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Fixed time stability and optimal stabilisation of discrete autonomous systems

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ABSTRACT

Unlike finite time stability, wherein the upper bound of the settling-time function capturing the finite settling time behaviour of the dynamical system depends on the system initial conditions, fixed time stability involves finite time stable systems for which the minimum bound of the settling-time function is guaranteed to be independent of the system initial conditions and can a priori be adjusted. In this paper, we develop several fixed time stability results for discrete autonomous systems including a fixed-time Lyapunov theorem that involves a Lyapunov difference that satisfies an exponential inequality of the Lyapunov function giving rise to a minimum bound on the settling-time function characterised by the primary and secondary branches of the Lambert $W$ function. Using these results, we develop an optimal control framework by exploiting connections between Lyapunov theory for fixed time stability and Bellman optimal control theory. In particular, we show that fixed time stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that serves as the solution to the steady state Bellman equation guaranteeing both fixed time stability and optimality.

1. Introduction

In a recent paper by Haddad and Lee (2020), the authors built on the results of Bhat and Bernstein (2000) to develop the notion of finite time stability for discrete-time autonomous systems. Specifically, Lyapunov and converse Lyapunov theorems for finite time stability involving scalar difference fractional inequalities and minimum operators were established and the regularity properties of the Lyapunov functions were shown to be dependent on the regularity properties of the settling-time function capturing the finite settling time behaviour of the discrete-time dynamical system. An inherent drawback of finite time stability is that the settling-time function depends on the system initial conditions, and hence, the time of convergence to a Lyapunov stable equilibrium point may increase (possibly unboundedly) as the vector norm of the initial condition increases.

For continuous-time dynamical systems, the stronger notion of fixed time stability was developed in Andrieu et al. (2008), Polyakov (2012), Polyakov et al. (2015), Rios et al. (2017), Sánchez-Torres et al. (2017), Jiménez-Rodriguez et al. (2017), and Song et al. (2019) to ensure convergence of the system trajectories to a finite-time stable equilibrium point in a fixed-time for any system initial condition. More specifically, the settling-time function of a fixed-time stable system is uniformly bounded regardless of the system initial conditions.

In this paper, we build on the work of Haddad and Lee (2020) to develop the notion of fixed time stability for discrete-time dynamical systems. Specifically, we develop Lyapunov theorems for fixed time stability of discrete autonomous systems involving scalar difference inequalities and minimum operators that allow for the adjustment of a guaranteed settling time that is independent of the system initial conditions. Our first fixed time stability theorem involves a Lyapunov difference inequality with parameters that explicitly define an upper bound on the system convergence time. Next, we construct a fixed-time stability theorem that involves a Lyapunov difference that satisfies an exponential inequality of the Lyapunov function giving rise to a minimum bound on the settling-time function characterised by the primary and secondary branches of the Lambert $W$ function (Euler, 1783).

Next, building on the fixed time stability analysis results in the first part of the paper, we develop optimal and inverse optimal fixed time controllers minimising a nonlinear-nonquadratic cost functional. In particular, an optimal fixed time control problem for discrete-time nonlinear systems is stated and sufficient Bellman optimality conditions are used to characterise an optimal feedback controller. The steady state solution of the Bellman equation serves as a Lyapunov function for the closed-loop, discrete-time system and additionally satisfies a difference inequality involving an exponential of the Lyapunov function guaranteeing both fixed time stability and optimality. Furthermore, we explore connections of our approach with inverse optimal control (see Freeman & Kokotovic, 1996; Jacobson, 1977; Molinari, 1973; Moylan & Anderson, 1973; Sepulchre et al., 1997), wherein we parametrise a family of fixed time stabilising controllers that minimise a derived cost functional involving a combination of quadratic and subquadratic terms. Finally, several illustrative numerical examples are provided to illustrate the proposed framework.

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2. Mathematical preliminaries

In this section, we present notation, definitions, and some key results needed for developing the main results of this paper. Let \( \mathbb{R} \) denote the set of real numbers, \( \mathbb{R}^n \) denote the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denote the set of \( n \times m \) real matrices, \( \mathbb{Z} \) denote the set of integers, \( \mathbb{Z}_+ \) (resp., \( \mathbb{Z}_+^\ast \)) denote the set of positive (resp., nonnegative) integers, and \( (\cdot)^\ast \) denote transpose. We write \( ||\cdot|| \) for the Euclidean vector norm in \( \mathbb{R}^n \), \( B_s(x) \) for the open ball centred at \( x \) with radius \( s \) in the Euclidean norm, \( \Delta V(x) \) for the difference operator of \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) at \( x \in \mathbb{R}^n \), and \([\alpha]\) for the ceiling function denoting the smallest integer greater than or equal to \( \alpha \). Finally, we write \( \mathcal{S} \) for the closure and \( \delta \mathcal{S} \) for the boundary of the set \( \mathcal{S} \subset \mathbb{R}^n \).

In this paper, we consider discrete-time nonlinear dynamical systems of the form

\[
x(k + 1) = f(x(k)), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \tag{1}
\]

where \( x(k) \in D \subseteq \mathbb{R}^n, k \in \mathbb{Z}_+ \), is the system state vector, \( D \) is an open set, \( 0 \in D \), \( f : D \rightarrow D \) is continuous, and \( f(0) = 0 \). We denote the solution to (1) with initial condition \( x(0) = x_0 \) by \( s_t(x_0) \) so that the map of the dynamical system given by \( s : \mathbb{Z}_+ \times D \rightarrow D \) is continuous on \( D \) and satisfies the consistency property \( s(0, x_0) = x_0 \) and the semigroup property \( s(k, s(k, x_0)) = s(k + \kappa, x_0) \) for all \( x_0 \in D \) and \( k, \kappa \in \mathbb{Z}_+ \). We use the notation \( s(k, x_0), k \in \mathbb{Z}_+, \) and \( x(k), k \in \mathbb{Z}_+ \), interchangeably to denote the solution of the nonlinear discrete-time dynamical system (1) with initial condition \( x(0) = x_0 \). By a solution to (1) with initial condition \( x(0) = x_0 \) we mean a function \( x : \mathbb{Z}_+ \rightarrow D \) that satisfies (1). Given \( k \in \mathbb{Z}_+ \) and \( x \in D \), we denote the map \( s(k, \cdot) : D \rightarrow D \) by \( s_k \) and the map \( s(\cdot, x) : \mathbb{Z}_+ \rightarrow D \) by \( s^x \).

The following definition introduces the notion of finite time stability for discrete-time systems.

**Definition 2.1** (Haddad & Lee, 2020): Consider the nonlinear dynamical system (1). The zero solution \( x(k) \equiv 0 \) to (1) is finite time stable if there exist an open neighbourhood \( N \subseteq D \) of the origin and a function \( K : N \setminus \{0\} \rightarrow \mathbb{Z}_+ \), called the settling-time function, such that the following statements hold:

(i) **Finite time convergence.** For every \( x \in N \setminus \{0\} \), \( s^x(k) \in N \setminus \{0\} \) is defined on \( k \in [0, \ldots, K(x) - 1] \), and \( s(k, x) = 0, k \geq K(x) \).

(ii) **Lyapunov stability.** For every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( B_{\delta}(0) \subset N \) and for every \( x \in B_{\delta}(0) \setminus \{0\} \), \( s(k, x) \in B_{\varepsilon} \) for all \( k \in [0, \ldots, K(x) - 1] \).

The zero solution \( x(k) \equiv 0 \) to (1) is globally finite time stable if it is finite time stable with \( N = D = \mathbb{R}^n \).

Since forward uniqueness is always guaranteed for finite time discrete-time systems, we can extend \( K(\cdot) \) to all of \( N \) by defining \( K(0) \equiv 0 \) for the equilibrium point \( x_0 = 0 \). For details, see Haddad and Lee (2020). Note that it follows from Definition 2.1 that

\[
K(x) = \min\{k \in \mathbb{Z}_+ : s(k, x) = 0\}, \quad x \in N. \tag{2}
\]

The following proposition shows that finite time stability implies that the settling-time function \( K(\cdot) \) is lower semicontinuous on \( N \).

**Proposition 2.1** (Haddad & Lee, 2020): Consider the nonlinear dynamical system (1). Assume that the zero solution \( x(k) \equiv 0 \) to (1) is finite time stable, let \( N \subseteq D \) be as in Definition 2.1, and let \( K : N \rightarrow \mathbb{Z}_+ \) be the settling-time function. Then \( K(\cdot) \) is lower semicontinuous on \( N \).

