Finite-time stabilization of nonlinear impulsive dynamical systems

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Abstract

Finite-time stability involves dynamical systems whose trajectories converge to a Lyapunov stable equilibrium state in finite time. For continuous-time dynamical systems finite-time convergence implies nonuniqueness of system solutions in reverse time, and hence, such systems possess non-Lipschitzian dynamics. For impulsive dynamical systems, however, it may be possible to reset the system states to an equilibrium state achieving finite-time convergence without requiring non-Lipschitzian system dynamics. In this paper, we develop sufficient conditions for finite-time stability of impulsive dynamical systems using both scalar and vector Lyapunov functions. Furthermore, we design hybrid finite-time stabilizing controllers for impulsive dynamical systems that are robust against full modelling uncertainty. Finally, we present a numerical example for finite-time stabilization of large-scale impulsive dynamical systems.

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1. Introduction

The mathematical descriptions of many hybrid dynamical systems can be characterized by impulsive differential equations [1–6]. Impulsive dynamical systems can be viewed as a subclass of hybrid systems and consist of three elements—namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset. Since impulsive systems can involve impulses at variable times, they are in general time-varying systems, wherein the resetting events are both a function of time and the system’s state. In the case where the resetting events are defined by a prescribed sequence of times which are independent of the system state, the equations are known as time-dependent differential equations [1,2,4,7–9]. Alternatively, in the case where the resetting events are
defined by a manifold in the state space that is independent of time, the equations are autonomous and are known as
state-dependent differential equations [1,2,4,7–9].

Finite-time stability implies Lyapunov stability and convergence of system trajectories to an equilibrium state in
finite-time, and hence, is a stronger notion than asymptotic stability. For continuous-time dynamical systems, finite-
Uniqueness of solutions in forward time, however, can be preserved in the case of finite-time convergence. Sufficient
conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in
[12–15]. Finite-time convergence to a Lyapunov stable equilibrium for continuous-time systems, that is, finite-time
stability, was rigorously studied in [11,16] using Hölder continuous Lyapunov functions.

Finite-time stability of impulsive dynamical systems has not been studied in the literature. For impulsive dynamical
systems, it may be possible to reset the system states to an equilibrium state, in which case finite-time convergence
of the system trajectories can be achieved without the requirement of non-Lipschitzian dynamics. In addition, due to
system resets, impulsive dynamical systems may exhibit nonuniqueness of solutions in reverse time even when the
continuous-time dynamics are Lipschitz continuous.

In this paper, we develop sufficient conditions for finite-time stability of nonlinear impulsive dynamical systems.
Furthermore, we present stability results using vector Lyapunov functions wherein finite-time stability of the impulsive
system is guaranteed via finite-time stability of a hybrid vector comparison system. We use these results to construct
hybrid finite-time stabilizing controllers for impulsive dynamical systems. In addition, we construct decentralized
finite-time stabilizers for large-scale impulsive dynamical systems. Finally, we present a numerical example to show
the utility of the proposed framework.

2. Mathematical preliminaries

In this section, we introduce notation and definitions needed for developing finite-time stability analysis and
synthesis results for nonlinear impulsive dynamical systems. Let \( \mathbb{R} \) denote the set of real numbers, \( \mathbb{Z}_+ \) denote
the set of nonnegative integers, \( \mathbb{R}^n \) denote the set of \( n \times 1 \) column vectors, and \((\cdot)^T\) denote transpose. For \( v \in \mathbb{R}^q \) we write
\( v \geq 0 \) (respectively, \( v \gg 0 \)) to indicate that every component of \( v \) is nonnegative (respectively, positive).
In this case, we say that \( v \) is nonnegative or positive, respectively. Let \( \mathbb{R}_+^q \) and \( \mathbb{R}_+^q \) denote the nonnegative and positive orthants of
\( \mathbb{R}^q \), that is, if \( v \in \mathbb{R}^q \), then \( v \in \mathbb{R}_+^q \) and \( v \in \mathbb{R}_+^q \) are equivalent, respectively, to \( v \geq 0 \) and \( v \gg 0 \). Furthermore,
let \( \overline{D} \), \( \partial D \), and \( \partial \overline{D} \) denote the interior, the closure, and the boundary of the set \( D \subseteq \mathbb{R}^n \), respectively. Finally, we write
\( \| \cdot \| \) for an arbitrary spatial vector norm in \( \mathbb{R}^q \), \( V'(x) \) for the Fréchet derivative of \( V \) at \( x \), \( B_{\varepsilon}(\alpha) \), \( \alpha \in \mathbb{R}^n \), \( \varepsilon > 0 \), for the open ball
centered at \( \alpha \) with radius \( \varepsilon \), \( e \in \mathbb{R}^q \) for the ones vector given by \( e = [1, \ldots, 1]^T \), \( A^T \) for Moore–Penrose
generalized inverse of \( A \in \mathbb{R}^{n \times m} \) [17], and \( x(t) \to M \) as \( t \to \infty \) to denote that \( x(t) \) approaches the set \( M \), that is, for every \( \varepsilon > 0 \) there exists \( T > 0 \) such that \( \text{dist}(x(t), M) < \varepsilon \) for all \( t > T \), where \( \text{dist}(p, M) = \inf_{z \in M} \| p - z \| \).

The following definition introduces the notion of class \( \mathcal{V}_c \) functions involving quasimonotone increasing functions.

**Definition 2.1** ([18]). A function \( w = [w_1, \ldots, w_q]^T : \mathbb{R}^q \to \mathbb{R}^q \) is of class \( \mathcal{V}_c \) if \( w_i(z') \leq w_i(z'') \), \( i = 1, \ldots, q \),
for all \( z', z'' \in \mathbb{R}^q \) such that \( z'_j \leq z''_j \), \( z'_j = z''_j \), \( j = 1, \ldots, q \), \( i \neq j \), where \( z_i \) denotes the \( i \)-th component of \( z \).

If \( w(\cdot) \in \mathcal{V}_c \), then we say that \( w \) satisfies the Kamke condition [19,20]. Note that if \( w(z) = W z \), where \( W \in \mathbb{R}^{q \times q} \),
then the function \( w(\cdot) \) is of class \( \mathcal{V}_c \) if and only if \( W \) is essentially nonnegative, that is, all the off-diagonal entries of
the matrix \( W \) are nonnegative. Furthermore, note that it follows from **Definition 2.1** that any scalar \( (q = 1) \) function
\( w(z) \) is of class \( \mathcal{V}_c \).

Finally, we introduce the notion of class \( \mathcal{V}_d \) functions involving nondecreasing functions.

