *the scalar function* $v\colon \mathcal{D} \to \overline{\mathbb{R}}_+$ *defined by* $v(x) \triangleq p^{\mathrm{T}} V(x)$, $x \in \mathcal{D}$, *is such that* $v(0) = 0$, $v(x) > 0$, $x \neq 0$, *and*

$$V'(x)F(x) \leq\leq w(V(x)), \quad x \in \mathcal{D} \tag{3}$$

where $w\colon \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ *is a class* $\mathcal{W}$ *function such that* $w(0) = 0$. *Then the stability properties of the zero solution* $r(t) \equiv 0$ *to*

$$\dot{r}(t) = w(r(t)), \quad r(t_0) = r_0, \quad t \geq t_0 \tag{4}$$

imply the corresponding stability properties of the zero solution $x(t) \equiv 0$ *to* (2). *That is, if the zero solution* $r(t) \equiv 0$ *to* (4) *is Lyapunov* (*respectively, asymptotically*) *stable, then the zero solution* $x(t) \equiv 0$ *to* (2) *is Lyapunov* (*respectively, asymptotically*) *stable. If, in addition,* $\mathcal{D} = \mathbb{R}^n$ *and* $v(x) \to \infty$ *as* $\|x\| \to \infty$, *then global asymptotic stability of the zero solution* $r(t) \equiv 0$ *to* (4) *implies global asymptotic stability of the zero solution* $x(t) \equiv 0$ *to* (2).

If $V\colon \mathcal{D} \to \overline{\mathbb{R}}_+^q$ satisfies the conditions of Theorem 1 we say that $V(x)$, $x \in \mathcal{D}$, is a *vector Lyapunov function* for the large-scale non-linear dynamical system (2). Finally, we recall the standard notions of dissipativity (Willems 1972 a) and exponential dissipativity (Chellaboina and Haddad 2003) for non-linear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{5}$$

$$y(t) = h(x(t)) + J(x(t))u(t) \tag{6}$$

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† In this paper it is important to distinguish between a square non-negative (respectively, positive) matrix and a non-negative-definite (respectively, positive-definite) matrix.

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^l$, $f\colon \mathcal{D} \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G\colon \mathcal{D} \to \mathbb{R}^{n\times m}$, $h\colon \mathcal{D} \to \mathbb{R}^l$ and satisfies $h(0) = 0$, and $J\colon \mathcal{D} \to \mathbb{R}^{l\times m}$. For the non-linear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $u(\cdot)$ satisfies sufficient regularity conditions such that (5) has a unique solution forward in time. For the non-linear dynamical system $\mathcal{G}$ given by (5) and (6) a function $s\colon \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ such that $s(0,0) = 0$ is called a *supply rate* (Willems 1972 a) if it is locally integrable for all input–output pairs satisfying (5), (6); that is, for all input–output pairs $u \in \mathcal{U}$, $y \in \mathcal{Y}$ satisfying (5), (6), $s(\cdot,\cdot)$ satisfies $\int_{t_1}^{t_2} |s(u(\sigma), y(\sigma))| \mathrm{d}\sigma < \infty$, $t_2 \geq t_1 \geq t_0$.

Definition 4 (Willems 1972 a, Chellaboina and Haddad 2003)**:** The non-linear dynamical system $\mathcal{G}$ given by (5), (6) is *exponentially dissipative* (respectively, *dissipative*) with respect to the supply rate $s(u, y)$ if there exist a continuous non-negative-definite function $v_{\mathrm{s}}\colon \mathbb{R}^n \to \overline{\mathbb{R}}_+$, called a *storage function*, and a scalar $\varepsilon > 0$ (respectively, $\varepsilon = 0$) such that $v_{\mathrm{s}}(0) = 0$ and the *dissipation inequality*

$$\mathrm{e}^{\varepsilon t_2}\, v_{\mathrm{s}}(x(t_2)) \leq \mathrm{e}^{\varepsilon t_1}\, v_{\mathrm{s}}(x(t_1)) + \int_{t_1}^{t_2} \mathrm{e}^{\varepsilon t}\, s(u(t), y(t))\, \mathrm{d}t, \quad t_2 \geq t_1 \tag{7}$$

is satisfied for all $t_1, t_2 \geq t_0$, where $x(t)$, $t \geq t_1$, is the solution of (5) with $u \in \mathcal{U}$. The non-linear dynamical system $\mathcal{G}$ given by (5), (6) is *lossless with respect to the supply rate* $s(u, y)$ if the dissipation inequality is satisfied as an equality with $\varepsilon = 0$ for all $t_2 \geq t_1 \geq t_0$.

If $v_{\mathrm{s}}(\cdot)$ is continuously differentiable, then an equivalent statement for exponential dissipativity (respectively, dissipativity) of the dynamical system (5), (6) is

$$\dot{v}_{\mathrm{s}}(x(t)) + \varepsilon v_{\mathrm{s}}(x(t)) \leq s(u(t), y(t)), \quad t \geq t_0, \quad u \in \mathcal{U}, \quad y \in \mathcal{Y} \tag{8}$$

where $\varepsilon > 0$ (respectively, $\varepsilon = 0$) and $\dot{v}_{\mathrm{s}}(x(t))$ denotes the total derivative of $v_{\mathrm{s}}(x)$ along the state trajectories $x(t)$, $t \geq t_0$, of (5).

3. Vector dissipativity theory for large-scale non-linear dynamical systems

In this section we extend the notion of dissipative dynamical systems to develop the generalized notion of vector dissipativity for large-scale non-linear dynamical systems. We begin by considering non-linear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{9}$$

$$y(t) = H(x(t), u(t)) \tag{10}$$

where $x \in \mathcal{D} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^l$, $F\colon \mathcal{D} \times \mathcal{U} \to \mathbb{R}^n$, $H\colon \mathcal{D} \times \mathcal{U} \to \mathcal{Y}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $F(0,0) = 0$. Here, we assume that $\mathcal{G}$ represents a large-scale dynamical system composed of q interconnected controlled subsystems $\mathcal{G}_i$ so that, for all $i = 1, \ldots, q$

$$F_i(x, u_i) = f_i(x_i) + \mathcal{I}_i(x) + G_i(x_i) u_i \tag{11}$$

$$H_i(x_i, u_i) = h_i(x_i) + J_i(x_i) u_i \tag{12}$$

where $x_i \in \mathcal{D}_i \subseteq \mathbb{R}^{n_i}$, $u_i \in \mathcal{U}_i \subseteq \mathbb{R}^{m_i}$, $y_i \stackrel{\triangle}{=} H_i(x_i, u_i) \in \mathcal{Y}_i \subseteq \mathbb{R}^{l_i}$, (u_i, y_i) is the input–output pair for the ith subsystem, $f_i\colon \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ and $\mathcal{I}_i\colon \mathcal{D} \to \mathbb{R}^{n_i}$ are Lipschitz continuous and satisfy $f_i(0) = 0$ and $\mathcal{I}_i(0) = 0$, $G_i\colon \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$ is continuous, $h_i\colon \mathbb{R}^{n_i} \to \mathbb{R}^{l_i}$ and satisfies $h_i(0) = 0$, $J_i\colon R^{n_i} \to R^{l_i \times m_i}$, $\sum_{i=1}^q n_i = n$, $\sum_{i=1}^q m_i = m$, and $\sum_{i=1}^q l_i = l$. Furthermore, for the system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, for every $i \in \{1, \ldots, q\}$, $u_i(\cdot)$ satisfies sufficient regularity conditions such that the system (9) has a unique solution forward in time. We define the composite input and composite output for the large-scale system $\mathcal{G}$ as $u \stackrel{\triangle}{=} [u_1^{\mathrm{T}}, \ldots, u_q^{\mathrm{T}}]^{\mathrm{T}}$ and $y \stackrel{\triangle}{=} [y_1^{\mathrm{T}}, \ldots, y_q^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Note that in this case the set $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_q$ contains the set of input values and $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_q$ contains the set of output values.

Definition 5: For the large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) a vector function $S = [s_1, \ldots, s_q]^{\mathrm{T}}\colon \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^q$ such that $S(u, y) \stackrel{\triangle}{=} [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^{\mathrm{T}}$ and $S(0,0) = 0$ is called a *vector supply rate* if it is componentwise locally integrable for all input–output pairs satisfying (9), (10); that is, for every $i \in \{1, \ldots, q\}$ and for all input–output pairs $(u_i, y_i) \in \mathcal{U}_i \times \mathcal{Y}_i$ satisfying (9), (10), $s_i(\cdot,\cdot)$ satisfies $\int_{t_1}^{t_2} |s_i(u_i(s), y_i(s))|\, \mathrm{d}s < \infty$, $t_2 \geq t_1 \geq t_0$.