Next, we provide sufficient Lyapunov conditions for finite time stability of the nonlinear dynamical system given by (1). For stating this result, as well as several of the results in this paper, define \( \Delta V(x) \equiv V(f(x)) - V(x) \) for a given continuous function \( V : D \rightarrow \mathbb{R} \).

**Theorem 2.1** (Haddad & Lee, 2020): Consider the nonlinear dynamical system (1). Assume that there exist a continuous function \( V : D \rightarrow \mathbb{R} \), real numbers \( \alpha \in (0, 1) \) and \( c > 0 \), and a neighbourhood \( M \subseteq D \) of the origin such that

\[
\begin{align*}
V(0) & = 0, \\
V(x) & > 0, \quad x \in M \setminus \{0\}, \\
\Delta V(x) & \leq -c \min \left\{ \frac{V(x)}{c}, V(x)^\alpha \right\}, \quad x \in M \setminus \{0\}.
\end{align*}
\]

Then the zero solution \( x(k) \equiv 0 \) to (1) is finite time stable. Moreover, there exist an open neighbourhood \( N \) of the origin and a settling-time function \( K : N \rightarrow \mathbb{Z}_+ \) such that either

\[
K(x_0) \leq \left( \frac{\log_\alpha \left( 1 - c \right)}{c V(x_0)^{\alpha - 1}} \right) + 1, \quad x_0 \in N, \tag{6}
\]

or

\[
K(x_0) = 1, \quad x_0 \in N \setminus \{0\}, \quad V(x_0) \leq c \frac{1}{1-\alpha}, \tag{7}
\]

where \( K(\cdot) \) is lower semicontinuous on \( N \). If, in addition, \( N = D = \mathbb{R}^n \), \( V(\cdot) \) is radially unbounded, and (5) holds on \( \mathbb{R}^n \), then the zero solution \( x(k) \equiv 0 \) to (1) is globally finite time stable.

**Remark 2.1:** It is important to note that Theorem 2.1 also holds for the case where (5) is replaced by

\[
\Delta V(x) \leq -\min \{V(x), c\}, \quad x \in M \setminus \{0\},
\]

and with the lower semicontinuous settling-time function \( K : N \rightarrow \mathbb{Z}_+ \) given by

\[
K(x_0) \leq \frac{V(x_0)}{c}, \quad x_0 \in N. \tag{9}
\]

For details of this fact, see Theorem 4.2 of Haddad and Lee (2020).

The following example shows that the settling-time function of a finite-time stable system depends on the system initial conditions and may increase (possibly unboundedly) as the vector norm of the initial condition increases.
Example 2.1: Consider the scalar discrete-time nonlinear dynamical system given by
\[ x(k+1) = x(k) - c \cdot \text{sign}(x(k)) \cdot \min \left\{ \frac{|x(k)|}{c}, |x(k)|^\alpha \right\}, \]
\[ x(0) = x_0, \quad k \geq 0, \tag{10} \]
where \( x(k) \in \mathbb{R}, \ k \in \mathbb{Z}^+, \ \text{sign}(x) \triangleq x/|x|, \ x \neq 0, \ \text{sign}(0) \triangleq 0, \ \alpha \in (0,1), \) and \( c > 0. \) To show that the zero solution \( x(k) \equiv 0 \) to (10) is globally finite-time stable using Theorem 2.1, consider the radially unbounded Lyapunov function candidate
\[ V(x) = |x| \text{ and let } |x_0| > c \frac{1}{1-\alpha}. \] (Note that if \( |x_0| \leq c \frac{1}{1-\alpha}, x_0 \neq 0, \) then the zero solution \( x(k) \equiv 0 \) to (10) is finite-time stable with \( K(x_0) = 1. \) Since \( |x| \geq c \min \left\{ \frac{|x|}{c}, |x|^\alpha \right\}, \) it follows that
\[ \Delta V(x) = \left| x - c \cdot \text{sign}(x) \cdot \min \left\{ \frac{|x|}{c}, |x|^\alpha \right\} \right| - |x| \]
\[ = - c \cdot \text{sign}(x) \cdot \min \left\{ \frac{|x|}{c}, |x|^\alpha \right\}, \]
\[ = - c \min \left\{ \frac{V(x)}{c}, V(x)^\alpha \right\}, \quad x \in \mathbb{R} \setminus \{0\}. \tag{11} \]
Hence, it follows from Theorem 2.1 that the zero solution \( x(k) \equiv 0 \) to (10) is globally finite-time stable with the settling-time function
\[ K(x_0) \leq \left[ \log_{\left[1 - c \cdot V(x_0)^{\alpha - 1}\right]} \left( \frac{1}{c \cdot V(x_0)^{\alpha}} \right) \right] + 1. \tag{12} \]
Figure 1 shows that the settling-time function of (10), with \( c = 2 \) and \( \alpha = 0.5, \) depends on the initial conditions of (10), and hence, \( K(x_0) \) for finite time stable systems can be an unbounded function of the system initial conditions.

3. Fixed time stability of discrete-time nonlinear dynamical systems

In this section, we develop the notion of fixed time stability for discrete-time nonlinear dynamical systems and develop sufficient conditions for fixed time stability using Lyapunov theorems satisfying a scalar difference fractional inequality and a minimum operator. The notion of fixed time stability involves finite time stability along with a bounded settling-time function as detailed in the next definition.

Definition 3.1: Consider the nonlinear dynamical system (1). The zero solution \( x(k) \equiv 0 \) to (1) is fixed time stable if there exist an open neighbourhood \( \mathcal{N} \subseteq \mathcal{D} \) of the origin and a settling-time function \( K : \mathcal{N} \rightarrow \mathbb{R}_+ \) such that the following statements hold:

(i) Finite time stability. The zero solution \( x(k) \equiv 0 \) to (1) is finite time stable.

(ii) Uniform boundedness of the settling time function. For every \( x \in \mathcal{N}, \) there exists \( K_{\text{max}} > 0 \) such that \( K(x) \leq K_{\text{max}}. \)

The zero solution \( x(k) \equiv 0 \) to (1) is globally fixed time stable if it is fixed time stable with \( \mathcal{N} = \mathcal{D} = \mathbb{R}^n. \)

Remark 3.1: The key difference between the notions of fixed time stability and finite time stability lies in the fact that in finite time stability the upper bound of the settling-time function depends on the system initial condition \( x_0, \) whereas the settling-time function for fixed time stable systems is uniformly upper bounded for all initial conditions \( x_0. \)

The next theorem provides sufficient conditions for fixed time stability of discrete-time dynamical systems.

Theorem 3.1: Consider the nonlinear dynamical system (1). Assume that there exist a continuous function \( V : \mathcal{D} \rightarrow \mathbb{R}, \) real
numbers $\alpha \in (0,1)$, $\beta > 1$, $c \in (0,1)$, and $d \in (0,1)$, and a neighbourhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(0) = 0,$$

$$V(x) > 0, \quad x \in \mathcal{M} \setminus \{0\},$$

$$\Delta V(x) \leq -\min \{ cV(x)^{\alpha} + dV(x)^{\beta}, V(x) \}, \quad x \in \mathcal{M}. \quad (15)$$

Then the zero solution $x(k) \equiv 0$ to (1) is fixed time stable. Moreover, there exist an open neighbourhood $\mathcal{N}$ of the origin and a settling-time function $K : \mathcal{N} \rightarrow \mathbb{Z}_+$ such that

$$K(x_0) \leq \left[ \log_{1-c} \left( \frac{1}{c^{1/\alpha}} \right) + \left| \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right| + 1 \right], \quad x_0 \in \mathcal{N}, \quad (16)$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (15) holds on $\mathbb{R}^n$, then the zero solution $x(k) \equiv 0$ to (1) is globally fixed time stable.

**Proof:** First note that since $V(\cdot)$ is positive definite and

$$\Delta V(x) \leq -\min \{ cV(x)^{\alpha} + dV(x)^{\beta}, V(x) \} \leq -\min \{ cV(x)^{\alpha}, V(x) \}, \quad x \in \mathcal{M}, \quad (17)$$

it follows from Theorem 2.1 that the zero solution $x(k) \equiv 0$ to (1) is finite time stable and $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$.