**Definition 2.2** ([21]). A function \( w = [w_1, \ldots, w_q]^T : \mathbb{R}^q \to \mathbb{R}^q \) is of class \( \mathcal{V}_d \) if \( w(z') \leq w(z'') \) for all \( z', z'' \in \mathbb{R}^q \) such that \( z' \leq z'' \).

If \( w(z) = W z \), where \( W \in \mathbb{R}^{q \times q} \), then the function \( w(\cdot) \) is of class \( \mathcal{V}_d \) if and only if \( W \) is nonnegative, that is, all
the entries of the matrix \( W \) are nonnegative. Note that if \( w(\cdot) \in \mathcal{V}_d \), then \( w(\cdot) \in \mathcal{V}_c \), however, the converse is not
necessarily true.
3. Finite-time stability of impulsive dynamical systems

Consider the nonlinear state-dependent impulsive dynamical system $\mathcal{G}$ [6] given by

\[
\begin{align*}
\dot{x}(t) &= f_c(x(t)), \quad x(0) = x_0, \quad x(t) \not\in \mathcal{Z}, \quad t \in \mathcal{I}_{x_0}, \\
\Delta x(t) &= f_d(x(t)), \quad x(t) \in \mathcal{Z},
\end{align*}
\]

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \in \mathcal{I}_{x_0}$, is the system state vector, $\mathcal{I}_{x_0}$ is the maximal interval of existence of a solution $x(t)$ to (1) and (2), $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f_c : \mathcal{D} \to \mathbb{R}^n$ is continuous and satisfies $f_c(0) = 0$, $f_d : \mathcal{D} \to \mathbb{R}^n$ is continuous, $\Delta x(t) \triangleq x(t^+) - x(t)$, $x(t^+) \triangleq x(t) + f_d(x(t)) = \lim_{t \to 0} x(t + \delta)$, $x(t) \in \mathcal{Z}$, and $\mathcal{Z} \subset \mathcal{D} \subseteq \mathbb{R}^n$ is the resetting set. A function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is a solution to the impulsive dynamical system (1) and (2) on the interval $\mathcal{I}_{x_0} \subseteq \mathbb{R}$ with initial condition $x(0) = x_0$ if $x(\cdot)$ is left-continuous and $x(t)$ satisfies (1) and (2) for all $t \in \mathcal{I}_{x_0}$. For a particular trajectory $x(t)$, $t \geq 0$, we let $t_k = t_k(x_0)$, $x_0 \in \mathcal{D}$, denote the $k$th instant of time at which $x(t)$ intersects $\mathcal{Z}$ and we let $x_k^+ \triangleq x(t_k^+(x_0)) \triangleq x(t_k(x_0)) + f_d(x(t_k(x_0)))$ denote the state of (1) and (2) after the $k$th resetting. To ensure well-posedness of the resetting times we make the following assumptions [6]:

A1. If $x(t) \in \mathcal{Z} \setminus \mathcal{Z}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $x(t + \delta) \not\in \mathcal{Z}$.

A2. If $x \in \mathcal{Z}$, then $x + f_d(x) \not\in \mathcal{Z}$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$; that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. It was shown in [6] that assumptions A1 and A2 ensure that the resetting times are well defined and distinct. Furthermore, note that if $x_0 \in \mathcal{Z}$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) \not\in \mathcal{Z}$, which serves as the initial condition for continuous-time dynamics (1). Finally, we note that if the origin is the equilibrium point of (1) and (2), then it follows from A2 that $0 \not\in \mathcal{Z}$. For details, see [6].

We assume that (1) possesses unique solutions in forward time for all initial conditions in $\mathcal{D}$ except possibly the origin in the following sense. For every $x \in \mathcal{D} \setminus \{0\}$ there exists $\tau_x > 0$ such that, if $y_1 : [0, \tau_1) \to \mathcal{D}$ and $y_2 : [0, \tau_2) \to \mathcal{D}$ are two solutions of (1) with $y_1(0) = y_2(0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that for each $x \in \mathcal{D}$, $\tau_x$ is chosen to be the largest such number in $\mathbb{R}_+$. Sufficient conditions for forward uniqueness of solutions to continuous-time dynamical systems in the absence of Lipschitz continuity of the system dynamics can be found in [12], [13, Sect. 10], [14], and [15, Sect. 1].

Since the resetting times are well defined and distinct, and since the solution to (1) exists and is unique, it follows that the solution of the impulsive dynamical system (1) and (2) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit Zenoess and beating, as well as confluence, wherein solutions exhibit infinitely many resetttings in a finite time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a certain point in time. In this paper we allow for the possibility of confluence and Zeno solutions; however, A2 precludes the possibility of beating. Furthermore, since not every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details, see [1,2,4]. Finally, we denote the trajectory or solution curve of (1) and (2) satisfying $x(0) = x$ by $s(\cdot, x)$ or $s^x(\cdot)$.

The following definition introduces the notion of finite-time stability for impulsive dynamical systems.

**Definition 3.1.** Consider the nonlinear impulsive dynamical system $\mathcal{G}$ given by (1) and (2). The zero solution $x(t) \equiv 0$ to (1) and (2) is finite-time stable if there exist an open neighbourhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T : \mathcal{N} \setminus \{0\} \to (0, \infty)$, called the settling-time function, such that the following statements hold:

(i) **Finite-time convergence.** For every $x \in \mathcal{N} \setminus \{0\}$, $s^x(t)$ is defined on $[0, T(x))$, $s^x(t) \in \mathcal{N} \setminus \{0\}$ for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} s(x, t) = 0$.

(ii) **Lyapunov stability.** For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s(t, x) \in \mathcal{B}_\delta(0)$ for all $t \in [0, T(x))$.