Definition 6: The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is *vector dissipative* (respectively, *exponentially vector dissipative*) *with respect to the vector supply rate* $S(u, y)$ if there exists a continuous, non-negative definite vector function $V_{\mathrm{s}} = [v_{\mathrm{s}1}, \ldots, v_{\mathrm{s}q}]^{\mathrm{T}}\colon \mathcal{D} \to \overline{\mathbb{R}}_+^q$, called a *vector storage function*, and a class $\mathcal{W}$ function $w\colon \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\mathrm{s}}(0) = 0$, $w(0) = 0$, the zero solution $r(t) \equiv 0$ to the comparison system

$$\dot{r}(t) = w(r(t)), \quad r(t_0) = r_0, \quad t \geq t_0 \tag{13}$$

is Lyapunov (respectively, asymptotically) stable, and the *vector dissipation inequality*

$$V_{\mathrm{s}}(x(T)) \leq\leq V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{T} w(V_{\mathrm{s}}(x(t)))\, \mathrm{d}t + \int_{t_0}^{T} S(u(t), y(t))\, \mathrm{d}t \tag{14}$$

is satisfied for all $T \geq t_0$, where $x(t)$, $t \geq t_0$, is the solution to (9) with $u \in \mathcal{U}$. The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is *vector lossless* *with respect to the vector supply rate* $S(u, y)$ if the vector dissipation inequality is satisfied as an equality with the zero solution $r(t) \equiv 0$ to (13) being Lyapunov stable.

Remark 1: If $V_{\rm s}(\cdot)$ is continuously differentiable, then (14) can be equivalently written as

$$\dot{V}_{\rm s}(x(t)) \leq\leq w(V_{\rm s}(x(t))) + S(u(t), y(t)), \quad t \geq t_0, \quad u \in \mathcal{U}. \tag{15}$$

Remark 2: If in Definition 6 the function $w\colon \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ is such that $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$, then W is essentially non-negative. In this case, the large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u, y)$ if and only if there exist a vector storage function $V_{\rm s} = [v_{{\rm s}1}, \ldots, v_{{\rm s}q}]^{\rm T}\colon \mathcal{D} \to \overline{\mathbb{R}}_+^q$ and an essentially non-negative *dissipation matrix* $W \in \mathbb{R}^{q\times q}$ such that $V_{\rm s}(0) = 0$, W is semistable† (respectively, asymptotically stable), and the vector dissipation inequality

$$V_{\rm s}(x(T)) \leq\leq {\rm e}^{W(T-t_0)} V_{\rm s}(x(t_0)) + \int_{t_0}^{T} {\rm e}^{W(T-t)} S(u(t), y(t))\, {\rm d}t, \quad T \geq t_0 \tag{16}$$

is satisfied, where $x(t)$, $t \geq t_0$, is the solution to (9) with $u \in \mathcal{U}$. The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is vector lossless with respect to the vector supply rate $S(u, y)$ if and only if the vector dissipation inequality (16) is satisfied as an equality with W semistable.

Note that if the subsystems $\mathcal{G}_i$ of $\mathcal{G}$ are *disconnected*; that is, $\mathcal{I}_i(x) \equiv 0$ for all $i = 1, \ldots, q$, and $w(V_{\rm s}) = WV_{\rm s}$, where $-W \in \mathbb{R}^{q\times q}$ is diagonal and non-negative definite, then it follows from Remark 2 that each of the disconnected subsystems $\mathcal{G}_i$ is dissipative (respectively, exponentially dissipative) in the sense of Definition 4. A similar remark holds in the case where $q = 1$. To state the main results of this section the following definition is required.

Definition 7 (Willems 1972 a)**:** The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is *completely reachable* if for all $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$, there exist a finite time $t_i < t_0$ and a square integrable input $u(\cdot)$ defined on $[t_i, t_0]$ such that the state $x(t)$, $t \geq t_i$, can be driven from $x(t_i) = 0$ to $x(t_0) = x_0$. A large-scale non-linear dynamical system $\mathcal{G}$ is *zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ imply $x(t) \equiv 0$.

Recall that if a disconnected subsystem $\mathcal{G}_i$ (i.e. $\mathcal{I}_i(x) \equiv 0$, $i \in \{1, \ldots, q\}$) of a large-scale dynamical system $\mathcal{G}$ is exponentially dissipative (respectively, dissipative) with respect to a supply rate $s_i(u_i, y_i)$, then there exist a storage function $v_{{\rm s}i}\colon \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$ and a constant $\varepsilon_i > 0$ (respectively, $\varepsilon_i = 0$) such that the dissipation inequality (7) holds. In the case where $v_{{\rm s}i}\colon \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$ is continuously differentiable, equation (7) yields

$$v'_{{\rm s}i}(x_i)(f_i(x_i) + G_i(x_i)u_i) \leq -\varepsilon_i v_{{\rm s}i}(x_i) + s_i(u_i, y_i), \quad x_i \in \mathbb{R}^{n_i}, \quad u_i \in \mathcal{U}_i. \tag{17}$$

The next result relates exponential dissipativity with respect to a scalar supply rate of each disconnected subsystem $\mathcal{G}_i$ of $\mathcal{G}$ with vector dissipativity (or, possibly, exponential vector dissipativity) of $\mathcal{G}$ with respect to a vector supply rate.

Proposition 2: *Consider the large-scale non-linear dynamical system* $\mathcal{G}$ *given by* (9), (10). *Assume that* $\mathcal{G}$ *is completely reachable and each disconnected subsystem* $\mathcal{G}_i$ *of* $\mathcal{G}$ *is exponentially dissipative with respect to the supply rate* $s_i(u_i, y_i)$ *and with a continuously differentiable storage function* $v_{{\rm s}i}\colon \mathbb{R}^{n_i} \to \overline{\mathbb{R}}_+$, $i = 1, \ldots, q$. *Furthermore, assume that the interconnection functions* $\mathcal{I}_i\colon \mathcal{D} \to \mathbb{R}^{n_i}$, $i = 1, \ldots, q$, *of* $\mathcal{G}$ *are such that*

$$v'_{{\rm s}i}(x_i)\mathcal{I}_i(x) \leq \sum_{j=1}^{q} \xi_{ij}(x) v_{{\rm s}j}(x_j), \quad x \in \mathcal{D}, \quad i = 1, \ldots, q \tag{18}$$

where $\xi_{ij}\colon \mathcal{D} \to \mathbb{R}$, $i, j = 1, \ldots, q$, *are given bounded functions. If* $W \in \mathbb{R}^{q\times q}$ *is semistable* (*respectively, asymptotically stable*), *with*

$$W_{(i,j)} = \begin{cases} -\varepsilon_i + \alpha_{ii}, & i = j \\ \alpha_{ij}, & i \neq j \end{cases} \tag{19}$$

where $\varepsilon_i > 0$ *and* $\alpha_{ij} \triangleq \max\{0, \sup_{x\in\mathcal{D}} \xi_{ij}(x)\}$, $i, j = 1, \ldots, q$, *then* $\mathcal{G}$ *is vector dissipative* (*respectively, exponentially vector dissipative*) *with respect to the vector supply rate* $S(u, y) \triangleq [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^{\rm T}$ *and with vector storage function* $V_{\rm s}(x) \triangleq [v_{{\rm s}1}(x_1), \ldots, v_{{\rm s}q}(x_q)]^{\rm T}$, $x \in \mathcal{D}$.

Proof: Since each disconnected subsystem $\mathcal{G}_i$ of $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $s_i(u_i, y_i)$, $i = 1, \ldots, q$, it follows from (17) and (18) that, for all $u_i \in \mathcal{U}_i$ and $i = 1, \ldots, q$

$$\begin{aligned} \dot{v}_{{\rm s}i}(x_i(t)) &= v'_{{\rm s}i}(x_i(t))[f_i(x_i(t)) + \mathcal{I}_i(x(t)) + G_i(x_i(t))u_i(t)] \\ &\leq -\varepsilon_i v_{{\rm s}i}(x_i(t)) + s_i(u_i(t), y_i(t)) \\ &\quad + \sum_{j=1}^{q} \xi_{ij}(x(t)) v_{{\rm s}j}(x_j(t)) \\ &\leq -\varepsilon_i v_{{\rm s}i}(x_i(t)) + s_i(u_i(t), y_i(t)) \\ &\quad + \sum_{j=1}^{q} \alpha_{ij} v_{{\rm s}j}(x_j(t)), \quad t \geq t_0. \end{aligned} \tag{20}$$

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† A matrix $W \in \mathbb{R}^{q\times q}$ is *semistable* if and only if $\lim_{t\to\infty} {\rm e}^{Wt}$ exists while W is *asymptotically stable* if and only if $\lim_{t\to\infty} {\rm e}^{Wt} = 0$.