Next, consider the scalar discrete-time nonlinear dynamical system given by

$$z(k + 1) = z(k) - d \text{sign}(z(k)) \min \left\{ \frac{|z(k)|}{d}, |z(k)|^{\beta} \right\},$$

$$z(0) = z_0, \quad k \in \mathbb{Z}_+, \quad (18)$$

where $z(k) \in \mathbb{R}$, $k \in \mathbb{Z}_+$, $\beta > 1$, and $d \in (0,1)$. Note that the right-hand side of (18) is continuous everywhere and, for every initial condition in $\mathbb{R}$, (18) has a unique solution in forward time. Furthermore, note that if $|z(k)| \geq d^{\frac{1}{\beta-1}}$, $k \in \mathbb{Z}_+$, then $z(k + 1) = 0$, and if $|z(k)| < d^{\frac{1}{\beta-1}}$, $k \in \mathbb{Z}_+$, then

$$|z(k)| = |z(k - 1)(1 - d|z(k - 1)|^{\beta-1})| < |z(k - 1)|,$$

$$k \in \mathbb{Z}_+. \quad (19)$$

Now, note that if $|z| > 1$, then, since $\beta > 1$,

$$1 - d|z|^{\beta-1} < 1 - d, \quad (20)$$

and hence, it follows from (19), (20), and $|z(k)| < d^{\frac{1}{\beta-1}}$, $k \in \mathbb{Z}_+$, that

$$|z(k)| \equiv |z(k - 1)| (1 - d|z(k - 1)|^{\beta-1})$$

$$= |z(k - 2)| (1 - d|z(k - 2)|^{\beta-1}) (1 - d|z(k - 1)|^{\beta-1})$$

$$\vdots$$

$$= |z_0| (1 - d|z_0|^{\beta-1}) \cdots (1 - d|z(k - 1)|^{\beta-1})$$

$$< |z_0|(1 - d)^k, \quad k \in \mathbb{Z}_+. \quad (21)$$

Next, note that if $k = \left\lfloor \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right\rfloor$, then, since $|z_0| < d^{\frac{1}{\beta-1}}$,

$$|z(k)| < |z_0|d^{\frac{1}{\beta-1}} < 1, \quad (22)$$

and hence,

$$|z(k)| < 1, \quad k \geq \left\lfloor \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right\rfloor. \quad (23)$$

Now, note that

$$\Delta V(x) \leq -\min \{ cV(x)^{\alpha} + dV(x)^{\beta}, V(x) \} \leq -\min \{ dV(x)^{\beta}, V(x) \}, \quad x \in \mathcal{M},$$

and hence, it follows from Corollary 2.1 of Haddad and Lee (2020) and (23), with $w(V(x)) = -\min \{dV(x)^{\beta}, V(x)\}$ and $z(k) = V(x(k))$, that

$$V(x(k)) \leq 1, \quad k \geq \left\lfloor \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right\rfloor, \quad (25)$$

Thus, it follows from (6), with $V(x(0)) = 1$, that the zero solution $x(k) \equiv 0$ to (1) is fixed time stable with a settling-time function $K : \mathcal{N} \rightarrow \mathbb{Z}_+$ satisfying

$$K(x_0) \leq \left[ \log_{1-c} \left( \frac{1}{c^{1/\alpha}} \right) + \left| \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right| + 1 \right], \quad x_0 \in \mathcal{N}, \quad c^{\frac{1}{\alpha}} < V(x_0) < d^{\frac{1}{\beta}}. \quad (26)$$

Alternatively, it follows from Theorem 2.1 that

$$K(x_0) = 1, \quad x_0 \in \mathcal{N} \setminus \{0\}, \quad V(x_0) \leq c^{\frac{1}{\alpha}}, \quad V(x_0) \geq d^{\frac{1}{\beta}}. \quad (27)$$

Since $\left\lfloor \log_{1-c} \left( \frac{1}{c^{1/\alpha}} \right) \right\rfloor + \left\lfloor \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right\rfloor + 1 \geq 1,

$$K(x) \leq K_{\text{max}} = \left[ \log_{1-c} \left( \frac{1}{c^{1/\alpha}} \right) + \left| \log_{1-d} \left( \frac{1}{d^{1/\beta}} \right) \right| + 1 \right], \quad x \in \mathcal{N}, \quad (28)$$

and hence, it follows from Definition 3.1 that the zero solution $x(k) \equiv 0$ to (1) is fixed time stable.

Finally, if $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then global fixed time stability follows using standard arguments.

The following example shows that the settling-time function for a fixed-time stable system is uniformly bounded for all system initial conditions.

**Example 3.1:** Consider the scalar discrete-time nonlinear dynamical system given by

$$x(k + 1) = x(k) - \text{sign}(x(k)) \min \{ c|x(k)|^\alpha$$

$$+ d|x(k)|^\beta, |x(k)| \}, \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad (29)$$

where $x(k) \in \mathbb{R}$, $k \in \mathbb{Z}_+$, $\alpha \in (0,1)$, $\beta > 1$, $c \in (0,1)$, and $d \in (0,1)$. Note that the right-hand side of (29) is continuous everywhere and, for every initial condition in $\mathbb{R} \setminus \{0\}$, (29) has a unique solution in forward time. Furthermore, note that if
Consider the Lyapunov function candidate $V(x) = |x|$ and, since $|x| \geq \min\{c|x|^\alpha + d|x|^\beta, |x|\}$, note that

$$
\Delta V(x) = |x - \text{sign}(x) \min\{c|x|^\alpha + d|x|^\beta, |x|\} - |x| \\
= - \min\{|c|x|^\alpha + d|x|^\beta, |x|\} \\
= - \min\{cV(x)^\alpha + dV(x)^\beta, V(x)\}, \quad x \in \mathbb{R}\setminus\{0\}.
$$

Hence, it follows from Theorem 3.1 that the zero solution $x(k) \equiv 0$ to (29) is globally fixed time stable with

$$K_{\text{max}} = \left[\log_{1-c} e^{\frac{1}{1-\alpha}}\right] + \left[\log_{1-d} d^{\frac{1}{1-\beta}}\right] + 1.
$$

Figure 2 shows that the settling time function of (29), with $c = 0.75, d = 0.45, \alpha = 0.1$, and $\beta = 1.1$, is uniformly bounded and satisfies $K(x) \leq 6$. In this case, (16) holds with

$$K_{\text{max}} = \left[\log_{1-c} e^{\frac{1}{1-\alpha}}\right] + \left[\log_{1-d} d^{\frac{1}{1-\beta}}\right] + 1 = 16.
$$

Note that it follows from Theorem 3.1 that if $V(x) \leq c^{\frac{1}{1-\alpha}}$ or $V(x) \geq d^{\frac{1}{1-\beta}}$, for a given $\alpha \in (0, 1)$, $\beta > 1$, $c \in (0, 1)$, and $d \in (0, 1)$, then $cV(x)^\alpha + dV(x)^\beta \geq V(x)$. In this case, (15) becomes $\Delta V(x) \leq -V(x), x \in \mathcal{M}$, and hence, $V(f(x)) = 0$. If $V(x) \geq d^{\frac{1}{1-\beta}}$, then $V(f(x)) = 0$ implies finite time convergence to the origin can be achieved in a single time step for a large enough initial condition. This is not particular to fixed time stability theory for discrete-time systems, but is also seen in the case of fixed time stability for continuous-time systems with the key difference being that instead of a global jump to the origin we have a limiting impulsive jump (i.e. a continuous albeit very sharp jump) to the origin (see Jiménez-Rodríguez et al., 2017).

The achieved settling-time clearly depends on the values of $\beta$ and $d$, and the global jump to the origin can be subverted by choosing different values of $\beta$ and $d$. Figure 3 shows the interplay between $\beta$ and $d$, and the achieved settling time to the origin of (29) for a fixed $K_{\text{max}}$ and different system initial conditions. Note that in Cases I and II, the achieved settling time increases as initial condition increases for $x_0 \in [0, 5000]$ and the chosen $\beta$ and $d$ with uniform guaranteed upper bounds given by $K_{\text{max}} = 51$ and $K_{\text{max}} = 19$, respectively. However, in Cases III and IV the uniform guaranteed upper bound of the settling time function is $K_{\text{max}} = 8$ and $K_{\text{max}} = 7$, respectively, for the chosen $\beta$ and $d$, while the achieved settling time decreases after the large initial conditions $x_0 = 2506$ and $x_0 = 459$, respectively. A similar trend holds for fixed time stability for continuous-time systems.

4. Fixed time stability using a Lambert function construction

In this section, we present an alternative Lyapunov sufficient condition for fixed time stability using a Lambert $W$ function construction.