The zero solution $x(t) \equiv 0$ to (1) and (2) is globally finite-time stable if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$. 
Consider the nonlinear impulsive dynamical system

\begin{equation}
\mathcal{G}
\end{equation}

and

\begin{equation}
\mathcal{H}
\end{equation}

Thus, since \( V(x) \) is radially unbounded, then the zero solution \( x(0) = 0 \) converges to the origin in finite time given by

\[ t = \frac{1}{c(1-\alpha)} \| V(s(x_0)) \|^{1-\alpha}. \]

Furthermore, note that \( \tau_1(x_0^+) < T_c(x_0^+) \), \( k = 0, 1, \ldots \), since (1) and (2) exhibit an infinite number of resettings, where \( T_c() \) is the settling-time function when \( \mathcal{Z} = \emptyset \). Moreover, as shown in [11],

\[ V(x(t, y)) \leq [(V(y))^{1-\alpha} - c(1-\alpha)\tau_1(x_0)]^{1-\alpha}, \quad t \in [0, T_c(y)), \ y \in B_{\delta}(0) \]

and hence, since \( \tau_1(x_0) < T_c(x_0) \), it follows that

\[ V(x_1) \leq [(V(x_0))^{1-\alpha} - c(1-\alpha)\tau_1(x_0)]^{1-\alpha}. \]

Thus, since \( V(x + f_d(x)) \leq V(x) \), it follows from (6) that

\[ \tau_1(x_1^+) < T_c(x_1^+) \]

\[ \leq \frac{1}{c(1-\alpha)} (V(x_1^+))^{1-\alpha} \]

\[ \leq \frac{1}{c(1-\alpha)} (V(x_1))^{1-\alpha} \]

\[ \leq \frac{1}{c(1-\alpha)} [(V(x_0))^{1-\alpha} - c(1-\alpha)\tau_1(x_0)]. \]

Similarly, it follows from (5) that, for \( y = x_2^+ \),

\[ \tau_1(x_2^+) < T_c(x_2^+) \]

\[ \leq \frac{1}{c(1-\alpha)} (V(x_2^+))^{1-\alpha} \]
Theorem 3.1. Consider the nonlinear impulsive dynamical system given by (1) and (2). Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \mathbb{R}^q_+$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, continuous functions $w_c : \mathcal{Q} \to \mathbb{R}^q$ and $w_d : \mathcal{Q} \to \mathbb{R}^q$, and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, $w_c(\cdot) \in \mathcal{W}_c$, $w_d(\cdot) \in \mathcal{W}_d$, $w_c(0) = 0$, $w_d(0) = 0$, the scalar function $p^TV(x)$, $x \in \mathcal{D}$, is positive definite, and

\begin{align*}
V'(x)f_c(x) & \leq w_c(V(x)), \quad x \notin \mathcal{Z}, \\
V(x + f_d(x)) & \leq V(x) + w_d(V(x)), \quad x \in \mathcal{Z}.
\end{align*}

Recursively repeating this procedure for $k = 3, 4, \ldots$, it follows that, with $\tau_1(x_0^+) = \tau_1(x_0)$,

$$
\tau_1(x_k^+) < \frac{1}{c(1 - \alpha)}[(V(x_0))^1-\alpha - c(1 - \alpha)\sum_{i=0}^{k-1} \tau_1(x_i^+)].
$$

Next, let $k \to \infty$ and note that since $x(t) \equiv 0$ is asymptotically stable, it follows that $\lim_{k \to \infty} \tau_1(x_k^+) = 0$. Hence,

$$
0 = \lim_{k \to \infty} \tau_1(x_k^+) < \frac{1}{c(1 - \alpha)}[(V(x_0))^1-\alpha - c(1 - \alpha)\sum_{i=0}^{\infty} \tau_1(x_i^+)].
$$

Thus,

$$
T(x_0) = \sum_{i=0}^{\infty} \tau_1(x_i^+) < \frac{1}{c(1 - \alpha)}(V(x_0))^1-\alpha < \infty,
$$

which implies that the trajectory $s(\cdot, x_0)$ is Zeno [6] and converges to the origin in finite time with an infinite number of resettings, that is, $s(t, x_0) \to 0$ as $t \to T(x_0)$.

Finally, suppose, ad absurdum, that $s(t', x_0) \neq 0$ for some $t' > T(x_0), x_0 \in \mathcal{B}_d(0)$. Then, since $V(\cdot)$ is positive definite, $V(s(t', x_0)) = \beta > 0$. Furthermore, since $s(t, x_0) \to 0$ as $t \to T(x_0)$, there exists $t'' < T(x_0)$ such that $V(s(t'', x_0)) < \beta$. Now, since $V(s(t, x_0))$ is a decreasing function of time, it follows that for $t'' < T(x_0) < t'$,

$$
\beta = V(s(t', x_0)) < V(s(t'', x_0)) < \beta,
$$

which leads to a contradiction. Hence, $s(t, x_0) = 0, t \geq T(x_0), x_0 \in \mathcal{B}_d(0)$, which implies convergence in finite time with $\mathcal{N} \triangleq \mathcal{B}_d(0)$. This completes the proof of finite-time stability.

Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then global finite-time stability follows using standard arguments. See, for instance, [6]. \qed

Remark 3.1. Conditions (3) and (4) are only sufficient conditions for guaranteeing finite-time stability of impulsive dynamical systems. Alternatively, finite-time stability can also be achieved by imposing additional conditions on the discrete-time dynamics. For example, if for some $x_0 \in \mathcal{D}, x(t_k) \in \mathcal{Z} \cap \{x \in \mathcal{D} : x - f_d(x) = 0\}$, then the trajectory $x(\cdot)$ resets to the origin and, since $f_c(0) = 0$, finite-time convergence is achieved. A convergent Zeno solution is yet another example of finite-time convergence to the equilibrium point.

The next theorem generalizes Theorem 3.1 to the case of vector Lyapunov functions involving a vector differential inequality.

Theorem 3.2. Consider the nonlinear impulsive dynamical system given by (1) and (2). Assume there exist a continuously differentiable vector function $V : \mathcal{D} \to \mathcal{Q} \cap \mathbb{R}^q_+$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, continuous functions $w_c : \mathcal{Q} \to \mathbb{R}^q$ and $w_d : \mathcal{Q} \to \mathbb{R}^q$, and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, $w_c(\cdot) \in \mathcal{W}_c$, $w_d(\cdot) \in \mathcal{W}_d$, $w_c(0) = 0$, $w_d(0) = 0$, the scalar function $p^TV(x)$, $x \in \mathcal{D}$, is positive definite, and

\begin{align*}
V'(x)f_c(x) & \leq w_c(V(x)), \quad x \notin \mathcal{Z}, \\
V(x + f_d(x)) & \leq V(x) + w_d(V(x)), \quad x \in \mathcal{Z}.
\end{align*}
In addition, assume that the hybrid vector comparison system
\begin{align}
\dot{z}(t) &= w_c(z(t)), \quad z(0) = 0, \quad x(t) \not\in Z, \\
\Delta z(t) &= w_d(z(t)), \quad x(t) \in Z,
\end{align}
has a unique solution \( z(t), t \geq 0, \) in forward time, and there exist a continuously differentiable function \( v : \mathcal{Q} \to \mathbb{R}, \) real numbers \( c > 0 \) and \( \alpha \in (0, 1), \) and a neighbourhood \( \mathcal{M} \subseteq \mathcal{Q} \) of the origin such that \( v(\cdot) \) is positive definite and
\begin{align}
v'(z)w_c(z) &\leq -c(v(z))^\alpha, \quad z \in \mathcal{M}, \\
v(z + w_d(z)) &\leq v(z), \quad z \in \mathcal{M}.
\end{align}
Then the zero solution \( x(t) \equiv 0 \) to (1) and (2) is finite-time stable. If, in addition, \( \mathcal{D} = \mathbb{R}^n, \) \( v(\cdot) \) is radially unbounded, and (17) and (18) hold on \( \mathbb{R}^q, \) then the zero solution \( x(t) \equiv 0 \) to (1) and (2) is globally finite-time stable.