Now, the result follows from Remark 1 by noting that for all subsystems $\mathcal{G}_i$ of $\mathcal{G}$

$$\dot{V}_{\rm s}(x(t)) \leq\leq WV_{\rm s}(x(t)) + S(u(t), y(t)), \quad t \geq t_0, \quad u \in \mathcal{U} \tag{21}$$

where W is essentially non-negative and, by assumption, semistable (respectively, asymptotically stable) and $V_{\rm s}(x) \triangleq [v_{{\rm s}1}(x_1), \ldots, v_{{\rm s}q}(x_q)]^{\rm T}$, $x \in \mathcal{D}$, is a vector storage function for $\mathcal{G}$. □

Next, we show that vector dissipativeness (respectively, exponential vector dissipativeness) of a large-scale non-linear dynamical system $\mathcal{G}$ of the form (9), (10) can be characterized in terms of the local subsystem functions $f_i(\cdot)$, $G_i(\cdot)$, $h_i(\cdot)$, and $J_i(\cdot)$, along with the interconnection structures $\mathcal{I}_i(\cdot)$ for $i = 1, \ldots, q$. For these results we consider the special case of dissipative systems with quadratic vector supply rates and set $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_i = \mathbb{R}^{m_i}$, and $\mathcal{Y}_i = \mathbb{R}^{l_i}$. Specifically, let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$ and $Q_i \in \mathbb{S}^{l_i}$ be given and assume $S(u, y)$ is such that $s_i(u_i, y_i) = y_i^{\rm T} Q_i y_i + 2y_i^{\rm T} S_i u_i + u_i^{\rm T} R_i u_i$, $i = 1, \ldots, q$. Furthermore, for the remainder of this paper we assume that there exists a continuously differentiable vector storage function $V_{\rm s}(x)$, $x \in \mathbb{R}^n$, for the large-scale non-linear dynamical system $\mathcal{G}$. For the statement of the next result recall that $x = [x_1^{\rm T}, \ldots, x_q^{\rm T}]^{\rm T}$, $u = [u_1^{\rm T}, \ldots, u_q^{\rm T}]^{\rm T}$, $y = [y_1^{\rm T}, \ldots, y_q^{\rm T}]^{\rm T}$, $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{l_i}$, $i = 1, \ldots, q$, $\sum_{i=1}^q n_i = n$, $\sum_{i=1}^q m_i = m$, and $\sum_{i=1}^q l_i = l$. Furthermore, for (9), (10) define $\mathcal{F}$: $\mathbb{R}^n \to \mathbb{R}^n$, G: $\mathbb{R}^n \to \mathbb{R}^{n \times m}$, h: $\mathbb{R}^n \to \mathbb{R}^l$, and J: $\mathbb{R}^n \to \mathbb{R}^{l \times m}$ by $\mathcal{F}(x) \triangleq [\mathcal{F}_1^{\rm T}(x), \ldots, \mathcal{F}_q^{\rm T}(x)]^{\rm T}$, where $\mathcal{F}_i(x) \triangleq f_i(x_i) + \mathcal{I}_i(x)$, $i = 1, \ldots, q$, $G(x) \triangleq {\rm diag}[G_1(x_1), \ldots, G_q(x_q)]$, $h(x) \triangleq [h_1^{\rm T}(x_1), \ldots, h_q^{\rm T}(x_q)]^{\rm T}$, and $J(x) \triangleq {\rm diag}[J_1(x_1), \ldots, J_q(x_q)]$. Finally, for all $i = 1, \ldots, q$, define $\hat{R}_i \in \mathbb{S}^m$, $\hat{S}_i \in \mathbb{R}^{l \times m}$, and $\hat{Q}_i \in \mathbb{S}^l$ such that each of these block matrices consists of zero blocks except, respectively, for the matrix blocks $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$, and $Q_i \in \mathbb{S}^{l_i}$ on (i, i) position.

Theorem 2: *Consider the large-scale non-linear dynamical system $\mathcal{G}$ given by* (9), (10). *Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$ and $Q_i \in \mathbb{S}^{l_i}$, $i = 1, \ldots, q$. Then $\mathcal{G}$ is vector dissipative* (*respectively*, *exponentially vector dissipative*) *with respect to the vector quadratic supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\rm T} R_i u_i + 2y_i^{\rm T} S_i u_i + y_i^{\rm T} Q_i y_i$, $i = 1, \ldots, q$, if and only if there exist functions $V_{\rm s} = [v_{{\rm s}1}, \ldots, v_{{\rm s}q}]^{\rm T}$: $\mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $w = [w_1, \ldots, w_q]^{\rm T}$: $\overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, ℓ_i: $\mathbb{R}^n \to \mathbb{R}^{s_i}$, and $\mathcal{Z}_i$: $\mathbb{R}^n \to \mathbb{R}^{s_i \times m}$, such that $v_{{\rm s}i}(\cdot)$ is continuously differentiable*, *$v_{{\rm s}i}(0) = 0$, $i = 1, \ldots, q$, $w \in \mathcal{W}$, $w(0) = 0$, the zero solution $r(t) \equiv 0$ to* (13) *is Lyapunov* (*respectively*, *asymptotically*) *stable*, *and*, *for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, q$*,

$$0 = v_{{\rm s}i}'(x)\mathcal{F}(x) - h^{\rm T}(x)\hat{Q}_i h(x) - w_i(V_{\rm s}(x)) + \ell_i^{\rm T}(x)\ell_i(x) \tag{22}$$

$$0 = \tfrac{1}{2} v_{{\rm s}i}'(x)G(x) - h^{\rm T}(x)(\hat{S}_i + \hat{Q}_i J(x)) + \ell_i^{\rm T}(x)\mathcal{Z}_i(x) \tag{23}$$

$$0 = \hat{R}_i + J^{\rm T}(x)\hat{S}_i + \hat{S}_i^{\rm T} J(x) + J^{\rm T}(x)\hat{Q}_i J(x) - \mathcal{Z}_i^{\rm T}(x)\mathcal{Z}_i(x). \tag{24}$$

Proof: First, suppose that there exist functions $v_{{\rm s}i}$: $\mathbb{R}^n \to \overline{\mathbb{R}}_+$, ℓ_i: $\mathbb{R}^n \to \mathbb{R}^{s_i}$, $\mathcal{Z}_i$: $\mathbb{R}^n \to \mathbb{R}^{s_i \times m}$, w: $\overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, such that $v_{{\rm s}i}(\cdot)$ is continuously differentiable and non-negative-definite, $v_{{\rm s}i}(0) = 0$, $i = 1, \ldots, q$, $w(0) = 0$, $w \in \mathcal{W}$, the zero solution $r(t) \equiv 0$ to (13) is Lyapunov (respectively, asymptotically) stable, and (22)–(24) are satisfied. Then for any $u \in \mathcal{U}$, t_1, $t_2 \in \mathbb{R}$, $t_2 \geq t_1 \geq t_0$, and $i = 1, \ldots, q$, it follows from (22)–(24) that

$$\begin{aligned}
&\int_{t_1}^{t_2} s_i(u_i(t), y_i(t))\,{\rm d}t \\
&\quad = \int_{t_1}^{t_2} [u^{\rm T}(t)\hat{R}_i u(t) + 2y^{\rm T}(t)\hat{S}_i u(t) + y^{\rm T}(t)\hat{Q}_i y(t)]\,{\rm d}t \\
&\quad = \int_{t_1}^{t_2} [h^{\rm T}(x(t))\hat{Q}_i h(x(t)) + 2h^{\rm T}(x(t))(\hat{S}_i + \hat{Q}_i J(x(t)))u(t) \\
&\qquad\qquad + u^{\rm T}(t)(J^{\rm T}(x(t))\hat{Q}_i J(x(t)) + J^{\rm T}(x(t))\hat{S}_i \\
&\qquad\qquad + \hat{S}_i^{\rm T} J(x(t)) + \hat{R}_i)u(t)]\,{\rm d}t \\
&\quad = \int_{t_1}^{t_2} [v_{{\rm s}i}'(x(t))(\mathcal{F}(x(t)) + G(x(t))u(t)) + \ell_i^{\rm T}(x(t))\ell_i(x(t)) \\
&\qquad\qquad + 2\ell_i^{\rm T}(x(t))\mathcal{Z}_i(x(t))u(t) \\
&\qquad\qquad + u^{\rm T}(t)\mathcal{Z}_i^{\rm T}(x(t))\mathcal{Z}_i(x(t))u(t) - w_i(V_{\rm s}(x(t)))]\,{\rm d}t \\
&\quad = \int_{t_1}^{t_2} [\dot{v}_{{\rm s}i}(x(t)) + [\ell_i(x(t)) + \mathcal{Z}_i(x(t))u(t)]^{\rm T} \\
&\qquad\qquad \times [\ell_i(x(t)) + \mathcal{Z}_i(x(t))u(t)] - w_i(V_{\rm s}(x(t)))]\,{\rm d}t \\
&\quad \geq v_{{\rm s}i}(x(t_2)) - v_{{\rm s}i}(x(t_1)) - \int_{t_1}^{t_2} w_i(V_{\rm s}(x(t)))\,{\rm d}t
\end{aligned} \tag{25}$$

where $x(t)$, $t \geq t_0$, satisfies (9). Now, the result follows from (25) with vector storage function $V_{\rm s}(x) = [v_{{\rm s}1}(x), \ldots, v_{{\rm s}q}(x)]^{\rm T}$, $x \in \mathbb{R}^n$.