**Definition 4.1:** Let $z \in \mathbb{C}$. The Lambert $W$ function $W(z)$ is a multivalued function satisfying $we^w = z$, where $w = W(z)$.

Note that if $z \in \mathbb{R}$ and $z < -1/e$, then $W(z)$ is a multivalued complex number. Alternatively, if $z \in \mathbb{R}$ and $-1/e \leq z < 0$, then there are two possible real values for $W(z)$; namely, the primary branch denoted by $W_0(z)$, where $W_0(z) \geq -1$, and the secondary branch denoted by $W_{-1}(z)$, where $W_{-1}(z) \leq -1$. Note that the two branches meet at the branch point $z = -1/e$ (see Figure 4).
Figure 3. Interplay between $\beta$ and $d$, and achieved finite time convergence for different values of initial conditions. (a) Case I, $K_{\text{max}} = 51$. (b) Case II, $K_{\text{max}} = 19$. (c) Case III, $K_{\text{max}} = 8$ and (d) Case IV, $K_{\text{max}} = 7$.

Figure 4. The primary and secondary branches of the Lambert $W$ function.

Remark 4.1: The form $z = w e^w$ of the Lambert $W$ function was first introduced by Euler (1783) as a special case of the Lambert transcendental equation (Lambert, 1758). Although the Lambert $W$ function cannot be expressed by elementary functions, it provides a useful closed-form expression for the solution to a large-class of transcendental functions involving exponentials. In particular, it is widely used in circuit analysis (see Batzelis et al., 2014; Bernardini et al., 2016) as well as system thermodynamics (Shafee, 2007), where it is used to define the system entropy.
The following example illustrates the notion of fixed time stability of the scalar system using the Lambert $W$ function.

**Example 4.1:** Consider the scalar discrete-time nonlinear dynamical system given by

\[ x(k + 1) = x(k) - \text{sign}(x(k)) \min \left\{ \frac{1}{\kappa_f} e^{\left| x(k) \right|}, |x(k)| \right\}, \]

\[ x(0) = x_0, \quad k \in \mathbb{Z}_+, \]  

(31)

where $x(k) \in \mathbb{R}$, $k \in \mathbb{Z}_+$, and $\kappa_f > e$. Note that the right-hand side of (31) is continuous everywhere and, for every initial condition in $\mathbb{R}$, (31) has a unique solution in forward time. Moreover, note that Lyapunov stability of the zero solution $x(k) \equiv 0$ to (31) follows by considering the radially unbounded Lyapunov function $V(x) = x^2$. Furthermore, note that if $1/\kappa_f e^{\left| x(k) \right|} \geq |x(k)|, k \in \mathbb{Z}_+$, then $x(k + 1) = 0$.

Alternatively, note that if $1/\kappa_f e^{\left| x \right|} \leq |x|$, then

\[ e^{-|x|}(-|x|) \leq -\frac{1}{\kappa_f}. \]  

(32)

Taking the Lambert $W$ function on both sides of (32) yields

\[ W \left( e^{-|x|}(-|x|) \right) \leq W \left( -\frac{1}{\kappa_f} \right), \]

and since $-1/\kappa_f \geq -1/e$, it follows from the definition of Lambert $W$ function that

\[ -W_0 \left( -\frac{1}{\kappa_f} \right) \leq |x| \leq -W_{-1} \left( -\frac{1}{\kappa_f} \right). \]  

(33)

Next, if $-W_0(-1/\kappa_f) < |x(k)| < -W_{-1}(-1/\kappa_f), k \in \mathbb{Z}_+$, then

\[ |x(k)| = |x(k - 1) - \text{sign}(x(k - 1)) \frac{1}{\kappa_f} e^{\left| x(k - 1) \right|}| \]

\[ = |x(k - 1) \left[ 1 - \frac{1}{\kappa_f} \exp \left| \frac{\left| x(k - 1) \right|}{\kappa_f} \right| \right]| \]

\[ \leq |x(k - 1) \left( 1 - \frac{e}{\kappa_f} \right)| \]

\[ \vdots \]

\[ \leq \left| x_0 \left( 1 - \frac{e}{\kappa_f} \right)^k \right|, \quad k \in \mathbb{Z}_+. \]  

(34)

Now, if $|x_0(1 - \frac{e}{\kappa_f})| \leq W_0(-1/\kappa_f), k \in \mathbb{Z}_+$, then $|x(k)| \leq -W_0(-1/\kappa_f), k \in \mathbb{Z}_+$, which implies $x(k + 1) = 0$ for

\[ k \geq \log_{1-e/\kappa_f} \frac{W_0(-1/\kappa_f)}{W_{-1}(-1/\kappa_f)}. \]

Therefore, the zero solution $x(k) \equiv 0$ to (31) is globally fixed time stable with $K_{\text{max}}$ given by

\[ K_{\text{max}} = \left[ \log_{1-e/\kappa_f} \frac{W_0(-1/\kappa_f)}{W_{-1}(-1/\kappa_f)} \right] + 1, \]

\[ -W_0(-1/\kappa_f) < |x| < -W_{-1}(-1/\kappa_f). \]

Finally, since $[\log_{1-e/\kappa_f} \frac{W_0(-1/\kappa_f)}{W_{-1}(-1/\kappa_f)}] + 1 \geq 1$, it follows that

\[ K_{\text{max}} = \left[ \log_{1-e/\kappa_f} \frac{W_0(-1/\kappa_f)}{W_{-1}(-1/\kappa_f)} \right] + 1, \quad x \in \mathbb{R}. \]  

\[ \Box \]

**Theorem 4.1:** Consider the nonlinear dynamical system (1). Assume that there exist a continuous function $V : \mathcal{D} \rightarrow \mathbb{R}$, real numbers $p \in (0, 1]$ and $\kappa_1 > e$, and a neighbourhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

\[ V(0) = 0, \]

\[ V(x) > 0, \quad x \in \mathcal{M} \setminus \{0\}, \]

\[ \Delta V(x) \leq -\min \left\{ \frac{1}{\kappa_f} \exp(V(x)^p) V(x)^{-p}, V(x) \right\}, \quad x \in \mathcal{M}. \]

Then the zero solution $x(k) \equiv 0$ to (1) is fixed time stable. Moreover, there exist an open neighbourhood $\mathcal{N}$ of the origin and a real number $K_{\text{max}} > 0$ such that a settling-time function $K : \mathcal{N} \rightarrow \mathbb{Z}_+$ satisfies

\[ K(x_0) \leq K_{\text{max}} = \left[ \frac{1}{p} \log_{1-e/\kappa_1} \frac{W_0(-1/\kappa_1)}{W_{-1}(-1/\kappa_1)} \right] + 1, \quad x_0 \in \mathcal{N}, \]  

(38)

where $K()$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n, V()$ is radially unbounded, and (37) holds on $\mathbb{R}^n$, then the zero solution $x(k) \equiv 0$ to (1) is globally fixed time stable.

**Proof:** Since $V()$ is positive definite and $\Delta V()$ takes negative values on $\mathcal{M} \setminus \{0\}$, it follows from Lyapunov stability that $x(k) \equiv 0$ is the unique solution of (1) for $k \in \mathbb{Z}_+$ satisfying $x(0) = 0$. Thus, for every initial condition in $\mathcal{D}, (1)$ has a unique solution in forward time.

Let $\mathcal{V} \subseteq \mathcal{M}$ be a bounded open set with $0 \in \mathcal{V}$ and $\overline{\mathcal{V}} \subset \mathcal{D}$, and note that $\partial \mathcal{V}$ is compact and $0 \notin \partial \mathcal{V}$. Now, it follows from Weierstrass’ theorem (Haddad & Chellaboina, 2008, p. 44) that since $V()$ is continuous, it attains a minimum on $\partial \mathcal{V}$ and, since $V()$ is positive definite, $\min_{x \in \partial \mathcal{V}} V(x) > 0$. Let $0 < \beta < \min_{x \in \partial \mathcal{V}} V(x)$ and define the set $\mathcal{D}_\beta$ to be the argwise connected component of the set $\{x \in \mathcal{V} : V(x) \leq \beta\}$. $\mathcal{D}_\beta$ is nonempty, closed, and bounded since $0 \in \mathcal{D}_\beta$, $V()$ is continuous, and $\mathcal{V}$ is bounded.