**Proof.** First, note that since the dynamics of the impulsive dynamical system (1) and (2), and the impulsive comparison system (15) and (16) are decoupled, the comparison system (15) and (16) is a time-dependent impulsive dynamical system [6]; that is, the resetting times for the trajectories of (15) and (16) are predetermined by the trajectories of (1) and (2). Hence, the stability analysis for the comparison system (15) and (16) involves a time-dependent impulsive dynamical system [6]. Since \( v(\cdot) \) is positive definite, there exist \( r > 0 \) and class \( \mathcal{K} \) functions \([22]\) \( \alpha, \beta : [0, r] \to \mathbb{R}_+ \) such that \( B_r(0) \subseteq \mathcal{M} \) and
\begin{equation}
\alpha(\|z\|) \leq v(z) \leq \beta(\|z\|), \quad z \in B_r(0),
\end{equation}
and hence,
\begin{equation}
v'(z)w_c(z) \leq -c(v(z))^\alpha \leq -c(\alpha(\|z\|))^\alpha, \quad z \in B_r(0).
\end{equation}
Thus, it follows from (18)–(20), and Theorem 2.6 of [6] that the zero solution \( z(t) \equiv 0 \) to the impulsive comparison system (15) and (16) is uniformly asymptotically stable. Furthermore, using identical arguments as in the proof of Theorem 3.1, it can be shown that the trajectories of (15) and (16) converge to the origin in finite time for all \( z_0 \in B_r(0), \) and hence, the zero solution \( z(t) \equiv 0 \) to (15) and (16) is finite-time stable. Moreover, it follows from (13) and (14), the asymptotic stability of the comparison system (15) and (16), and Theorem 2.11 of [6] that the nonlinear impulsive dynamical system \( \mathcal{G} \) given by (1) and (2) is asymptotically stable. Hence, it remains to be shown that there exists a neighbourhood \( \mathcal{N} \subseteq \mathcal{D} \) of the origin such that the trajectories of (1) and (2) converge to the origin in finite time for all \( x_0 \in \mathcal{N}. \)

Since \( V(\cdot) \) is continuous there exists \( \delta > 0 \) such that \( V(x_0) \in B_\delta(0) \) for all \( x_0 \in B_\delta(0). \) Let \( x_0 \in B_\delta(0) \) and \( z_0 = V(x_0) \in B_r(0). \) It follows from (13) and (14), and the hybrid comparison principle [6, Theorem 2.10] that
\begin{equation}
0 \leq V(x(t)) \leq \zeta(t), \quad t \in [0, \tau],
\end{equation}
where \( x(t), t \geq t_0, \) is the solution to (1) and (2) with the initial condition \( x_0 \in B_\delta(0), \) \( z(t), t \geq t_0, \) is the solution to (15) and (16) with the initial condition \( z_0 = V(x_0) \in B_r(0), \) and \([0, \tau] \) is any arbitrarily large compact time interval. Now, forming \( p^TV(21) \) yields
\begin{equation}
0 \leq p^TV(x(t)) \leq p^Tz(t), \quad t \in [0, \tau].
\end{equation}
Since \( z(\cdot) \) converges to the origin in finite time and \( p^TV(\cdot) \) is positive definite it follows that \( x(t), t \geq t_0, \) converges to the origin in finite time. Hence, the nonlinear impulsive dynamical system \( \mathcal{G} \) given by (1) and (2) is finite-time stable with \( \mathcal{N} \triangleq B_\delta(0). \)

Finally, if \( \mathcal{D} = \mathbb{R}^n, \) \( v(\cdot) \) is radially unbounded, and (17) and (18) hold on \( \mathbb{R}^q, \) then global finite-time stability follows using standard arguments. \( \square \)

The next result is a specialization of Theorem 3.2 to the case where the structure of the comparison dynamics directly guarantees finite-time stability of the impulsive comparison system. That is, there is no need to require the existence of a scalar function \( v(\cdot) \) such that (17) and (18) hold in order to guarantee finite-time stability of the nonlinear impulsive dynamical system (1) and (2).
Corollary 3.1. Consider the nonlinear impulsive dynamical system given by (1) and (2). Assume there exist a continuously differentiable vector function $V : \mathcal{D} \rightarrow \mathbb{Q} \cap \mathbb{R}^q_+$, where $\mathcal{Q} \subset \mathbb{R}^q$ and $0 \in \mathcal{Q}$, and a positive vector $p \in \mathbb{R}^q_+$ such that $V(0) = 0$, the scalar function $p^T V(x)$, $x \in \mathcal{D}$, is positive definite, and
\[
V'(x) f(x) \leq W(V(x))^{[\alpha]}, \quad x \not\in \mathcal{Z},
\]
\[
V(x + f_a(x)) \leq V(x), \quad x \in \mathcal{Z},
\]
where $\alpha \in (0, 1)$, $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative and Hurwitz, and $(V(x))^{[\alpha]} \triangleq [(V_1(x))^{\alpha}, \ldots, (V_q(x))^{\alpha}]^T$. Then the zero solution $x(t) \equiv 0$ to (1) and (2) is finite-time stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$, then the zero solution $x(t) \equiv 0$ to (1) and (2) is globally finite-time stable.

