**Abstract**

In this paper, we develop $\mathcal{H}_2$ semistability theory for linear dynamical systems. Using this theory, we design $\mathcal{H}_2$ optimal semistable controllers for linear dynamical systems. Unlike the standard $\mathcal{H}_2$ optimal control problem, a complicating feature of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. An interesting feature of the proposed approach, however, is that a least squares solution over all possible semistabilizing solutions corresponds to the $\mathcal{H}_2$ optimal solution. It is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

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1. Introduction

Dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and
space vehicle formations, and congestion control in communication networks, to cite but a few examples. A unique feature of the closed-loop dynamics under any control algorithm in dynamical networks is the existence of a continuum of equilibria representing a desired state of convergence. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well [1,2].

The dependence of the limiting state on the initial state is not limited to dynamical network systems, it is also seen in the dynamics of compartmental systems [3] which arise in chemical kinetics [4], and biomedical [5], environmental [6], economic [7], power [8], and thermodynamic systems [9]. In all such systems possessing a continuum of equilibria, semistability, and not asymptotic stability, is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability then implies Lyapunov stability, and is implied by asymptotic stability.

Semistability was first introduced in [10] for linear systems, and applied to matrix second-order systems in [11]. Nonlinear extensions were considered in [12,13], which give several stability results for systems having a continuum of equilibria based on nontangency and arc length of trajectories, respectively. Refs. [1,2] build on the results of [12,13] and give semistable stabilization results for nonlinear network dynamical systems. Optimal semistable stabilization, however, has never been considered in the literature.

In this paper, we use linear matrix inequalities (LMIs) to develop $H_2$ optimal semistable controllers for linear dynamical systems. Linear matrix inequalities provide a powerful design framework for linear control problems [14]. Since LMIs lead to convex or quasiconvex optimization problems, they can be solved very efficiently using interior-point algorithms. Unlike the standard $H_2$ optimal control problem, a complicating feature of the $H_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. An interesting feature of the proposed approach, however, is that a least squares solution over all possible semistabilizing solutions corresponds to the $H_2$ optimal solution. It is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

2. $H_2$ semistability theory

In this section, we establish notation along with several key results on $H_2$ semistability theory involving the notions of semistability, semicon trollability, and semiobservability. The notion we use in this paper is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n \times m}$ (resp., $\mathbb{C}^{n \times m}$) denotes the set of $n \times m$ real (resp., complex) matrices, $(\cdot)^T$ denotes transpose, $(\cdot)^*$ denotes complex conjugate transpose, $(\cdot)^+$ denotes the group generalized inverse, and $I_n$ or $I$ denotes the $n \times n$ identity matrix. Furthermore, we write $\| \cdot \|$ for the Euclidean vector norm, $\| \cdot \|_F$ for the Frobenius matrix norm, $S^\perp$ for the orthogonal complement of a set $S$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix $A$, respectively, $\text{spec}(A)$ for the spectrum of the square matrix $A$, $\det A$ for the determinant of the square matrix $A$, $\text{rank} A$ for the rank of a matrix $A$, $\text{tr}(\cdot)$ for the trace operator, $\mathbb{E}$ for the expectation operator, and $A \geq 0$ (resp., $A > 0$) to denote the fact that the Hermitian matrix $A$ is nonnegative (resp., positive) definite. Finally, we write $B_r(x)$, $x \in \mathbb{R}^n$,
\( \varepsilon > 0 \), for the open ball with radius \( \varepsilon \) and center \( x \), \( \otimes \) for the Kronecker product, \( \oplus \) for the Kronecker sum, and \( \text{vec}(\cdot) \) for the column stacking operator.

The following definition for semistability for a dynamical system is needed. For this definition, consider the nonlinear dynamical system given by

\[
\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0,
\]

where \( x(t) \in \mathcal{D} \subseteq \mathbb{R}^n \), \( t \geq 0 \), and \( f : \mathcal{D} \rightarrow \mathbb{R}^n \) is locally Lipschitz continuous on \( \mathcal{D} \).