Furthermore, it follows from (37) that $\mathcal{D}_\beta \subset \mathcal{M}$ is positively invariant with respect to (1). In addition, since $V()$ is positive definite and (37) holds, it follows from standard Lyapunov arguments that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_\beta \subset \mathcal{M}$ and

\[ \|x(k)\| \leq \varepsilon, \quad \|x_0\| < \delta, \quad k \in \mathcal{I}_{x_0}, \]  

(39)

where $\mathcal{I}_{x_0} \subseteq \mathbb{Z}$ is the maximal interval of existence of a solution $z(k)$ to (1). Moreover, since the solution $x(k)$ to (1) is bounded for all $k \in \mathcal{I}_{x_0}$, it can be extended on $\mathbb{Z}_+$, and hence, $x(k)$ is defined for all $k \geq k_0$. 

Now, consider the scalar discrete-time nonlinear dynamical system given by

$$z(k + 1) = z(k) - \text{sign}(z(k)) \min \left\{ \frac{1}{\kappa f} e^{z(k)} |z(k)|^{1 - p}, |z(k)| \right\},$$

$$z(0) = z_0, \quad k \in \mathbb{Z}_+,$$  \hspace{1cm} (40)

where $z(k) \in \mathbb{R}, \ k \in \mathbb{Z}_+, \ p \in (0, 1], \text{ and } \kappa f > e$. Note that the right-hand side of (40) is continuous everywhere and, for every initial condition in $\mathbb{R}$, (40) has a unique solution in forward time. Moreover, note that Lyapunov stability of the zero solution $z(k) \equiv 0$ to (40) follows by considering the radially unbounded Lyapunov function $V(z) = z^2$. Furthermore, note that if $1/\kappa f e^{\lambda z(k)} |z(k)|^{1 - p} \geq |z(k)|, \ k \in \mathbb{Z}_+$, then $z(k + 1) = 0$.

Alternatively, note that if $1/\kappa f e^{\lambda z(k)} |z(k)|^{1 - p} \leq |z|$, then

$$e^{-\lambda |z|} (-|z|)^p \leq -\frac{1}{\kappa f}.$$  \hspace{1cm} (41)

Taking the Lambert $W$ function on both sides of (41) yields

$$W\left( e^{-\lambda |z|} (-|z|)^p \right) \leq W\left( -\frac{1}{\kappa f} \right),$$

and since $-1/\kappa f \geq -1/e$, it follows from the definition of Lambert $W$ function that

$$\left[-W_0\left( -\frac{1}{\kappa f} \right)^{1/p} \right] \leq |z| \leq \left[-W_{-1}\left( -\frac{1}{\kappa f} \right)^{1/p} \right].$$  \hspace{1cm} (42)

Thus, if $-W_0\left( -\frac{1}{\kappa f} \right)^{1/p} < |z(k)| < -W_{-1}\left( -\frac{1}{\kappa f} \right)^{1/p}, \ k \in \mathbb{Z}_+$, then

$$|z(k)| = \left| z(k - 1) - \text{sign}(z(k - 1)) \frac{1}{\kappa f} e^{z(k - 1)} |z(k - 1)|^{1 - p} \right|$$

$$\leq \left| z(k - 1) \left( 1 - \frac{1}{\kappa f} \right) \right|$$

$$\leq \left| z_0 \left( 1 - \frac{e}{\kappa f} \right)^k \right|.$$  \hspace{1cm} (43)

Next, if $|z_0(1 - e/k)| \leq -W_0\left( -1/\kappa f \right)^{1/p}, \ k \in \mathbb{Z}_+$, then $|z(k)| \leq -W_0\left( -1/\kappa f \right)^{1/p}, \ k \in \mathbb{Z}_+$, which implies $z(k + 1) = 0$ for

$$k \geq \frac{1}{p} \log_{1 - e/\kappa f} \left\{ W_0\left( -\frac{1}{\kappa f} \right) \right\},$$

and hence, there exists a settling-time function for (40) such that $\hat{K}(z) \leq \hat{K}_{\text{max}}$, where

$$\hat{K}(z) \leq \hat{K}_{\text{max}} = \left[ \frac{1}{p} \log_{1 - e/\kappa f} \left\{ \frac{W_0\left( -\frac{1}{\kappa f} \right)}{W_{-1}\left( -\frac{1}{\kappa f} \right)} \right\} \right] + 1, \hspace{1cm} (44)$$

$$[-W_0\left( -1/\kappa f \right)^{1/p} \leq |z| < -W_{-1}\left( -1/\kappa f \right)^{1/p}.$$

Alternatively, if $|z| \leq -W_0\left( -1/\kappa f \right)^{1/p}$ or $|z| \geq -W_{-1}\left( -1/\kappa f \right)^{1/p}$, then

$$\hat{K}(z) = 1 \leq \left[ \frac{1}{p} \log_{1 - e/\kappa f} \left\{ \frac{W_0\left( -\frac{1}{\kappa f} \right)}{W_{-1}\left( -\frac{1}{\kappa f} \right)} \right\} \right] + 1,$$

and hence, the zero solution $z(k) \equiv 0$ to (40) is globally fixed time stable with

$$\hat{K}_{\text{max}} = \left[ \frac{1}{p} \log_{1 - e/\kappa f} \left\{ \frac{W_0\left( -\frac{1}{\kappa f} \right)}{W_{-1}\left( -\frac{1}{\kappa f} \right)} \right\} \right] + 1, \quad z \in \mathbb{R}.$$  \hspace{1cm} (45)

Now, it follows from Corollary 2.1 of Haddad and Lee (2020), with

$$w(V(x)) = V(x) - \min \left\{ \frac{1}{\kappa f} e^{V(x)} |V(x)|^{1 - p}, |V(x)| \right\},$$

that

$$V(x(k)) \leq s(k, V(x_0)), \quad x_0 \in B_{\delta}(0), \quad k \in \mathbb{Z}_+,$$  \hspace{1cm} (46)

where $s(\cdot, \cdot)$ is the solution to (40). Hence, it follows from (44), (45), and the positive definiteness of $V(\cdot)$ that

$$x(k) = 0, \quad k \geq \left[ \frac{1}{p} \log_{1 - e/\kappa f} \left\{ \frac{W_0\left( -\frac{1}{\kappa f} \right)}{W_{-1}\left( -\frac{1}{\kappa f} \right)} \right\} \right] + 1,$$

$$x_0 \in B_{\delta}(0),$$  \hspace{1cm} (46)

which implies fixed time convergence of the trajectories of (1) for all $x_0 \in B_{\delta}(0)$. This along with (39) implies fixed time stability of the zero solution $x(k) \equiv 0$ to (1) with $\mathcal{N} \triangleq B_{\delta}(0)$. Moreover, since fixed time stability implies finite time stability, it follows from Proposition 2.1 that the settling time function $K(\cdot)$ is lower semicontinuous.

Finally, if $\mathcal{N} = D = \mathbb{R}^p$ and $V(\cdot)$ is radially unbounded, then global fixed time stability follows using standard arguments. \hfill \blacksquare

**Example 4.2:** Consider the scalar discrete-time nonlinear dynamical system given by

$$x(k + 1) = x(k) - \text{sign}(x(k)) \min \left\{ \frac{1}{\kappa f} e^{x(k)p} \right\},$$

$$\cdot \left| x(k) \right|^{1 - p}, \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad x \in \mathbb{R}, \hspace{1cm} (47)$$

where $x(k) \in \mathbb{R}, \ k \in \mathbb{Z}_+, \ p \in (0, 1], \text{ and } \kappa f > e$. For this system, we show that the zero solution $x(k) \equiv 0$ to (47) is globally fixed time stable using Theorem 4.1. To see this, consider the Lyapunov function candidate $V(x) = |x|$ and let $[-W_0\left( -1/\kappa f \right)^{1/p} \leq |x_0| < -W_{-1}\left( -1/\kappa f \right)^{1/p}$. Since $|x| \geq$
Consider the spacecraft with one axis of symmetry.