Conversely, suppose that $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\rm T} R_i u_i + 2y_i^{\rm T} S_i u_i + y_i^{\rm T} Q_i y_i$, $i = 1, \ldots, q$. Then there exist a vector storage function $V_{\rm s} = [v_{{\rm s}1}, \ldots, v_{{\rm s}q}]^{\rm T}$: $\mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ and a class $\mathcal{W}$ function $w = [w_1, \ldots, w_q]^{\rm T}$: $\overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $V_{\rm s}(0) = 0$, $w(0) = 0$, the zero solution $r(t) \equiv 0$ to (13) is Lyapunov (respectively, asymptotically) stable, and, for all $i = 1, \ldots, q$, $t_2 \geq t_1$, and $u \in \mathcal{U}$,

$$\begin{aligned}
v_{{\rm s}i}(x(t_2)) \leq v_{{\rm s}i}(x(t_1)) &+ \int_{t_1}^{t_2} s_i(u_i(t), y_i(t))\,{\rm d}t \\
&+ \int_{t_1}^{t_2} w_i(V_{\rm s}(x(t)))\,{\rm d}t.
\end{aligned} \tag{26}$$

Since, by assumption, $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, equation (26) is equivalent to

$$\dot{v}_{\mathrm{s}i}(x(t)) \le s_i(u_i(t), y_i(t)) + w_i(V_{\mathrm{s}}(x(t))), \quad t \ge t_0 \tag{27}$$

where $x(t)$, $t \ge t_0$, satisfies (9). Now, with $t = t_0$ it follows from (27) that

$$v'_{\mathrm{s}i}(x_0)(\mathcal{F}(x_0) + G(x_0)u(t_0)) \le s_i(u_i(t_0), y_i(t_0)) + w_i(V_{\mathrm{s}}(x_0)) \tag{28}$$

for all $u(t_0) \in \mathbb{R}^m$, $y(t_0) \in \mathbb{R}^l$, $x_0 \in \mathbb{R}^n$ and $i = 1, \dots, q$. Next, let $d_i\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be such that for all $i = 1, \dots, q$

$$\begin{aligned} d_i(x,u) \triangleq{}& -v'_{\mathrm{s}i}(x)(\mathcal{F}(x) + G(x)u) \\ &+ s_i(u_i, h_i(x_i) + J_i(x_i)u_i) + w_i(V_{\mathrm{s}}(x)). \end{aligned} \tag{29}$$

Now, since $x_0 \in \mathcal{D}$ is arbitrary, it follows from (28) that $d_i(x,u) \ge 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $i = 1, \dots, q$. Furthermore, note that $d_i(x,u)$ given by (29) is quadratic in u and hence there exist functions $\ell_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i}$ and $\mathcal{Z}_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i \times m}$ such that, for all $i = 1, \dots, q$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$

$$\begin{aligned} d_i(x,u) &= [\ell_i(x) + \mathcal{Z}_i(x)u]^{\mathrm{T}}[\ell_i(x) + \mathcal{Z}_i(x)u] \\ &= -v'_{\mathrm{s}i}(x)(\mathcal{F}(x) + G(x)u) + s_i(u_i, h_i(x_i) + J_i(x_i)u_i) \\ &\quad + w_i(V_{\mathrm{s}}(x)) \\ &= -v'_{\mathrm{s}i}(x)(\mathcal{F}(x) + G(x)u) + u^{\mathrm{T}}\hat{R}_i u + 2(h(x) \\ &\quad + J(x)u)^{\mathrm{T}}\hat{S}_i u + (h(x) + J(x)u)^{\mathrm{T}}\hat{Q}_i(h(x) \\ &\quad + J(x)u) + w_i(V_{\mathrm{s}}(x)). \end{aligned} \tag{30}$$

Now, equating coefficients of equal powers yields (22)–(24). □

Using (22)–(24) it follows that for $T \ge t_0 \ge 0$ and $i = 1, \dots, q$

$$\begin{aligned} &\int_{t_0}^{T} s_i(u_i(t), y_i(t))\,\mathrm{d}t + \int_{t_0}^{T} w_i(V_{\mathrm{s}}(x(t)))\,\mathrm{d}t \\ &\quad = v_{\mathrm{s}i}(x(T)) - v_{\mathrm{s}i}(x(t_0)) \\ &\qquad + \int_{t_0}^{T} [\ell_i(x(t)) + \mathcal{Z}_i(x(t))u(t)]^{\mathrm{T}} \\ &\qquad\quad \times [\ell_i(x(t)) + \mathcal{Z}_i(x(t))u(t)]\,\mathrm{d}t \end{aligned} \tag{31}$$

where $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x), \dots, v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, which can be interpreted as a *generalized energy* balance equation for the ith subsystem of $\mathcal{G}$ where $v_{\mathrm{s}i}(x(T)) - v_{\mathrm{s}i}(x(t_0))$ is the stored or accumulated generalized energy of the ith subsystem, the two path-dependent terms on the left are, respectively, the external supplied energy to the ith subsystem and the energy gained by the ith subsystem from the net energy flow between all subsystems due to subsystem coupling, and the second path-dependent term on the right corresponds to the dissipated energy from the ith subsystem. Equivalently, equation (31) can be rewritten for all $i = 1, \dots, q$ as

$$\begin{aligned} \dot{v}_{\mathrm{s}i}(x(t)) &= s_i(u_i(t), y_i(t)) + w_i(V_{\mathrm{s}}(x(t))) - [\ell_i(x(t)) \\ &\quad + \mathcal{Z}_i(x(t))u(t)]^{\mathrm{T}}[\ell_i(x(t)) + \mathcal{Z}_i(x(t))u(t)], \quad t \ge t_0 \end{aligned} \tag{32}$$

which yields a set of q generalized energy conservation equations for the large-scale dynamical system $\mathcal{G}$. Specifically, equation (32) shows that the rate of change in generalized energy, or generalized power, of the ith subsystem of $\mathcal{G}$ is equal to the generalized system power input to the ith subsystem plus the instantaneous rate of energy supplied to the ith subsystem from the net energy flow between all subsystems minus the internal generalized system power dissipated from the ith subsystem.

Remark 3: Note that if $\mathcal{G}$ with $u(t) \equiv 0$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector quadratic supply rate where $Q_i \le 0$, $i = 1, \dots, q$, then it follows from the vector dissipation inequality that

$$\begin{aligned} \dot{V}_{\mathrm{s}}(x(t)) &\le\le w(V_{\mathrm{s}}(x(t))) + S(0, y(t)) \\ &\le\le w(V_{\mathrm{s}}(x(t))), \quad t \ge t_0 \end{aligned} \tag{33}$$

where $S(0,y) = [s_1(0,y_1), \dots, s_q(0,y_q)]^{\mathrm{T}}$, $s_i(0, y_i(t)) = y_i^{\mathrm{T}}(t)Q_i y_i(t) \le 0$, $t \ge t_0$, $i = 1, \dots, q$, and $x(t)$, $t \ge t_0$, is the solution to (9) with $u(t) \equiv 0$. If, in addition, there exists $p \in \mathbb{R}_+^q$ such that $p^{\mathrm{T}}V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is positive definite, then it follows from Theorem 1 that the undisturbed ($u(t) \equiv 0$) large-scale non-linear dynamical system (9) is Lyapunov (respectively, asymptotically) stable.

Next, we extend the notions of passivity and non-expansivity to vector passivity and vector non-expansivity.

Definition 8: The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) with $m_i = l_i$, $i = 1, \dots, q$, is *vector passive* (respectively, *vector exponentially passive*) if it is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u,y)$, where $s_i(u_i, y_i) = 2y_i^{\mathrm{T}}u_i$, $i = 1, \dots, q$.

Definition 9: The large-scale non-linear dynamical system $\mathcal{G}$ given by (9), (10) is *vector non-expansive* (respectively, *vector exponentially non-expansive*) if it is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = \gamma_i^2 u_i^{\mathrm{T}} u_i - y_i^{\mathrm{T}} y_i$, $i = 1, \ldots, q$, and $\gamma_i > 0$, $i = 1, \ldots, q$, are given.

Remark 4: Note that a mixed vector passive-non-expansive formulation of $\mathcal{G}$ can also be considered. Specifically, one can consider large-scale non-linear dynamical systems $\mathcal{G}$ which are vector dissipative with respect to vector supply rates $S(u, y)$, where $s_i(u_i, y_i) = 2y_i^{\mathrm{T}} u_i$, $i \in \mathcal{N}_{\mathrm{p}}$, $s_j(u_j, y_j) = \gamma_j^2 u_j^{\mathrm{T}} u_j - y_j^{\mathrm{T}} y_j$, $\gamma_j > 0$, $j \in \mathcal{N}_{\mathrm{ne}}$, $\mathcal{N}_{\mathrm{p}} \cap \mathcal{N}_{\mathrm{ne}} = \emptyset$, and $\mathcal{N}_{\mathrm{p}} \cup \mathcal{N}_{\mathrm{ne}} = \{1, \ldots, q\}$. Furthermore, supply rates for vector input strict passivity, vector output strict passivity, and vector input–output strict passivity generalizing the passivity notions given in Hill and Moylan (1977) can also be considered. However, for simplicity of exposition we do not do so here.

The next result presents constructive sufficient conditions guaranteeing vector dissipativity of $\mathcal{G}$ with respect to a vector quadratic supply rate for the case where the vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, is component decoupled; that is, $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$.