Proof. Consider the impulsive comparison system given by
\[
\dot{z}(t) = W(z(t))^{[\alpha]}, \quad z(0) = z_0, \quad x(t) \not\in \mathcal{Z},
\]
\[
\Delta z(t) = 0, \quad x(t) \in \mathcal{Z},
\]
where $z_0 \in \mathbb{R}^q$. Note that the right-hand side of (25) is of class $\mathcal{W}_c$ and is essentially nonnegative and, hence, the solutions to (25) and (26) are nonnegative for all nonnegative initial conditions [23]. Since $W \in \mathbb{R}^{q \times q}$ is essentially nonnegative and Hurwitz, it follows from Theorem 3.2 of [24] that there exist positive vectors $\hat{p} \in \mathbb{R}^q_+$ and $r \in \mathbb{R}^q_+$ such that
\[
0 = W^T \hat{p} + r.
\]

Now, consider the Lyapunov function candidate $v(z) = \hat{p}^T z$, $z \in \mathbb{R}^q_+$. Note that $v(0) = 0$, $v(z) > 0$, $z \in \mathbb{R}^q_+$, $z \neq 0$, and $v(\cdot) \in \mathcal{C}$ is radially unbounded. Let $\beta \triangleq \min_{i=1, \ldots, q} r_i$ and $\gamma \triangleq \max_{i=1, \ldots, q} \hat{p}_i^{\alpha}$, where $r_i$ and $\hat{p}_i$ are the $i$th components of $r \in \mathbb{R}^q_+$ and $\hat{p} \in \mathbb{R}^q_+$, respectively. Then
\[
\dot{v}(z) = \hat{p}^T W^T z^{[\alpha]}
\]
\[
= -r^T z^{[\alpha]}
\]
\[
\leq -\frac{\beta}{\gamma} \left( \sum_{i=1}^q z_i^{\alpha} \right)
\]
\[
\leq -\frac{\beta}{\gamma} \left( \sum_{i=1}^q \hat{p}_i^{\alpha} z_i^{\alpha} \right)
\]
\[
\leq -\frac{\beta}{\gamma} \left( \sum_{i=1}^q \hat{p}_i^{\alpha} \right)^{\alpha}
\]
\[
\leq -\frac{\beta}{\gamma} (v(z))^{\alpha}, \quad z \in \mathbb{R}^q_+, \tag{27}
\]
\[
\Delta v(z) = 0, \quad z \in \mathbb{R}^q_+, \tag{28}
\]
and
\[
0 = W^T \hat{p} + r.
\]

where $c \triangleq \frac{\beta}{\gamma}$. Thus, it follows from Theorem 3.1 that the impulsive comparison system (25) and (26) is finite-time stable. In addition, it follows from Theorem 3.2 of [25] that the nonlinear impulsive dynamical system (1) and (2) is asymptotically stable with the domain of attraction $\mathcal{N} \subset \mathcal{D}$. Now, the result is a direct consequence of Theorem 3.2. $\square$

4. Finite-time stabilization of impulsive dynamical systems

In this section, we design hybrid finite-time stabilizing controllers for nonlinear affine in the control impulsive dynamical systems. In addition, for large-scale impulsive dynamical systems we design decentralized hybrid finite-time stabilizers predicated on a control vector Lyapunov function [26]. Consider the controlled nonlinear impulsive
dynamical system given by
\[
\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \quad x(t) \notin Z, t \geq t_0, \\
\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad x(t) \in Z,
\]
where \( f_c : \mathbb{R}^n \to \mathbb{R}^n \) satisfying \( f_c(0) = 0 \) and \( G_c : \mathbb{R}^n \to \mathbb{R}^{n \times m_c} \) are continuous functions, \( f_d : \mathbb{R}^n \to \mathbb{R}^n \) and \( G_d : \mathbb{R}^n \to \mathbb{R}^{n \times m_d} \) are continuous, \( u_c(t) \in \mathbb{R}^{m_c}, t \geq t_0, \) and \( u_d(t_k) \in \mathbb{R}^{m_d}, k \in \mathbb{Z}_+ \).

**Theorem 4.1.** Consider the controlled nonlinear impulsive dynamical system given by (30) and (31). If there exist a continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) and continuous functions \( P_{1u} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_d} \) and \( P_{2u} : \mathbb{R}^n \to \mathbb{R}^{m_d \times m_d} \) such that \( V(\cdot) \) is positive definite and

\[
V(x + f_d(x)) - V(x) - \frac{1}{4} P_{1u}(x) P_{2u}^T(x) P_{1u}^T(x) \leq 0, \quad x \in Z,
\]

where \( c > 0, \alpha \in (0, 1), \) and \( R \triangleq \{ x \in \mathbb{R}^n, x \notin Z : V'(x)G_c(x) = 0 \}, \) then the nonlinear impulsive dynamical system (30) and (31) with the hybrid feedback control law \( (u_c, u_d) = (\phi_c(\cdot), \phi_d(\cdot)) \) given by

\[
\phi_c(x) = \begin{cases} 
0, & \beta(x) = 0, x \notin Z, \\
\beta(x) \neq 0, x \notin Z, \\
c_0 + (\alpha(x) - w_c(V(x))) + \sqrt{(\alpha(x) - w_c(V(x)))^2 + (\beta^T(x)\beta(x))^2} \beta(x), & \text{otherwise} 
\end{cases}
\]

and

\[
\phi_d(x) = -\frac{1}{2} P_{2u}^T(x) P_{1u}^T(x), \quad x \in Z,
\]

where \( \alpha(x) \triangleq V'(x)f_c(x), x \in \mathbb{R}^n, \beta(x) \triangleq G_c^T(x)V'(x), x \in \mathbb{R}^n, w_c(V(x)) \triangleq -c(V(x))^a, x \in \mathbb{R}^n, \) and \( c_0 > 0, \) is finite-time stable.

**Proof.** Note that between resettings the time derivative of \( V(\cdot) \) along the trajectories of (30), with \( u_c = \phi_c(x), x \in \mathbb{R}^n, \) given by (35), is given by

\[
\dot{V}(x) = V'(x)(f_c(x) + G_c(x)\phi_c(x)) \\
= \alpha(x) + \beta^T(x) \phi_c(x) \\
= \begin{cases} 
-\alpha(x), & \beta(x) = 0, \\
0, & \beta(x) \neq 0 
\end{cases}
\]

\[
\leq w_c(V(x)), \quad x \notin Z.
\]

In addition, using (32) and (34), the difference of \( V(\cdot) \) at the resetting instants, with \( u_d = \phi_d(x), x \in Z, \) given by (36), is given by

\[
\Delta V(x) = V(x + f_d(x)) - V(x) \\
= V(x + f_d(x)) - V(x) - P_{1u}(x)\phi_d(x) + \phi_d(x) P_{2u}^T(x) P_{1u}^T(x) \\
= V(x + f_d(x)) - V(x) - \frac{1}{4} P_{1u}(x) P_{2u}^T(x) P_{1u}^T(x) \\
\leq 0, \quad x \in Z.
\]