Example 4.3: Consider the spacecraft with one axis of symmetry given by

\[
\begin{align*}
\dot{\omega}_1(t) &= I_{23} \omega_3(t) \omega_2(t) + u_1(t), & \omega_1(0) = \omega_{10}, & t \geq 0, \quad (49) \\
\dot{\omega}_2(t) &= -I_{23} \omega_3(t) \omega_1(t) + u_2(t), & \omega_2(0) = \omega_{20}, \quad (50) \\
\dot{\omega}_3(t) &= u_3(t), & \omega_3(0) = \omega_{30}, \quad (51)
\end{align*}
\]

where \( I_{23} \triangleq (I_2 - I_3)/I_1, \) \( I_1, \) \( I_2, \) and \( I_3 \) are the principal moments of inertia of the spacecraft satisfying \( 0 < I_1 = I_2 < I_3, \) \( \omega_1 : [0, \infty) \to \mathbb{R}, \) \( \omega_2 : [0, \infty) \to \mathbb{R}, \) and \( \omega_3 : [0, \infty) \to \mathbb{R} \) denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and \( u_1, u_2, \) and \( u_3 \) are the spacecraft control moments. We use a simple Euler discretisation scheme to discretise (49)–(51), which yields

\[
\begin{align*}
\omega_1(k+1) &= \omega_1(k) + h \left[ I_{23} \omega_3 \omega_2(k) + u_1(k) \right], \\
\omega_1(0) &= \omega_{10}, \quad k \in \mathbb{Z}_+, \quad (52) \\
\omega_2(k+1) &= \omega_2(k) + h \left[ -I_{23} \omega_3 \omega_1(k) + u_2(k) \right], \\
\omega_2(0) &= \omega_{20}, \quad (53) \\
\omega_3(k+1) &= \omega_3(k) + hu_3(k), \quad \omega_3(0) = \omega_{30}, \quad (54)
\end{align*}
\]

where \( h > 0 \) denotes the sampling time.

Now, let \( \omega = [\omega_1, \omega_2, \omega_3]^T \) and let the control law be given by

\[
\begin{align*}
u_1(\omega) &= -I_{23} \omega_3 \omega_2 - \frac{1}{h} \text{sign}(\omega_1) \min \left\{ \frac{1}{\kappa_f} \exp(|\omega_1|) + |\omega_2| + |\omega_3|^p |\omega_1|^{1-p}, |\omega_1| \right\}, \quad (55) \\
u_2(\omega) &= I_{23} \omega_3 \omega_1 - \frac{1}{h} \text{sign}(\omega_2) \min \left\{ \frac{1}{\kappa_f} \exp(|\omega_1| + |\omega_2|) + |\omega_3|^p |\omega_2|^{1-p}, |\omega_2| \right\}, \quad (56) \\
u_3(\omega) &= -\frac{1}{h} \text{sign}(\omega_3) \min \left\{ \frac{1}{\kappa_f} \exp(|\omega_1| + |\omega_2|) + |\omega_3|^p |\omega_3|^{1-p}, |\omega_3| \right\}, \quad (57)
\end{align*}
\]

Next, we provide an illustrative numerical example to demonstrate how Theorem 4.1 can be used for fixed time stabilisation.

**Figure 5.** Solutions \( x(k) \) of (47) for different initial conditions.
where $p \in (0, 1]$ and $\kappa_f > e$. Next, consider the Lyapunov function candidate $V(\omega) = |\omega_1| + |\omega_2| + |\omega_3|$. Since

$$|\omega_i| \geq \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_i| \right), \right.$$  

$$i = 1, 2, 3,$$

it follows that

$$\Delta V(\omega) = |\omega_1 - \operatorname{sign}(\omega_1) \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_1| \right) 
$$

$$- |\omega_2 - \operatorname{sign}(\omega_2) \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_2| \right) 
$$

$$- |\omega_3 - \operatorname{sign}(\omega_3) \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_3| \right) 
$$

$$\leq - \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_1| \right) 
$$

$$+ |\omega_2|^{1-p} + |\omega_3|^{1-p}, |\omega_1| + |\omega_2| + |\omega_3| \right) 
$$

$$\leq - \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, |\omega_1| + |\omega_2| + |\omega_3| \right) 
$$

$$\leq - \min \left\{ \frac{1}{\kappa_f} \exp\left(\|\omega_1| + |\omega_2| + |\omega_3|\|^{1-p}, V(\omega)^{1-p}, V(\omega) \right), 
$$

$$\omega \in \mathbb{R}^3 \setminus \{0\}, \right.$$  

(58)

and hence, it follows from Theorem 4.1 that the zero solution $(\omega_1(k), \omega_2(k), \omega_3(k)) \equiv (0, 0, 0)$ to (52)–(54) is globally fixed time stable with

$$K_{max} = \left\lceil \frac{1}{p} \log_{1-e/\kappa_f} \left( \frac{W_0(-\frac{1}{\kappa_f})}{W_{-1}(-\frac{1}{\kappa_f})} \right) + 1. \right.$$  

Let $I_1 = I_2 = 4$ kgm$^2$, $I_3 = 20$ kgm$^2$, $\omega_{10} = 2$ Hz, $\omega_{20} = -2$ Hz, and $\omega_{30} = 1$ Hz. The controlled system trajectory and control profile, with $h = 0.1$, $p = 0.5$, and $\kappa_f = 12$, are shown in Figure 6. Note that $(\omega_1(k), \omega_2(k), \omega_3(k)) = (0, 0, 0)$ for $k = 16 < K_{max} = 31$. It is clear from Figure 6 that the feedback controller (55)–(57) guarantees fixed time stabilisation. The parameters $p$ and $\kappa_f$ in (55)–(57) can be varied to reduce the conservatism between the guaranteed settling-time upper bound $K_{max}$ and the achieved fixed time convergence. However, achieving faster fixed time convergence comes at the expense of higher controller effort.

5. Optimal fixed time stabilisation

In this section, we use the results of Section 4 to extend the results of Haddad and Lee (2021) to obtain optimal feedback controllers that guarantee closed-loop, fixed time stabilisation. Specifically, sufficient conditions for optimality are given using a steady-state version of the Bellman equation. To address this problem, consider the controlled discrete-time nonlinear dynamical system given by

$$x(k + 1) = F(x(k), u(k)), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+,$$

(59)

where $x(k) \in \mathbb{D} \subseteq \mathbb{R}^n$, $u \in \mathbb{Z}_+$, is the state vector, $D$ is an open set with $0 \in D$, $u(k) \in U \subseteq \mathbb{Z}_+$, is the control input with $0 \in U$, $F : \mathbb{D} \times U \to \mathbb{R}^n$ is continuous in $x$ and $u$, and $F(0, 0) = 0$. The control $u(\cdot)$ in (59) is restricted to the class of admissible controls consisting of functions $u(\cdot)$ such that $u(k) \in U$ for all $k \in \mathbb{Z}_+$, where the control constraint set $U$ is given.

A control law is a continuous function $\phi : \mathbb{D} \to U$ satisfying $\phi(0) = 0$. If $u(k) = \phi(x(k))$, $k \in \mathbb{Z}_+$, where $\phi(\cdot)$ is a control law and $x(\cdot)$ satisfies (59), then $u(\cdot)$ is a feedback control law. A feedback control law is an admissible control since $\phi(\cdot)$ takes values in $U$. Given a feedback control law $u(k) = \phi(x(k))$, the closed-loop system has the form

$$x(k + 1) = F(x(k), \phi(x(k))), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+.$$  

(60)

Next, we provide sufficient conditions for optimality that characterise feedback controllers that guarantee fixed time stability for a nonlinear discrete-time system and minimise a nonlinear-nonquadratic performance criterion. For the statement of this result, let $L : D \times U \to \mathbb{R}$ and define the set of regulation control input signals for the nonlinear discrete-time system (59) by

$$\mathcal{S}(x_0) \triangleq \left\{ u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by } \right.$$  

$$\lim_{k \to \infty} x(k) = 0 \},$$

(59) satisfies

(59) satisfies

Since fixed time convergence is a stronger condition than asymptotic convergence, the set of regulation controllers $\mathcal{S}(x_0)$ is a superset of the set of all fixed time convergent controllers.