Theorem 3: *Consider the large-scale non-linear dynamical system $\mathcal{G}$ given by* (9), (10). *Assume that there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, \ldots, v_{\mathrm{s}q}]^{\mathrm{T}}\colon \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $w = [w_1, \ldots, w_q]^{\mathrm{T}}\colon \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$, $\ell_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i}$, $\mathcal{Z}_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i \times m_i}$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, $v_{\mathrm{s}i}(0) = 0$, $i = 1, \ldots, q$, $w \in \mathcal{W}$, $w(0) = 0$, the zero solution $r(t) \equiv 0$ to* (13) *is Lyapunov* (*respectively*, *asymptotically*) *stable*, *and*, *for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, q$*

$$
\begin{aligned}
0 &\geq v'_{\mathrm{s}i}(x_i)[f_i(x_i) + \mathcal{I}_i(x)] - h_i^{\mathrm{T}}(x_i) Q_i h_i(x_i) \\
&\quad - w_i(V_{\mathrm{s}}(x)) + \ell_i^{\mathrm{T}}(x_i)\ell_i(x_i) \qquad (34)
\end{aligned}
$$

$$
0 = \tfrac{1}{2} v'_{\mathrm{s}i}(x_i) G_i(x_i) - h_i^{\mathrm{T}}(x_i)(S_i + Q_i J_i(x_i)) + \ell_i^{\mathrm{T}}(x_i)\mathcal{Z}_i(x_i) \qquad (35)
$$

$$
\begin{aligned}
0 &\leq R_i + J_i^{\mathrm{T}}(x_i) S_i + S_i^{\mathrm{T}} J_i(x_i) + J_i^{\mathrm{T}}(x_i) Q_i J_i(x_i) \\
&\quad - \mathcal{Z}_i^{\mathrm{T}}(x_i)\mathcal{Z}_i(x_i). \qquad (36)
\end{aligned}
$$

Then $\mathcal{G}$ is vector dissipative (*respectively*, *exponentially vector dissipative*) *with respect to the vector supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, \ldots, q$.*

Proof: For any admissible input $u = [u_1^{\mathrm{T}}, \ldots, u_q^{\mathrm{T}}]^{\mathrm{T}}$ such that $u_i \in \mathbb{R}^{m_i}$, $t_1, t_2 \in \mathbb{R}$, $t_2 \geq t_1 \geq t_0$, and $i = 1, \ldots, q$, it follows from (34)–(36) that

$$
\begin{aligned}
&\int_{t_1}^{t_2} s_i(u_i(t), y_i(t))\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} [u_i^{\mathrm{T}}(t) R_i u_i(t) + 2y_i^{\mathrm{T}}(t) S_i u_i(t) + y_i^{\mathrm{T}}(t) Q_i y_i(t)]\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} [h_i^{\mathrm{T}}(x_i(t)) Q_i h_i(x_i(t)) \\
&\qquad + 2h_i^{\mathrm{T}}(x_i(t))(S_i + Q_i J_i(x_i(t))) u_i(t) \\
&\qquad + u_i^{\mathrm{T}}(t)(J_i^{\mathrm{T}}(x_i(t)) Q_i J_i(x_i(t)) + J_i^{\mathrm{T}}(x_i(t)) S_i \\
&\qquad + S_i^{\mathrm{T}} J_i(x_i(t)) + R_i) u_i(t)]\,\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} [v'_{\mathrm{s}i}(x_i(t))[f_i(x_i(t)) + \mathcal{I}_i(x(t)) + G_i(x_i(t)) u_i(t)] \\
&\qquad + \ell_i^{\mathrm{T}}(x_i(t))\ell_i(x_i(t)) + 2\ell_i^{\mathrm{T}}(x_i(t))\mathcal{Z}_i(x_i(t)) u_i(t) \\
&\qquad + u_i^{\mathrm{T}}(t)\mathcal{Z}_i^{\mathrm{T}}(x_i(t))\mathcal{Z}_i(x_i(t)) u_i(t) - w_i(V_{\mathrm{s}}(x(t)))]\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} [\dot{v}_{\mathrm{s}i}(x_i(t)) + [\ell_i(x_i(t)) + \mathcal{Z}_i(x_i(t)) u_i(t)]^{\mathrm{T}} \\
&\qquad \times [\ell_i(x_i(t)) + \mathcal{Z}_i(x_i(t)) u_i(t)] - w_i(V_{\mathrm{s}}(x(t)))]\,\mathrm{d}t \\
&\geq v_{\mathrm{s}i}(x_i(t_2)) - v_{\mathrm{s}i}(x_i(t_1)) - \int_{t_1}^{t_2} w_i(V_{\mathrm{s}}(x(t)))\,\mathrm{d}t \qquad (37)
\end{aligned}
$$

where $x(t)$, $t \geq t_0$, satisfies (9). Now, the result follows from (37) with vector storage function $V_{\mathrm{s}}(x) = [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$. ☐

Finally, we provide necessary and sufficient conditions for the case where the large-scale non-linear dynamical system $\mathcal{G}$ is vector lossless with respect to a vector quadratic supply rate.

Theorem 4: *Consider the large-scale non-linear dynamical system $\mathcal{G}$ given by* (9), (10). *Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i \times m_i}$ and $Q_i \in \mathbb{S}^{l_i}$, $i = 1, \ldots, q$. Then $\mathcal{G}$ is vector lossless with respect to the vector quadratic supply rate $S(u, y)$, where $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, \ldots, q$, if and only if there exist functions $V_{\mathrm{s}} = [v_{\mathrm{s}1}, \ldots, v_{\mathrm{s}q}]^{\mathrm{T}}\colon \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$ and $w = [w_1, \ldots, w_q]^{\mathrm{T}}\colon \overline{\mathbb{R}}_+^q \to \mathbb{R}^q$ such that $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable, $v_{\mathrm{s}i}(0) = 0$, $i = 1, \ldots, q$, $w \in \mathcal{W}$, $w(0) = 0$, the zero solution $r(t) \equiv 0$ to* (13) *is Lyapunov stable*, *and*, *for all $x \in \mathbb{R}^n$ and $i = 1, \ldots, q$*

$$
0 = v'_{\mathrm{s}i}(x)\mathcal{F}(x) - h^{\mathrm{T}}(x)\hat{Q}_i h(x) - w_i(V_{\mathrm{s}}(x)) \qquad (38)
$$

$$
0 = \tfrac{1}{2} v'_{\mathrm{s}i}(x) G(x) - h^{\mathrm{T}}(x)(\hat{S}_i + \hat{Q}_i J(x)) \qquad (39)
$$

$$
0 = \hat{R}_i + J^{\mathrm{T}}(x)\hat{S}_i + \hat{S}_i^{\mathrm{T}} J(x) + J^{\mathrm{T}}(x)\hat{Q}_i J(x). \qquad (40)
$$

Proof: The proof is analogous to the proof of Theorem 2. ☐

4. Specialization to large-scale linear dynamical systems

In this section we specialize the results of § 3 to the case of large-scale linear dynamical systems. Specifically, we assume that $w \in \mathcal{W}$ is linear so that $w(r) = Wr$, where $W \in \mathbb{R}^{q\times q}$ is essentially non-negative, and consider the large-scale linear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \quad t \geq t_0 \tag{41}$$

$$y(t) = Cx(t) + Du(t) \tag{42}$$

where $A \in \mathbb{R}^{n\times n}$ and A is partitioned as $A \triangleq [A_{ij}]$, $i, j = 1,\ldots,q$, $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $\sum_{i=1}^{q} n_i = n$, $B = \text{block}-\text{diag}[B_1,\ldots,B_q]$, $C = \text{block}-\text{diag}[C_1,\ldots,C_q]$, $D = \text{block}-\text{diag}[D_1,\ldots,D_q]$, $B_i \in \mathbb{R}^{n_i\times m_i}$, $C_i \in \mathbb{R}^{l_i\times n_i}$, $D_i \in \mathbb{R}^{l_i\times m_i}$ and $i = 1,\ldots,q$.

Theorem 5: *Consider the large-scale linear dynamical system $\mathcal{G}$ given by* (41), (42). *Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, $Q_i \in \mathbb{S}^{l_i}$, $i = 1,\ldots,q$. Then $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u,y)$, where $s_i(u_i,y_i) = u_i^{\mathrm{T}}R_iu_i + 2y_i^{\mathrm{T}}S_iu_i + y_i^{\mathrm{T}}Q_iy_i$, $i = 1,\ldots,q$, if and only if there exist $W \in \mathbb{R}^{q\times q}$, $P_i \in \mathbb{N}^n$, $L_i \in \mathbb{R}^{s_i\times n}$ and $Z_i \in \mathbb{R}^{s_i\times m}$, $i = 1,\ldots,q$, such that W is essentially non-negative and semistable* (*respectively, asymptotically stable*), *and, for all $i = 1,\ldots,q$*

$$0 = A^{\mathrm{T}}P_i + P_iA - C^{\mathrm{T}}\hat{Q}_iC - \sum_{j=1}^{q} W_{(i,j)}P_j + L_i^{\mathrm{T}}L_i \tag{43}$$

$$0 = P_iB - C^{\mathrm{T}}(\hat{S}_i + \hat{Q}_iD) + L_i^{\mathrm{T}}Z_i \tag{44}$$

$$0 = \hat{R}_i + D^{\mathrm{T}}\hat{S}_i + \hat{S}_i^{\mathrm{T}}D + D^{\mathrm{T}}\hat{Q}_iD - Z_i^{\mathrm{T}}Z_i. \tag{45}$$