Hence, it follows from Theorem 3.1 that the zero solution \( x(t) \equiv 0 \) to (30) and (31) with \( u_c = \phi_c(x), x \notin Z, \) given by (35) and \( u_d = \phi_d(x), x \in Z, \) given by (36), is finite-time stable, which proves the result. \( \square \)
In the next result we develop decentralized finite-time stabilizing hybrid controllers for a large-scale impulsive dynamical system composed of \( q \) interconnected subsystems given by

\[
\dot{x}_i(t) = f_{ci}(x_i(t)) + G_{ci}(x_i(t))u_{ci}(t), \quad x(t) \notin Z, \ i = 1, \ldots, q, \\
\Delta x_i(t) = f_{di}(x_i(t)) + G_{di}(x_i(t))u_{di}(t), \quad x(t) \in Z, \ i = 1, \ldots, q,
\]

where \( t \geq 0, \ f_{ci} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i} \) satisfying \( f_{ci}(0) = 0 \) and \( G_{ci} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i \times m_{ci}} \) are continuous functions for all \( i = 1, \ldots, q, \ f_{di} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( G_{di} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i \times m_{di}} \) are continuous for all \( i = 1, \ldots, q, \ u_{ci}(t) \in \mathbb{R}^{m_{ci}}, \ t \geq 0, \) and \( u_{di}(t_k) \in \mathbb{R}^{m_{di}}, \ k \in Z_+ \), for all \( i = 1, \ldots, q \).

**Theorem 4.2.** Consider the controlled nonlinear impulsive dynamical system given by (39) and (40). Assume there exist a continuously differentiable, component decoupled vector function \( V = [V_1(x_1), \ldots, V_q(x_q)]^T : \mathbb{R}^n \rightarrow \mathbb{R}^q \), continuous functions \( P_{1ui} : \mathbb{R}^n \rightarrow \mathbb{R}_+^{1 \times m_{di}}, P_{2ui} : \mathbb{R}^n \rightarrow \mathbb{R}_+^{m_{di} \times m_{di}}, \ i = 1, \ldots, q, \ w_c = [w_{c1}, \ldots, w_{cq}]^T : \mathbb{R}_+^q \rightarrow \mathbb{R}^q, \ w_d = [w_{d1}, \ldots, w_{dq}]^T : \mathbb{R}_+^q \rightarrow \mathbb{R}^q, \) and a positive vector \( p \in \mathbb{R}_+^q \) such that \( r^T V(x) = 0, \) the scalar function \( p^T V(x), \ x \in \mathbb{R}^n, \) is positive definite, \( w_c(\cdot) \in \mathcal{W}_c, \ w_d(\cdot) \in \mathcal{W}_d, \) \( w_c(0) = 0, \ w_d(0) = 0, \) and, for all \( i = 1, \ldots, q, \)

\[
V_i(x_i + f_{di}(x_i) + G_{di}(x_i)u_{di}(t)) = V_i(x_i + f_{di}(x_i)) + P_{1ui}(x_i)u_{di}(t) + u_{di}(t)P_{2ui}(x_i)u_{di}(t), \quad x \in \mathbb{R}^n, \ u_{di} \in \mathbb{R}^{m_{di}},
\]

\[
V_i'(x_i) - V_i(x_i) \leq \frac{1}{4} P_{1ui}(x_i)P_{2ui}^T(x_i)P_{1ui}^T(x_i) \leq w_d(V_i(x_i)), \quad x \in Z.
\]

where \( \mathcal{R}_i = \{ x \in \mathbb{R}^n, x \notin Z : V_i'(x_i)G_{ci}(x_i) = 0 \} \), \( i = 1, \ldots, q \). In addition, assume there exist a positive definite function \( v : \mathbb{R}_+^q \rightarrow \mathbb{R}_+, \) real numbers \( c > 0 \) and \( \alpha_i \in (0, 1), \) and a neighbourhood \( \mathcal{M} \subseteq \mathbb{R}_+^q \) of the origin such that

\[
v'(z)w_c(z) \leq -c(v(z))^{\alpha_i}, \quad z \in \mathcal{M} \cap \mathbb{R}_+^q, \quad (44)
\]

\[
v(z + w_d(z)) \leq v(z), \quad z \in \mathcal{M} \cap \mathbb{R}_+^q, \quad (45)
\]

Then the nonlinear impulsive dynamical system (39) and (40) is finite-time stable with the hybrid feedback control law \( u_c = \phi_c(x) = [\phi_{c1}(x), \ldots, \phi_{cq}(x)]^T, \) \( x \notin Z, \) and \( u_d = \phi_d(x) = [\phi_{d1}(x), \ldots, \phi_{dq}(x)]^T, \) \( x \in Z, \) where for \( i = 1, \ldots, q, \)

\[
\phi_{ci}(x) = \begin{cases} 
-c_0i + \frac{(\alpha_i(x) - w_{ci}(V(x))) + \sqrt{(\alpha_i(x) - w_{ci}(V(x)))^2 + (\beta_i(x))^2}}{\beta_i(x)} & \beta_i(x) \neq 0, \\
0 & \beta_i(x) = 0,
\end{cases}
\]

for \( x \notin Z, \) and

\[
\phi_{di}(x) = -\frac{1}{2} P_{2ui}^T(x)P_{1ui}^T(x), \quad x \in Z,
\]

where \( \alpha_i(x) \triangleq V_i'(x_i)f_{ci}(x_i), x \in \mathbb{R}^n, \beta_i(x) \triangleq G_{ci}^T(x)V_i^T(x_i), x \in \mathbb{R}^n, \) and \( c_0i > 0, \ i = 1, \ldots, q. \)

**Proof.** Using identical arguments as in the proof of Theorem 4.1 it can be shown that for the closed-loop system (39), (40), (46) and (47) the time derivative and the difference of the vector function \( V(\cdot) \) between resettings and resetting instants satisfy, respectively,

\[
\dot{V}(x) \leq w_c(V(x)), \quad x \notin Z, \quad (48)
\]

\[
V(x + f_d(x) + G_d(x)\phi_d(x)) \leq V(x) + w_d(V(x)), \quad x \in Z, \quad (49)
\]

where \( G_d(x) \triangleq \text{block-diag}[G_{d1}(x), \ldots, G_{dq}(x)], x \in Z. \) The result now follows immediately from (44) and (45), and Theorem 3.2. \( \square \)
Remark 4.1. If, in Theorem 4.2, \( \mathcal{R}_i = \emptyset, i = 1, \ldots, q, \) and (43) is satisfied with \( w_d(z) \equiv 0, \) then the function \( w_c(\cdot) \) in (46) can be chosen to be
\[
 w_c(z) = Wz^{[\alpha]}, \quad z \in \mathbb{R}^q, 
\]
where \( W \in \mathbb{R}^{q \times q} \) is essentially nonnegative and asymptotically stable, \( \alpha \in (0, 1), \) and \( z^{[\alpha]} \triangleq [z_1^{\alpha}, \ldots, z_q^{\alpha}]^T. \) In this case, conditions (44) and (45) need not be verified and it follows from Corollary 3.1 that the close-loop system (39), (40), (46) and (47) with \( w_c(\cdot) \) given by (50) is finite-time stable and, hence, the hybrid controller (46) and (47) is a finite-time stabilizing controller for (39) and (40).