Theorem 5.1: Consider the nonlinear controlled dynamical system (59) with performance functional

$$J(x_0, u(\cdot)) \triangleq \sum_{k=0}^{\infty} L(x(k), u(k)),$$

(61)

where $u(\cdot)$ is an admissible control. Assume there exist a continuous function $V : \mathbb{D} \to \mathbb{R}$, real numbers $p \in (0, 1]$ and $\kappa_f > e,$
Finally, if \( D \), a neighbourhood \( \mathcal{M} \subseteq D \) of the origin, and a control law \( \phi : D \rightarrow U \) such that
\[
\begin{align*}
\phi(0) &= 0, \\
V(0) &= 0, \\
V(x) &> 0, \quad x \in \mathcal{M} \setminus \{0\}, \\
V(F(x, \phi(x))) &- V(x) \\
& \leq - \min \left\{ \frac{1}{\kappa_f} \exp(V(x)^{\beta}) V(x)^{1-\beta}, V(x) \right\}, \quad x \in \mathcal{M \setminus \{0\}}, \\
L(x, \phi(x)) + V(F(x, \phi(x))) &- V(x) = 0, \quad x \in \mathcal{D}, \\
L(x, u) + V(F(x, u)) &- V(x) \geq 0, \quad (x, u) \in D \times U.
\end{align*}
\]

Then, with the feedback control \( u = \phi(x) \), the zero solution \( x(k) \equiv 0, k \in \mathbb{Z}_+ \), to (59) is fixed time stable. Moreover, there exist an open neighbourhood \( D_0 \subset \mathcal{M} \) of the origin and a lower semicontinuous settling-time function \( K : D_0 \rightarrow \mathbb{Z}_+ \) such that
\[
K(x_0) \leq K_{\text{max}} = \left[ \frac{1}{p} \log_{\left[ \left(1 - \epsilon/\kappa_f \right) \left( -\frac{1}{\kappa_f} \right) \left( W_0 \left( -\frac{1}{\kappa_f} \right) \right) \left( W_{-1} \left( -\frac{1}{\kappa_f} \right) \right) \right] + 1, \quad x_0 \in D_0, \right.
\]
and
\[
F(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in D_0. \quad (69)
\]

In addition, if \( x_0 \in D_0 \), then the feedback control \( u(\cdot) = \phi(x(\cdot)) \) minimises \( J(x_0, u(\cdot)) \) in the sense that
\[
J(x_0, \phi(\cdot)) = \min_{u(\cdot) \in S(x_0)} J(x_0, u(\cdot)). \quad (70)
\]

Finally, if \( D = \mathbb{R}^n \), \( U = \mathbb{R}^m \), \( V(\cdot) \) is radially unbounded, and (65) holds on \( \mathbb{R}^n \setminus \{0\} \), then the closed-loop system (60) is globally fixed time stable.

**Proof:** Local and global fixed time stability along with the existence of a lower semicontinuous settling-time function \( K : D_0 \rightarrow \mathbb{Z}_+ \) satisfying (68) follow from (63)–(65) using Theorem 4.1. The remainder of the proof is similar to the proof of Theorem 3.1 of Haddad and Lee (2021) and, hence, is omitted.

**Remark 5.1:** It is important to emphasise that Theorems 5.1 and 6.1 (below), for the affine control case, are constructive in nature rather than existential. Our sufficient conditions provide explicit formulae for fixed time stability and optimality when the conditions in Theorems 5.1 and 6.1 have a solution, and in this case our constructive conditions are complementary to existential results on classical optimal control.

**Remark 5.2:** Note that Theorem 5.1 also holds for the case where (65) is replaced by
\[
V(F(x, \phi(x))) - V(x) \leq - \min \left\{ cV(x)^{\alpha} + dV(x)^{\beta}, V(x) \right\}, \quad x \in \mathcal{M \setminus \{0\}}, \quad (71)
\]
where \( ac(0, 1), \beta > 1, dc(0, 1), \) and \( de(0, 1) \), and with the lower semicontinuous settling-time function \( K : \mathcal{N} \rightarrow \mathbb{Z}_+ \) given by
\[
K(x_0) \leq \left[ \log_{1-c} e^{\frac{1}{1-\alpha}} \right] + \left[ \log_{1-d} e^{\frac{1}{1-\beta}} \right] + 1, \quad x_0 \in \mathcal{N}. \quad (72)
\]
A similar remark holds for Theorems 6.1 and 6.2 below.

Equation (66) is the steady-state Bellman equation for the controlled discrete-time system (59) with performance criterion (61). Furthermore, conditions (65)–(67) guarantee optimality as well as fixed time stability with respect to the set of admissible stabilising controllers \( \mathcal{S}(x_0) \). It is important to stress that an explicit description of \( \mathcal{S}(x_0) \) is not needed. In order to ensure fixed time stability of the closed-loop system (60), Theorem 5.1 requires that \( V(\cdot) \) satisfy (63)–(65), which implies that \( V(\cdot) \) is a Lyapunov function for the closed-loop system (60). For optimality, however, \( V(\cdot) \) does not need to satisfy (64) and (65). In particular, if \( V(\cdot) \) is continuous, (63) is satisfied, and \( \phi(\cdot) \in \mathcal{S}(x_0) \), then (66) and (67) imply (69) and (70).

**6. Fixed-time stabilisation for affine dynamical systems and connections to inverse optimal control**

In this section, we specialise the results of Section 5 to discrete-time nonlinear affine in the control dynamical systems of the form
\[
x(k + 1) = f(x(k)) + G(x(k))u(k), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad (73)
\]
where, for every \( k \in \mathbb{Z}_+ \), \( x(k) \in \mathbb{R}^n \), \( u(k) \in \mathbb{R}^m \), and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m} \) are such that \( f(\cdot) \) and \( G(\cdot) \) are continuous in \( x \) and \( f(0) = 0 \). Furthermore, we consider performance
Moreover, there exists a lower semicontinuous settling-time function, so that (61) becomes

\[ J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \left[ L_1(x(k)) + L_2(x(k))u(k) + u^T(k)R_2(x(k))u(k) \right]. \]  

(75)

**Theorem 6.1:** Consider the controlled discrete-time nonlinear affine system (73) with performance measure (75). Assume that there exist functions \( V : \mathbb{R}^n \to \mathbb{R}, L_2 : \mathbb{R}^n \to \mathbb{R}^{1 \times m}, P_1 : \mathbb{R}^n \to \mathbb{R}^{1 \times 1}, \) and \( P_2 : \mathbb{R}^n \to \mathbb{R}^{m \times m}, \) and real numbers \( p \in (0, 1) \) and \( \kappa_f > \epsilon \) such that \( V(\cdot) \) is continuous, \( P_2(\cdot) \) is nonnegative definite,

\[ L_2(0) = 0, \]
\[ P_{12}(0) = 0, \]
\[ V(0) = 0, \]
\[ V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \]

\[ V \left( f(x) - \frac{1}{2} G(x)[R_2(x) + P_2(x)]^{-1}[L_2(x) + P_{12}(x)] \right) \]
\[ - V(x) \leq - \min \left\{ \frac{1}{\kappa_f} \exp(V(x)^p) V(x)^{-p}, V(x) \right\}, \]

(80)

\[ x \neq 0, \]
\[ V(f(x) + G(x)u) = V(f(x)) + P_{12}(x)u + u^T P_2(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \]

(81)

\[ 0 = L_1(x) - \frac{1}{4}[L_2(x) + P_{12}(x)][R_2(x) + P_2(x)]^{-1} \]
\[ \times [L_2(x) + P_{12}(x)]^T + V(f(x)) - V(x), \]

(82)

and

\[ V(x) \to \infty \quad \text{as} \|x\| \to \infty. \]

(83)

Then the zero solution \( x(k) \equiv 0 \) of the closed-loop system

\[ x(k + 1) = f(x(k)) + G(x(k))\phi(x(k)), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \]

is globally fixed time stable with the feedback control

\[ u = \phi(x) = - \frac{1}{2} [R_2(x) + P_2(x)]^{-1}[L_2(x) + P_{12}(x)]^T. \]

(85)

Moreover, there exists a lower semicontinuous settling-time function \( K : \mathbb{R}^n \to \mathbb{Z}_+ \) such that

\[ K(x_0) \leq K_{\text{max}} = \left[ \frac{1}{p} \log \left[ \frac{1 - e^{-\epsilon}}{\epsilon} \right] \right]_+ + 1, \quad x_0 \in \mathbb{R}^n, \]

(86)

and the performance measure (75) is minimised in the sense of (70). Finally,

\[ J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \]

(87)

**Proof:** The proof is similar to the proof of Theorem 4.1 of Haddad and Lee (2021) and, hence, is omitted.

**Remark 6.1:** Condition (81) requires that \( V(f(x) + G(x)u) \) is quadratic in \( u \). In the local case, this condition is without loss of generality if the Lyapunov function \( V \) has a nondegenerate minimum at the origin. In the global case, a sufficient (but not necessary) condition for (81) holding is the case when \( V \) is quadratic. For further details, see Byrnes and Lin (1994).