Proof: Sufficiency follows from Theorem 2 with $\mathcal{F}(x) = Ax$, $G(x) = B$, $h(x) = Cx$, $J(x) = D$, $w(r) = Wr$, $\ell_i(x) = L_ix$, $\mathcal{Z}_i(x) = Z_i$ and $v_{\mathrm{s}i}(x) = x^{\mathrm{T}}P_ix$, $i = 1,\ldots,q$. To show necessity, suppose $\mathcal{G}$ is vector dissipative with respect to the vector supply rate $S(u,y)$, where $s_i(u_i,y_i) = u_i^{\mathrm{T}}R_iu_i + 2y_i^{\mathrm{T}}S_iu_i + y_i^{\mathrm{T}}Q_iy_i$, $i = 1,\ldots,q$. Then it follows from Theorem 2, with $w(r) = Wr$, that there exist $V_{\mathrm{s}}\colon \mathbb{R}^n \to \overline{\mathbb{R}}_+^q$, $\ell_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i}$, and $\mathcal{Z}_i\colon \mathbb{R}^n \to \mathbb{R}^{s_i\times m}$, such that W is essentially non-negative and semistable (respectively, asymptotically stable), $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x),\ldots,v_{\mathrm{s}q}(x)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$, $V_{\mathrm{s}}(0) = 0$, and (22)–(24) hold for all $i = 1,\ldots,q$ with $\mathcal{F}(x) = Ax$, $G(x) = B$, $h(x) = Cx$, $J(x) = D$ and $w(r) = Wr$. Since $v_{\mathrm{s}i}(\cdot)$ is non-negative-definite and $v_{\mathrm{s}i}(0) = 0$, $i = 1,\ldots,q$, it follows that there exists $P_i \in \mathbb{N}^n$, $i = 1,\ldots,q$, such that

$$v_{\mathrm{s}i}(x) = x^{\mathrm{T}}P_ix + v_{\mathrm{sr}i}(x), \quad x \in \mathbb{R}^n, \quad i = 1,\ldots,q \tag{46}$$

where $v_{\mathrm{sr}i}\colon \mathbb{R}^n \to \mathbb{R}$ contains the higher-order terms of $v_{\mathrm{s}i}(x)$. Next, note that it follows from (22) that $\ell_i(0) = 0$ and hence there exists $L_i \in \mathbb{R}^{s_i\times n}$ such that $\ell_i(x) = L_ix + \ell_{\mathrm{r}i}(x)$, $x \in \mathbb{R}^n$, where $\ell_{\mathrm{r}i}(\cdot)$ contains higher order terms. Furthermore, it follows from (24) that $\mathcal{Z}_i = Z_i$, $Z_i \in \mathbb{R}^{s_i\times m}$, $i = 1,\ldots,q$, which implies (45). Using the above expressions, equations (22) and (23) can be written as

$$0 = x^{\mathrm{T}}\Big(A^{\mathrm{T}}P_i + P_iA - C^{\mathrm{T}}\hat{Q}_iC - \sum_{j=1}^{q} W_{(i,j)}P_j + L_i^{\mathrm{T}}L_i\Big)x + \gamma_i(x) \tag{47}$$

$$0 = x^{\mathrm{T}}(P_iB - C^{\mathrm{T}}(\hat{S}_i + \hat{Q}_iD) + L_i^{\mathrm{T}}Z_i) + \Gamma_i(x) \tag{48}$$

where

$$\gamma_i(x) = v'_{\mathrm{sr}i}(x)Ax - \sum_{j=1}^{q} W_{(i,j)}v_{\mathrm{sr}j}(x) + 2x^{\mathrm{T}}L_i^{\mathrm{T}}\ell_{\mathrm{r}i}(x) + \ell_{\mathrm{r}i}^{\mathrm{T}}(x)\ell_{\mathrm{r}i}(x) \tag{49}$$

$$\Gamma_i(x) = \tfrac{1}{2}v'_{\mathrm{sr}i}(x)B + \ell_{\mathrm{r}i}^{\mathrm{T}}(x)Z_i. \tag{50}$$

Now, viewing (47) and (48) as the Taylor's series expansion of (22) and (24), respectively, about $x = 0$ and noting that $\lim_{\|x\|\to 0}(|\gamma_i(x)|/\|x\|^2) = 0$ and $\lim_{\|x\|\to 0}(|\Gamma_i(x)|/\|x\|) = 0$, $i = 1,\ldots,q$, it follows that P_i, $i = 1,\ldots,q$, satisfy (43) and (44). ☐

Remark 5: Note that (43)–(45) are equivalent to

$$\begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{B}_i^{\mathrm{T}} & \mathcal{C}_i \end{bmatrix} = -\begin{bmatrix} L_i^{\mathrm{T}} \\ Z_i^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} L_i & Z_i \end{bmatrix} \leq 0, \quad i = 1,\ldots,q \tag{51}$$

where, for all $i = 1,\ldots,q$

$$\mathcal{A}_i = A^{\mathrm{T}}P_i + P_iA - C^{\mathrm{T}}\hat{Q}_iC - \sum_{j=1}^{q} W_{(i,j)}P_j \tag{52}$$

$$\mathcal{B}_i = P_iB - C^{\mathrm{T}}(\hat{S}_i + \hat{Q}_iD) \tag{53}$$

$$\mathcal{C}_i = -(\hat{R}_i + D^{\mathrm{T}}\hat{S}_i + \hat{S}_i^{\mathrm{T}}D + D^{\mathrm{T}}\hat{Q}_iD). \tag{54}$$

Hence, vector dissipativity of large-scale linear dynamical systems with respect to vector quadratic supply rates can be characterized via (cascade) linear matrix inequalities (LMIs) (Boyd *et al.* 1994). A similar remark holds for Theorem 6 below.

The next result presents sufficient conditions guaranteeing vector dissipativity of $\mathcal{G}$ with respect to a vector quadratic supply rate in the case where the vector storage function is component decoupled.

Theorem 6: *Consider the large-scale linear dynamical system $\mathcal{G}$ given by* (41), (42). *Let $R_i \in \mathbb{S}^{m_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, $Q_i \in \mathbb{S}^{l_i}$, $i = 1,\ldots,q$, be given. Assume there exist matrices $W \in \mathbb{R}^{q\times q}$, $P_i \in \mathbb{N}^{n_i}$, $L_{ii} \in \mathbb{R}^{s_{ii}\times n_i}$, $\mathrm{Z}_{ii} \in \mathbb{R}^{s_{ii}\times m_i}$, $i = 1,\ldots,q$, $L_{ij} \in \mathbb{R}^{s_{ij}\times n_i}$ and $\mathrm{Z}_{ij} \in \mathbb{R}^{s_{ij}\times n_j}$, $i, j = 1,\ldots,q$, $i \neq j$, such that W is essentially non-negative and semistable* (*respectively, asymptotically stable*), *and,* *for all* $i = 1, \ldots, q$

$$0 \geq A_{ii}^{\mathrm{T}} P_i + P_i A_{ii} - C_i^{\mathrm{T}} Q_i C_i - W_{(i,i)} P_i + L_{ii}^{\mathrm{T}} L_{ii} + \sum_{j=1,\, j \neq i}^{q} L_{ij}^{\mathrm{T}} L_{ij} \tag{55}$$

$$0 = P_i B_i - C_i^{\mathrm{T}} S_i - C_i^{\mathrm{T}} Q_i D_i + L_{ii}^{\mathrm{T}} Z_{ii} \tag{56}$$

$$0 \leq R_i + D_i^{\mathrm{T}} S_i + S_i^{\mathrm{T}} D_i + D_i^{\mathrm{T}} Q_i D_i - Z_{ii}^{\mathrm{T}} Z_{ii} \tag{57}$$

and for $j = 1, \ldots, q,\ j \neq i$

$$0 = P_i A_{ij} + L_{ij}^{\mathrm{T}} Z_{ij} \tag{58}$$

$$0 \leq W_{(i,j)} P_j - Z_{ij}^{\mathrm{T}} Z_{ij}. \tag{59}$$

Then $\mathcal{G}$ *is vector dissipative* (*respectively*, *exponentially vector dissipative*) *with respect to the vector supply rate* $S(u, y) \triangleq [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^{\mathrm{T}}$, *where* $s_i(u_i, y_i) = u_i^{\mathrm{T}} R_i u_i + 2 y_i^{\mathrm{T}} S_i u_i + y_i^{\mathrm{T}} Q_i y_i$, $i = 1, \ldots, q$.

Proof: Since $P_i \in \mathbb{N}^{n_i}$, the function $v_{\mathrm{s}i}(x_i) \triangleq x_i^{\mathrm{T}} P_i x_i$, $x_i \in \mathbb{R}^{n_i}$, is non-negative definite and $v_{\mathrm{s}i}(0) = 0$. Moreover, since $v_{\mathrm{s}i}(\cdot)$ is continuously differentiable it follows from (55)–(59) that for all $u_i \in \mathbb{R}^{m_i}$, $i = 1, \ldots, q$, and $t \geq t_0$

$$\begin{aligned}
\dot{v}_{\mathrm{s}i}(x_i(t)) &= 2 x_i^{\mathrm{T}}(t) P_i \left[\sum_{j=1}^{q} A_{ij} x_j(t) + B_i u_i(t) \right] \\
&\leq x_i^{\mathrm{T}}(t) \left[W_{(i,i)} P_i + C_i^{\mathrm{T}} Q_i C_i - L_{ii}^{\mathrm{T}} L_{ii} - \sum_{j=1,\, j \neq i}^{q} L_{ij}^{\mathrm{T}} L_{ij} \right] x_i(t) \\
&\quad - \sum_{j=1,\, j \neq i}^{q} 2 x_i^{\mathrm{T}}(t) L_{ij}^{\mathrm{T}} Z_{ij} x_j(t) + 2 x_i^{\mathrm{T}}(t) C_i^{\mathrm{T}} S_i u_i(t) \\
&\quad + 2 x_i^{\mathrm{T}}(t) C_i^{\mathrm{T}} Q_i D_i u_i(t) - 2 x_i^{\mathrm{T}}(t) L_{ii}^{\mathrm{T}} Z_{ii} u_i(t) \\
&\quad + \sum_{j=1,\, j \neq i}^{q} x_j^{\mathrm{T}}(t) [W_{(i,j)} P_j - Z_{ij}^{\mathrm{T}} Z_{ij}] x_j(t) \\
&\quad + u_i^{\mathrm{T}}(t) R_i u_i(t) + 2 u_i^{\mathrm{T}}(t) D_i^{\mathrm{T}} S_i u_i(t) \\
&\quad + u_i^{\mathrm{T}}(t) D_i^{\mathrm{T}} Q_i D_i u_i(t) - u_i^{\mathrm{T}}(t) Z_{ii}^{\mathrm{T}} Z_{ii} u_i(t) \\
&= \sum_{j=1}^{q} W_{(i,j)} v_{\mathrm{s}j}(x_j(t)) + u_i^{\mathrm{T}}(t) R_i u_i(t) \\
&\quad + 2 y_i^{\mathrm{T}}(t) S_i u_i(t) + y_i^{\mathrm{T}}(t) Q_i y_i(t) \\
&\quad - [L_{ii} x_i(t) + Z_{ii} u_i(t)]^{\mathrm{T}} [L_{ii} x_i(t) + Z_{ii} u_i(t)] \\
&\quad - \sum_{j=1,\, j \neq i}^{q} (L_{ij} x_i(t) + Z_{ij} x_j(t))^{\mathrm{T}} (L_{ij} x_i(t) + Z_{ij} x_j(t)) \\
&\leq s_i(u_i(t), y_i(t)) + \sum_{j=1}^{q} W_{(i,j)} v_{\mathrm{s}j}(x_j(t))
\end{aligned} \tag{60}$$