Since \( f_{ci}(\cdot) \) and \( G_{ci}(\cdot) \) are continuous and \( V_i(\cdot) \) is continuously differentiable for all \( i = 1, \ldots, q, \) it follows that \( \alpha_i(x) \) and \( \beta_i(x), x \in \mathbb{R}^n, i = 1, \ldots, q, \) are continuous functions, and hence, \( \phi_{ci}(x) \) given by (46) is continuous for all \( x \in \mathbb{R}^n \) if either \( \beta_i(x) \neq 0 \) or \( \alpha_i(x) - w_{ci}(V(x)) < 0 \) for all \( i = 1, \ldots, q. \) Hence, the feedback control law given by (46) is continuous everywhere except for the origin. The following result provides necessary and sufficient conditions under which the feedback control law given by (46) is guaranteed to be continuous at the origin in addition to being continuous everywhere else.

Proposition 4.1 ([26]). The feedback control law \( \phi_c(x) \) given by (46) is continuous on \( \mathbb{R}^n \) if and only if for every \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that for all \( 0 < \|x\| < \delta \) there exists \( u_{ci} \in \mathbb{R}^{n_{ci}} \) such that \( \|u_{ci}\| < \varepsilon \) and \( \alpha_i(x) + \beta_i^T(x)u_{ci} < w_{ci}(V(x)), i = 1, \ldots, q. \)

Remark 4.2. Identical necessary and sufficient conditions apply in the case where \( q = 1 \) to ensure continuity of the feedback control law given by (35).

Remark 4.3. If the conditions of Proposition 4.1 are satisfied, then the feedback control law \( \phi_c(x) \) given by (46) is continuous on \( \mathbb{R}^n. \) However, it is important to note that for a particular trajectory \( x(t), t \geq 0, \) of (39) and (40), \( \phi_c(x(t)) \) is left-continuous on \([0, \infty)\) and is continuous everywhere on \([0, \infty)\) except on an unbounded closed discrete set of times when the resettings occur for \( x(t), t \geq 0.\)

5. Finite-time stabilizing control for large-scale impulsive dynamical systems

In this section, we apply the proposed hybrid control framework to decentralized control of large-scale nonlinear impulsive dynamical systems. Specifically, we consider the large-scale dynamical system \( \mathcal{G} \) shown in Fig. 5.1 involving energy exchange between \( n \) inter-connected subsystems. Let \( x_i : [0, \infty) \rightarrow \mathbb{R}_+^n \) denote the energy (and hence a nonnegative quantity) of the \( i \)th subsystem, let \( u_{ci} : [0, \infty) \rightarrow \mathbb{R} \) denote the control input to the \( i \)th subsystem, let \( \sigma_{cij} : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^+, i \neq j, i, j = 1, \ldots, n, \) denote the instantaneous rate of energy flow from the \( j \)th subsystem to the \( i \)th subsystem between resettings, let \( \sigma_{dij} : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^+, i \neq j, i, j = 1, \ldots, n, \) denote the amount of energy transferred from the \( j \)th subsystem to the \( i \)th subsystem at the resetting instant, and let \( \mathcal{Z} \subset \mathbb{R}_+^n \) be a resetting set for the large-scale impulsive dynamical system \( \mathcal{G}. \)

An energy balance for each subsystem \( \mathcal{G}_i, i = 1, \ldots, q, \) yields [27,6]
\[
 \dot{x}_i(t) = \sum_{j=1, j \neq i}^{n} [\sigma_{cij}(x(t)) - \sigma_{cji}(x(t))] + G_{ci}(x_i(t))u_{ci}(t), \quad x(t_0) = x_0, x(t) \notin \mathcal{Z}, t \geq t_0, 
\]
\[
 \Delta x_i(t) = \sum_{j=1, j \neq i}^{n} [\sigma_{dij}(x(t)) - \sigma_{dji}(x(t))] + G_{di}(x_i(t))u_{di}(t), \quad x(t) \in \mathcal{Z}, 
\]
or, equivalently, in vector form
\[
 \dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \quad x(t_0) = x_0, \quad x(t) \notin \mathcal{Z}, t \geq t_0, 
\]
\[
 \Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad x(t) \in \mathcal{Z}, 
\]
where \( x(t) = [x_1(t), \ldots, x_n(t)]^T, t \geq t_0, \) \( f_c(x) = \sum_{j=1, j \neq i}^{n} \phi_{cij}(x), \) where \( \phi_{cij}(x) \triangleq \sigma_{cij}(x) - \sigma_{cji}(x), x \in \mathbb{R}_+^n, \) \( i \neq j, i, j = 1, \ldots, q, \) denotes the net energy flow from the \( j \)th subsystem to the \( i \)th subsystem between resettings,
G_c(x) = diag[G_c1(x_1), \ldots, G_cn(x_n)], x \in \mathbb{R}^n_+, G_{ci} : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n, is such that G_{ci}(x_i) = 0 if and only if x_i = 0 for all i = 1, \ldots, n, G_c(x) = \text{diag}(G_{c1}(x), \ldots, G_{cn}(x)), x \in \mathbb{Z}, G_{ci} : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, n, u_c(t) \in \mathbb{R}^n, t \geq t_0, u_d(t_k) \in \mathbb{R}^n, k \in \mathbb{Z}_+, and f_{di}(x) = \sum_{j=1,j\neq i}^n \phi_{dij}(x), where \phi_{dij}(x) \leq \sigma_{dij}(x) - \sigma_{dij}(x), x \in \mathbb{Z}, i \neq j, i = 1, \ldots, q, denotes the net amount of energy transferred from the jth subsystem to the ith subsystem at the instant of resetting. Here, we assume that \sigma_{cij}(0) = 0, i \neq j, i = 1, \ldots, n, and u_c = [u_{c1}, \ldots, u_{cn}]^T : \mathbb{R} \to \mathbb{R}^n is such that u_{ci} : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n, are bounded piecewise continuous functions of time. Furthermore, we assume that \sigma_{cij}(x) = 0, x \in \mathbb{R}^n_+, whenever x_j = 0, i \neq j, i = 1, \ldots, n, and x_i + \sum_{j=1,j\neq i}^n \phi_{dij}(x) \geq 0, x \in \mathbb{Z}. In this case, f(x) is essentially nonnegative [24,27] (i.e., f_{ci}(x) \geq 0 for all x \in \mathbb{R}^n_+ such that x_i = 0, i = 1, \ldots, n) and x + f_d(x), x \in \mathbb{Z} \subset \mathbb{R}^n_+, is nonnegative (i.e., x_i + f_{di}(x) \geq 0 for all x \in \mathbb{Z}, i = 1, \ldots, n).