Next, we specialise Theorem 6.1 to an inverse optimal control problem (see Freeman & Kokotovic, 1996; Jacobson, 1977; Molyan & Anderson, 1973; Sepulchre et al., 1997). Specifically, we construct nonlinear feedback controllers that minimise a derived nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the mapping of the difference operator of the Lyapunov function guaranteeing closed-loop stability is negative definite along the closed-loop system trajectories while providing sufficient conditions for the existence of fixed time stabilising solutions to the Bellman equation. Hence, by varying the parameters of the Lyapunov function and the performance summand, the next result provides a family of globally stabilising feedback controllers parameterised by the cost functional that is minimised.

**Theorem 6.2:** Consider the controlled discrete-time nonlinear affine dynamical system (73) with performance criterion (75). Assume that there exist a continuous, radially unbounded function \( V : \mathbb{R}^n \to \mathbb{R} \), functions \( L_2 : \mathbb{R}^n \to \mathbb{R}^{1 \times m}, P_{12} : \mathbb{R}^n \to \mathbb{R}^{m \times m}, \) a nonnegative-definite function \( P_2 : \mathbb{R}^n \to \mathbb{R}^{m \times m}, \) and real numbers \( p \in (0, 1) \) and \( \kappa_f > \epsilon \) such that (76)–(81) hold. Then, with the feedback control

\[ u = \phi(x) = - \frac{1}{2} [R_2(x) + P_2(x)]^{-1}[L_2(x) + P_{12}(x)]^T, \]

(88)

the zero solution \( x(k) \equiv 0 \) to (84) is globally fixed time stable. Moreover, there exists a lower semicontinuous settling-time function \( K : \mathbb{R}^n \to \mathbb{Z}_+ \) such that (86) hold and the performance measure (75), with

\[ L_1(x) = \phi^T(x)(R_2(x) + P_2(x))\phi(x) - V(f(x)) + V(x), \]

(89)

is minimised in the sense of (70). Finally, (87) holds.

**Proof:** The proof is similar to the proof of Theorem 6.1 and, hence, is omitted.

Next, we provide an illustrative numerical example to demonstrate how Theorem 6.2 can be used for inverse optimal fixed time stabilisation using the spacecraft system addressed in Example 4.3.

**Example 6.1:** Consider the spacecraft with one axis of symmetry given by (52)–(54). Note that (52)–(54) can be cast in the form of (73) with \( n = 3, m = 3, x = \omega = [\omega_1, \omega_2, \omega_3]^T, \)

\[ f(\omega) = [\omega_1 + h_{123}\omega_3, \omega_2 - h_{123}\omega_3, \omega_3]^T, \]

and \( G(\omega) = hI_3. \)
To construct an inverse optimal fixed time controller for (52)–(54), let \( V(\omega) = \omega^2 \), \( L(\omega, u) \) be given by (74), \( R_2(\omega) = h^2 I_3 \), \( P_2(\omega) = h^2 I_3 \),

\[
P_{12}(\omega) = [2h\omega_1 + h^2 I_{23}\omega_3\omega_2, 2h\omega_2 - h^2 I_{23}\omega_3\omega_2, 2h\omega_3],
\]

and

\[
L_2(\omega) = \left[ -2h\omega_1 + h^2 I_{23}\omega_3\omega_2 + 4h \text{sign}(\omega_1) - 2h\omega_2 - h^2 I_{23}\omega_3\omega_2 + 4h \text{sign}(\omega_2) - 2h\omega_3 + 4h \text{sign}(\omega_3) \right],
\]

where \( p \in (0, 1], \kappa_f > e, \text{sign}(z) \triangleq z/|z|, z \neq 0, \) and \( \text{sign}(0) \triangleq 0 \). Now, the inverse optimal control law (88) is given by

\[
u = \phi(\omega) - \frac{1}{2}[R_2(\omega) + P_2(\omega)]^{-1} [L_2(\omega) + P_{12}(\omega)]^T
\]

\[
= -\frac{1}{2h^2} [L_2(\omega) + P_{12}(\omega)]^T
\]

\[
[ -I_{23}\omega_3\omega_2 - \frac{1}{h} \text{sign}(\omega_1) - \frac{1}{h} \text{sign}(\omega_2) - \frac{1}{h} \text{sign}(\omega_3) \\
\min \left\{ \frac{1}{\sqrt{\kappa_f}} \exp \left( \frac{1}{2} [\omega_1^2 + \omega_2^2 + \omega_3^2 P] \right) |\omega_1|^{1-p}, |\omega_1| \right\} - \frac{1}{h} \text{sign}(\omega_1) \\
\min \left\{ \frac{1}{\sqrt{\kappa_f}} \exp \left( \frac{1}{2} [\omega_1^2 + \omega_2^2 + \omega_3^2 P] \right) |\omega_2|^{1-p}, |\omega_2| \right\} - \frac{1}{h} \text{sign}(\omega_2) \\
\min \left\{ \frac{1}{\sqrt{\kappa_f}} \exp \left( \frac{1}{2} [\omega_1^2 + \omega_2^2 + \omega_3^2 P] \right) |\omega_3|^{1-p}, |\omega_3| \right\}
\]

and the performance functional (75), with

\[
L_1(\omega) = \phi^T(\omega)\phi(\omega) - V(f(\omega)) + V(\omega),
\]

is minimised in the sense of (70).

Furthermore, note that (78) and (79) hold and, since

\[
\Delta V(\omega) = \left[ \omega_1 - \text{sign}(\omega_1) \min \left\{ \frac{1}{\sqrt{\kappa_f}} \exp \left( \frac{1}{2} [\omega_1^2 + \omega_2^2 + \omega_3^2 P] \right) |\omega_1|^{1-p}, |\omega_1| \right\} \right]^2 - \omega_1^2
\]

\[
+ \left[ \omega_2 - \text{sign}(\omega_2) \min \left\{ \frac{1}{\sqrt{\kappa_f}} \exp \left( \frac{1}{2} [\omega_1^2 + \omega_2^2 + \omega_3^2 P] \right) |\omega_2|^{1-p}, |\omega_2| \right\} \right]^2 - \omega_2^2
\]

it follows from Theorem 4.1 that the zero solution \((\omega_1(k), \omega_2(k), \omega_3(k)) \equiv (0, 0, 0)\) to (52)–(54) is globally fixed time stable with

\[
K_{\text{max}} = \left[ \frac{1}{p} \log_{1-e^{|\kappa_f|}} \left( \frac{W_0(-\frac{1}{\kappa_f})}{W_{-1}(-\frac{1}{\kappa_f})} \right) + 1. \right]
\]
Let $I_1 = I_2 = 4 \text{ kg} \cdot \text{m}^2$, $I_3 = 20 \text{ kg} \cdot \text{m}^2$, $\omega_{10} = 1.5 \text{ Hz}$, $\omega_{20} = -1.5 \text{ Hz}$, and $\omega_{30} = 1 \text{ Hz}$. The controlled system trajectory and control profile, with $h = 0.2$, $p = 0.75$, and $\kappa_f = 100$, are shown in Figure 7. Note that $(\omega_1(k), \omega_2(k), \omega_3(k)) = (0, 0, 0)$ for $k \geq 11$. Finally, $J(x_0, \phi(x(.))) = 5.5 \text{ Hz}^2$. It is clear from Figure 7 that the feedback controller (92) guarantees fixed time stabilisation. The parameters $p$ and $\kappa_f$ in (92) can be varied to reduce the conservatism between the guaranteed settling-time upper bound $K_{\text{max}}$ and the achieved fixed time convergence. However, achieving faster fixed time convergence comes at the expense of higher controller effort.

7. Conclusion

This paper addresses the notion of fixed time stability and optimal stabilisation for discrete-time nonlinear systems. Specifically, we developed Lyapunov theorems for fixed time stability of discrete autonomous systems involving scalar difference inequalities and minimum operators that allow for the adjustment of a guaranteed settling time that is independent of the system initial conditions. This framework was then used to design optimal fixed time controllers for nonlinear dynamical systems. Specifically, we utilised a steady-state Bellman framework to develop optimal nonlinear feedback fixed time controllers with a notion of optimality that is directly related to a given Lyapunov function satisfying a difference inequality involving a minimum operator of the Lyapunov function and that allows for the adjustment of a guaranteed settling time that is independent of the system initial conditions and guarantees closed-loop fixed time stability and optimality.

Future research will focus on extending these results to data-driven reinforcement learning control for learning the optimal...
control policy online and in fixed time as well as to stochastic optimal control problems involving fixed time stabilisation for nonlinear discrete-time stochastic dynamical systems.

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