or, equivalently, in vector form

$$\dot{V}_{\mathrm{s}}(x(t)) \leq\leq W V_{\mathrm{s}}(x(t)) + S(u, y), \quad u \in \mathcal{U}, \quad t \geq t_0 \tag{61}$$

where $V_{\mathrm{s}}(x) \triangleq [v_{\mathrm{s}1}(x_1), \ldots, v_{\mathrm{s}q}(x_q)]^{\mathrm{T}}$, $x \in \mathbb{R}^n$. Now, it follows from Remark 1 that $\mathcal{G}$ is vector dissipative (respectively, exponentially vector dissipative) with respect to the vector supply rate $S(u, y)$ and with vector storage function $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$. □

5. Stability of feedback interconnections of large-scale non-linear dynamical systems

In this section we consider stability of feedback interconnections of large-scale non-linear dynamical systems. Specifically, for the large-scale dynamical system $\mathcal{G}$ given by (9), (10) we consider either a dynamic or static large-scale feedback system $\mathcal{G}_{\mathrm{c}}$. Then by appropriately combining vector storage functions for each system we show stability of the feedback interconnection. The use of vector storage functions as vector Lyapunov functions in the stability of feedback interconnections offers a very flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single scalar Lyapunov function (Willems 1972 a, Hill and Moylan 1977). Weakening the hypothesis on the Lyapunov function enlarges the class of Lyapunov functions that can be used for analysing stability of feedback systems. In particular, each component of a vector Lyapunov function need not be positive definite with a negative or even negative-semidefinite derivative. Alternatively, the time derivative of the vector Lyapunov function need only satisfy an element-by-element vector inequality involving a vector field of a certain comparison system.

We begin by considering the large-scale non-linear dynamical system (9), (10) with the large-scale feedback system $\mathcal{G}_{\mathrm{c}}$ given by

$$\dot{x}_{\mathrm{c}}(t) = F_{\mathrm{c}}(x_{\mathrm{c}}(t), u_{\mathrm{c}}(t)), \quad x_{\mathrm{c}}(t_0) = x_{\mathrm{c}0}, \quad t \geq t_0 \tag{62}$$

$$y_{\mathrm{c}}(t) = H_{\mathrm{c}}(x_{\mathrm{c}}(t), u_{\mathrm{c}}(t)) \tag{63}$$

where $F_{\mathrm{c}}\colon \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{c}} \to \mathbb{R}^{n_{\mathrm{c}}}$, $H_{\mathrm{c}}\colon \mathbb{R}^{n_{\mathrm{c}}} \times \mathcal{U}_{\mathrm{c}} \to \mathcal{Y}_{\mathrm{c}}$, $F_{\mathrm{c}} \triangleq [F_{\mathrm{c}1}^{\mathrm{T}}, \ldots, F_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, $H_{\mathrm{c}} \triangleq [H_{\mathrm{c}1}^{\mathrm{T}}, \ldots, H_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{U}_{\mathrm{c}} \subseteq \mathbb{R}^l$, $\mathcal{Y}_{\mathrm{c}} \subseteq \mathbb{R}^m$. Moreover, for all $i = 1, \ldots, q$, we assume that

$$F_{\mathrm{c}i}(x_{\mathrm{c}}, u_{\mathrm{c}i}) = f_{\mathrm{c}i}(x_{\mathrm{c}i}) + \mathcal{I}_{\mathrm{c}i}(x_{\mathrm{c}}) + G_{\mathrm{c}i}(x_{\mathrm{c}i}) u_{\mathrm{c}i} \tag{64}$$

$$H_{\mathrm{c}i}(x_{\mathrm{c}i}, u_{\mathrm{c}i}) = h_{\mathrm{c}i}(x_{\mathrm{c}i}) + J_{\mathrm{c}i}(x_{\mathrm{c}i}) u_{\mathrm{c}i} \tag{65}$$

where $u_{\mathrm{c}i} \in \mathcal{U}_{\mathrm{c}i} \subseteq \mathbb{R}^{l_i}$, $y_{\mathrm{c}i} \triangleq H_{\mathrm{c}i}(x_{\mathrm{c}i}, u_{\mathrm{c}i}) \in \mathcal{Y}_i \subseteq \mathbb{R}^{m_i}$, $(u_{\mathrm{c}i}, y_{\mathrm{c}i})$ is the input–output pair for the ith subsystem of $\mathcal{G}_{\mathrm{c}}$, $f_{\mathrm{c}i}\colon \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i}}$ and $\mathcal{I}_{\mathrm{c}i}\colon \mathbb{R}^{n_{\mathrm{c}}} \to \mathbb{R}^{n_{\mathrm{c}i}}$ satisfy $f_{\mathrm{c}i}(0) = 0$ and $\mathcal{I}_{\mathrm{c}i}(0) = 0$, $G_{\mathrm{c}i}\colon \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{n_{\mathrm{c}i} \times l_i}$, $h_{\mathrm{c}i}\colon \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_i}$ and satisfies $h_{\mathrm{c}i}(0) = 0$, $J_{\mathrm{c}i}\colon \mathbb{R}^{n_{\mathrm{c}i}} \to \mathbb{R}^{m_i \times l_i}$, and $\sum_{i=1}^{q} n_{\mathrm{c}i} = n_{\mathrm{c}}$. Furthermore, we define the composite input and composite output for the system $\mathcal{G}_{\mathrm{c}}$ as $u_{\mathrm{c}} \triangleq [u_{\mathrm{c}1}^{\mathrm{T}}, \ldots, u_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$ and $y_{\mathrm{c}} \triangleq [y_{\mathrm{c}1}^{\mathrm{T}}, \ldots, y_{\mathrm{c}q}^{\mathrm{T}}]^{\mathrm{T}}$, respectively. In this case, $\mathcal{U}_{\mathrm{c}} = \mathcal{U}_{\mathrm{c}1} \times \cdots \times \mathcal{U}_{\mathrm{c}q}$ and $\mathcal{Y}_{\mathrm{c}} = \mathcal{Y}_{\mathrm{c}1} \times \cdots \times \mathcal{Y}_{\mathrm{c}q}$.

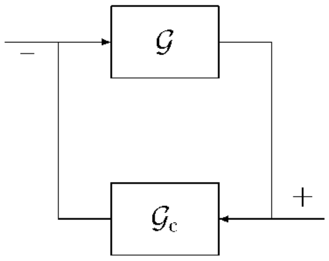

Figure 1. Feedback interconnection of large-scale systems $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$.

Note that with the feedback interconnection given by figure 1, $u_\mathrm{c} = y$ and $y_\mathrm{c} = -u$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is well posed; that is, $\det(I_{m_i} + J_{\mathrm{c}i}(x_{\mathrm{c}i})J_i(x_i)) \neq 0$ for all $x_i \in \mathbb{R}^{n_i}$, $x_{\mathrm{c}i} \in \mathbb{R}^{n_{\mathrm{c}i}}$, and $i = 1, \ldots, q$. Furthermore, we assume that for the large-scale systems $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$, the conditions of Theorem 3.3 of Haddad *et al.* (2003) are satisfied; that is, if $V_\mathrm{s}(x)$, $x \in \mathbb{R}^n$, and $V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, are vector storage functions for $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$, respectively, then there exist $p \in \mathbb{R}^q_+$ and $p_\mathrm{c} \in \mathbb{R}^q_+$ such that the functions $v_\mathrm{s}(x) = p^\mathrm{T} V_\mathrm{s}(x)$, $x \in \mathbb{R}^n$, and $v_\mathrm{cs}(x_\mathrm{c}) = p_\mathrm{c}^\mathrm{T} V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, are positive definite. The following result gives sufficient conditions for Lyapunov and asymptotic stability of the feedback interconnection given by figure 1.