The above constraints imply that if the energy of the jth subsystem of G is zero, then this subsystem cannot supply any energy to its surroundings between resettings and the ith subsystem of G cannot transfer more energy to its surroundings than it already possesses at the instant of resetting. Finally, in order to ensure that the trajectories of the closed-loop system remain in the nonnegative orthant of the state space for all nonnegative initial conditions, we seek a hybrid feedback control law \((u_c(\cdot), u_d(\cdot))\) that guarantees the continuous-time closed-loop system dynamics (53) are essentially nonnegative and the closed-loop system states after resettings are nonnegative [6,23].

For the dynamical system G, consider the Lyapunov function candidate \(V(x) = e^T x, x \in \mathbb{R}^n_+\). Note that \(V(0) = 0\) and \(V(x) > 0, x \neq 0, x \in \mathbb{R}^n_+\). Also, note that since \(V(x) = e^T x, x \in \mathbb{R}^n_+\), is a linear function of x, it follows from (32).
that $P_{1u}(x) = e^T G_d(x), x \in \mathbb{R}_+^n$, and $P_{2u}(x) \equiv 0$, and hence, by (36), $\phi_d(x) \equiv 0$. Define $u_c(V(x)) \triangleq -(V(x))^{1/2}, x \in \mathbb{R}_+^n$, and note that $\mathcal{R} \triangleq \{x \in \mathbb{R}_+^n, x \not\in \mathcal{Z} : V'(x)G_c(x) = 0\} = \{x \in \mathbb{R}_+^n, x \not\in \mathcal{Z} : x = 0\} = \{0\}$, since $0 \not\in \mathcal{Z}$, and hence, condition (33) is satisfied. In addition, note that since $V(\cdot)$ is linear in $x$, condition (32) is trivially satisfied and inequality (34) is satisfied as an equality.

Next, with $\alpha(x) = V'(x)f_c(x) = \sum_{i=1}^n \sum_{j=1, j \neq i}^n c_{ij}(x) = 0$, $\beta(x) = [G_{c1}(x_1), \ldots, G_{cn}(x_n)]^T, x \in \mathbb{R}_+^n$, and $c_0 > 0$, we construct a finite-time stabilizing hybrid feedback controller for (53) and (54) given by

$$
\phi_c(x) = \begin{cases} 
-c_0 + \left(\frac{(e^T x)^{1/2} + \sqrt{(e^T x) + (\beta^T \beta(x))}}{\beta^T \beta(x)}\right) \beta(x), & x \neq 0, \\
0, & x = 0,
\end{cases} 
$$

(55)

and

$$
\phi_d(x) \equiv 0. 
$$

(56)

It can be seen from the structure of the feedback control law (55) and (56) that the continuous-time closed-loop system dynamics are essentially nonnegative and the closed-loop system states after resettings are nonnegative. Furthermore, since $\alpha(x) - w_c(V(x)) = (V(x))^{1/2}, x \in \mathbb{R}_+^n$, the continuous-time feedback controller $\phi_c(\cdot)$ is fully independent from $f_c(x)$ which represents the internal interconnections of the large-scale system dynamics, and hence, is robust against full modeling uncertainty in $f_c(x)$. Finally, it follows from Theorem 4.1 that the closed-loop system (53)–(56) is finite-time stable.

For the following simulation we consider (53) and (54) with $\sigma_{cij}(x) = \sigma_{cij}x_ix_j$ and $\sigma_{dij}(x) = \sigma_{dij}x_j$, and $G_{ci}(x_i) = x_i^{1/4}$, where $\sigma_{cij} \geq 0, i \neq j, i, j = 1, \ldots, n, \sigma_{dij} \geq 0, i \neq j, i, j = 1, \ldots, n$, and $1 \geq \sum_{j=1, j \neq i}^n \sigma_{dij}, i = 1, \ldots, n$. To show that the conditions of Proposition 4.1 are satisfied for the case when $q = 1$, let $u_c = [-x_1^{1/4}, \ldots, -x_n^{1/4}]^T$ and note that

$$
\alpha(x) + \beta(x)u_c = -\sum_{i=1}^n x_i^{1/2} < -\left(\sum_{i=1}^n x_i^{1/2}\right) = -(V(x))^{1/2} = w_c(V(x)), x \in \mathbb{R}_+^n, x \neq 0.
$$

(57)

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $0 < \|x\| < \delta$ there exists $u_c \in \mathbb{R}_+^n$ such that $\|u_c\| < \varepsilon$ and $\alpha(x) + \beta^T(x)u_c < w_c(V(x))$, and hence, the continuous-time feedback control law (55) is continuous on $\mathbb{R}_+^n$. For our simulation we set $n = 2, \sigma_{c12} = 2, \sigma_{c21} = 1.5, \sigma_{d12} = 0.25, \sigma_{d21} = 0.33, c_0 = 1,$ and
\( Z = \{ x \in \mathbb{R}^2, x \neq 0 : x_2 - \frac{1}{3} x_1 = 0 \} \), with initial condition \( x_0 = [4, 6]^T \). Fig. 5.2 shows the states of the closed-loop system versus time and Fig. 5.3 shows continuous-time control signal as a function of time. Fig. 5.4 shows the phase portrait of the closed-loop system where the dashed line denotes the resetting set \( Z \).

6. Conclusion

The notion of finite-time stability was extended to impulsive dynamical systems using scalar and vector Lyapunov functions. These results were used to develop finite-time stabilizing controllers for nonlinear impulsive systems using scalar and vector control Lyapunov functions. A hybrid decentralized feedback stabilizer was also developed for a class of large-scale impulsive dynamical systems with robustness guarantees against full modelling uncertainty. Finally, it is important to note that finite-time stability and stabilization can also be achieved by imposing additional conditions on the discrete-time dynamics. In this case, deadbeat controllers \([28]\) can be used to achieve finite-time stabilization for impulsive dynamical systems.

References