Theorem 7: *Consider the large-scale non-linear dynamical systems $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ given by* (9), (10) *and* (62), (63), *respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are vector dissipative with respect to the vector supply rates $S(u, y)$ and $S_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c})$, and with continuously differentiable vector storage functions $V_\mathrm{s}(\cdot)$ and $V_\mathrm{cs}(\cdot)$ and dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_\mathrm{c} \in \mathbb{R}^{q\times q}$, respectively.*

(i) *If there exists $\Sigma \overset{\Delta}{=} \mathrm{diag}[\sigma_1, \ldots, \sigma_q] > 0$ such that $S(u, y) + \Sigma S_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) \leq\leq 0$ and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable* (*respectively*, *asymptotically stable*), *where $\tilde{W}_{(i,j)} \overset{\Delta}{=} \max\{W_{(i,j)}, (\Sigma W_\mathrm{c}\Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, (\sigma_i/\sigma_j)W_{\mathrm{c}(i,j)}\}$, $i, j = 1, \ldots, q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is Lyapunov* (*respectively*, *asymptotically*) *stable.*

(ii) *Let $Q_i \in \mathbb{S}^{l_i}$, $S_i \in \mathbb{R}^{l_i\times m_i}$, $R_i \in \mathbb{S}^{m_i}$, $Q_{\mathrm{c}i} \in \mathbb{S}^{m_i}$, $S_{\mathrm{c}i} \in \mathbb{R}^{m_i\times l_i}$, and $R_{\mathrm{c}i} \in \mathbb{S}^{l_i}$, and suppose $S(u, y) = [s_1(u_1, y_1), \ldots, s_q(u_q, y_q)]^\mathrm{T}$ and $S_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) = [s_{\mathrm{c}1}(u_{\mathrm{c}1}, y_{\mathrm{c}1}), \ldots, s_{\mathrm{c}q}(u_{\mathrm{c}q}, y_{\mathrm{c}q})]^\mathrm{T}$, where $s_i(u_i, y_i) = u_i^\mathrm{T} R_i u_i + 2y_i^\mathrm{T} S_i u_i + y_i^\mathrm{T} Q_i y_i$ and $s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = u_{\mathrm{c}i}^\mathrm{T} R_{\mathrm{c}i} u_{\mathrm{c}i} + 2y_{\mathrm{c}i}^\mathrm{T} S_{\mathrm{c}i} u_{\mathrm{c}i} + y_{\mathrm{c}i}^\mathrm{T} Q_{\mathrm{c}i} y_{\mathrm{c}i}$, $i = 1, \ldots, q$. If there exists $\Sigma \overset{\Delta}{=} \mathrm{diag}[\sigma_1, \ldots, \sigma_q] > 0$ such that for all $i = 1, \ldots, q$*

$$\tilde{Q}_i \overset{\Delta}{=} \begin{bmatrix} Q_i + \sigma_i R_{\mathrm{c}i} & -S_i + \sigma_i S_{\mathrm{c}i}^\mathrm{T} \\ -S_i^\mathrm{T} + \sigma_i S_{\mathrm{c}i} & R_i + \sigma_i Q_{\mathrm{c}i} \end{bmatrix} \leq 0 \qquad (66)$$

and $\tilde{W} \in \mathbb{R}^{q\times q}$ is semistable (*respectively*, *asymptotically stable*), *where $\tilde{W}_{(i,j)} \overset{\Delta}{=} \max\{W_{(i,j)}, (\Sigma W_\mathrm{c}\Sigma^{-1})_{(i,j)}\} = \max\{W_{(i,j)}, (\sigma_i/\sigma_j)W_{\mathrm{c}(i,j)}\}$, $i, j = 1, \ldots, q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is Lyapunov* (*respectively*, *asymptotically*) *stable.*

Proof:

(i) Consider the vector Lyapunov function candidate $V(x, x_\mathrm{c}) = V_\mathrm{s}(x) + \Sigma V_\mathrm{cs}(x_\mathrm{c})$, $(x, x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}$, and note that the corresponding vector Lyapunov derivative satisfies

$$\begin{aligned} \dot{V}(x, x_\mathrm{c}) &= \dot{V}_\mathrm{s}(x) + \Sigma \dot{V}_\mathrm{cs}(x_\mathrm{c}) \\ &\leq\leq S(u, y) + \Sigma S_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) + W V_\mathrm{s}(x) \\ &\quad + \Sigma W_\mathrm{c} V_\mathrm{cs}(x_\mathrm{c}) \\ &\leq\leq W V_\mathrm{s}(x) + \Sigma W_\mathrm{c} \Sigma^{-1} \Sigma V_\mathrm{cs}(x_\mathrm{c}) \\ &\leq\leq \tilde{W}(V_\mathrm{s}(x) + \Sigma V_\mathrm{cs}(x_\mathrm{c})) \\ &= \tilde{W} V(x, x_\mathrm{c}), \quad (x, x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}. \end{aligned} \qquad (67)$$

Next, since for $V_\mathrm{s}(x)$, $x \in \mathbb{R}^n$, and $V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, there exist, by assumption, $p \in \mathbb{R}^q_+$ and $p_\mathrm{c} \in \mathbb{R}^q_+$ such that the functions $v_\mathrm{s}(x) = p^\mathrm{T} V_\mathrm{s}(x)$, $x \in \mathbb{R}^n$, and $v_\mathrm{cs}(x_\mathrm{c}) = p_\mathrm{c}^\mathrm{T} V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, are positive definite and noting that $v_\mathrm{cs}(x_\mathrm{c}) \leq \max_{i=1,\ldots,q}\{p_{\mathrm{c}i}\} \mathbf{e}^\mathrm{T} V_\mathrm{cs}(x_\mathrm{c})$, where $p_{\mathrm{c}i}$ is the ith element of p_c and $\mathbf{e} \overset{\Delta}{=} [1, \ldots, 1]^\mathrm{T}$, it follows that $\mathbf{e}^\mathrm{T} V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, is positive definite. Now, since $\min_{i=1,\ldots,q}\{p_i \sigma_i\} \mathbf{e}^\mathrm{T} V_\mathrm{cs}(x_\mathrm{c}) \leq p^\mathrm{T} \Sigma V_\mathrm{cs}(x_\mathrm{c})$, it follows that $p^\mathrm{T} \Sigma V_\mathrm{cs}(x_\mathrm{c})$, $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, is positive definite. Hence, the function $v(x, x_\mathrm{c}) = p^\mathrm{T} V(x, x_\mathrm{c})$, $(x, x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}$, is positive definite. Now, the result is a direct consequence of Theorem 1.

(ii) The proof follows from (i) by noting that, for all $i = 1, \ldots, q$,

$$s_i(u_i, y_i) + \sigma_i s_{\mathrm{c}i}(u_{\mathrm{c}i}, y_{\mathrm{c}i}) = \begin{bmatrix} y \\ y_\mathrm{c} \end{bmatrix}^\mathrm{T} \tilde{Q}_i \begin{bmatrix} y \\ y_\mathrm{c} \end{bmatrix} \qquad (68)$$

and hence $S(u, y) + \Sigma S_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) \leq\leq 0$. □

Corollary 1: *Consider the large-scale non-linear dynamical systems $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ given by* (9), (10) *and* (62), (63), *respectively. Assume that $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are zero-state observable with dissipation matrices $W \in \mathbb{R}^{q\times q}$ and $W_\mathrm{c} \in \mathbb{R}^{q\times q}$, respectively. Then the following statements hold:*

(i) *If $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are vector passive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \overset{\Delta}{=} \max\{W_{(i,j)}, W_{\mathrm{c}(i,j)}\}$, $i, j = 1, \ldots, q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is asymptotically stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are vector non-expansive and $\tilde{W} \in \mathbb{R}^{q\times q}$ is asymptotically stable, where $\tilde{W}_{(i,j)} \overset{\Delta}{=} \max\{W_{(i,j)}, W_{\mathrm{c}(i,j)}\}$, $i, j = 1, \ldots, q$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is asymptotically stable.*

Proof: The proof is a direct consequence of Theorem 7. Specifically, (i) follows from Theorem 7 with $R_i = 0$, $S_i = I_{m_i}$, $Q_i = 0$, $R_{ci} = 0$, $S_{ci} = I_{m_i}$, $Q_{ci} = 0$, $i = 1, \cdots, q$, and $\Sigma = I_q$; while (ii) follows from Theorem 7 with $R_i = \gamma_i^2 I_{m_i}$, $S_i = 0$, $Q_i = -I_{l_i}$, $R_{ci} = \gamma_{ci}^2 I_{l_i}$, $S_{ci} = 0$, $Q_{ci} = -I_{m_i}$, $i = 1, \ldots, q$, and $\Sigma = I_q$. □

6. Conclusion

In this paper we have extended the notion of dissipativity theory to vector dissipativity theory. Specifically, using vector storage functions and vector supply rates, dissipativity properties of composite large-scale dynamical systems are shown to be determined from the dissipativity properties of the individual subsystems and the nature of their interconnections. Furthermore, extended Kalman–Yakubovich–Popov conditions, in terms of the local subsystem dynamics and the subsystem interconnection constraints, characterizing vector dissipativeness via vector storage functions are derived. In addition, general stability criteria were given for feedback interconnections of large-scale non-linear dynamical systems in terms of vector storage functions serving as vector Lyapunov functions.

The extended Kalman–Yakubovich–Popov conditions developed in this paper predicated on vector storage functions and vector supply rates provide a generalization to the classical Kalman–Yakubovich–Popov conditions in that dissipativity of a large-scale system can be deduced from the individual subsystems and the nature of the interconnections. Furthermore, the stability of feedback interconnection results based on vector storage functions developed in the paper are formally more general than the standard feedback stability interconnection results based on scalar storage functions in that they require weaker assumptions. Extensions of vector dissipativity theory to the relevant problems of large-scale hybrid systems involving continuous-time dynamics coupled to abstract decision-making units will be explored in a future paper.

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