**Definition 2.1.** Let \( \mathcal{D} \subseteq \mathbb{R}^n \) be positively invariant under Eq. (1). The equilibrium solution \( x(t) \equiv x_0 \in \mathcal{D} \) of Eq. (1) is Lyapunov stable with respect to \( \mathcal{D} \) if, for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( x_0 \in B_\delta(x_0) \cap \mathcal{D} \), then \( x(t) \in B_\varepsilon(x_0) \cap \mathcal{D}, \ t \geq 0 \). The equilibrium solution \( x(t) \equiv x_0 \in \mathcal{D} \) of Eq. (1) is semistable with respect to \( \mathcal{D} \) if it is Lyapunov stable with respect to \( \mathcal{D} \) and there exists \( \delta > 0 \) such that if \( x_0 \in B_\delta(x_0) \cap \mathcal{D} \), then \( \lim_{t \rightarrow \infty} x(t) \) exists and corresponds to a Lyapunov stable equilibrium point in \( \mathcal{D} \). Finally, the system (1) is said to be semistable with respect to \( \mathcal{D} \) if every equilibrium point in \( \mathcal{D} \) is semistable with respect to \( \mathcal{D} \).

Note that if in Eq. (1) \( f(x) = Ax \), where \( A \in \mathbb{R}^{n \times n} \), then Eq. (1) is semistable if and only if \( A \) is semistable, that is, \( \text{spec}(A) \subset \{ s \in \mathbb{C} : \text{Re} s < 0 \} \cup \{ 0 \} \) and, if \( 0 \in \text{spec}(A) \), then 0 is semisimple. In this case, it can be shown that for every \( x_0 \in \mathbb{R}^n \), \( \lim_{t \rightarrow \infty} x(t) \) exists or, equivalently, \( \lim_{t \rightarrow \infty} e^{At} \) exists and is given by \( \lim_{t \rightarrow \infty} e^{At} = I_n - AA^\# \) [15, pp. 437–438].

Next, we present the notions of semicontrollability and semiobservability. For these definitions let \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{l \times n} \), and consider the linear dynamical system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0,
\]

\[ y(t) =Cx(t), \]

with state \( x(t) \in \mathbb{R}^n \), input \( u(t) \in \mathbb{R}^m \), and output \( y(t) \in \mathbb{R}^l \), where \( t \geq 0 \).

**Definition 2.2.** Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times n} \). The pair \((A,B)\) is semicon trollable if

\[
\bigcap_{k=1}^n \mathcal{N}(B^T(A^{k-1})^T) = [\mathcal{N}(A^T)]^\perp,
\]

where \( A^0 \triangleq I_n \).

**Definition 2.3.** Let \( A \in \mathbb{R}^{n \times n} \) and \( C \in \mathbb{R}^{l \times n} \). The pair \((A,C)\) is semiobservable if

\[
\bigcap_{k=1}^n \mathcal{N}(CA^{k-1}) = \mathcal{N}(A).
\]

Semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to equilibrium controllability, whereas semiobservability is an extension of zero-state observability to equilibrium observability. It is important to note here that since Definitions 2.2 and 2.3 are dual, dual results to the semiobservability results that we establish in this section also hold for semicontrollability.
Definition 2.4. Let $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{l \times n}$, and $K \in \mathbb{R}^{m \times n}$. The pair $(A, C)$ is semiobservable with respect to $K$ if

$$
\mathcal{N}(K) \cap \left( \bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}) \right) = \mathcal{N}(K) \cap \mathcal{N}(A). \tag{6}
$$

The following result shows that semiobservability is unchanged by full state feedback.

Proposition 2.1. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $K \in \mathbb{R}^{m \times n}$, and $R \in \mathbb{R}^{n \times n}$, where $R$ is positive definite. If the pair $(A, C)$ is semiobservable, then the pair $(A + BK, C^T C + K^T R K)$ is semiobservable with respect to $K$.

Proof. Note that

$$
\mathcal{N}(C^T C + K^T R K) = \mathcal{N}(C) \cap \mathcal{N}(K). \tag{7}
$$

which implies that the pair $(A + BK, C^T C + K^T R K)$ is semiobservable with respect to $K$. \(\square\)

Next, we connect semistability with Lyapunov theory and semiobservability to arrive at a characterization of the $\mathcal{H}_2$ norm of semistable systems. For this result, we consider the linear dynamical system

$$
\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{8}
$$

where $A \in \mathbb{R}^{n \times n}$, with output equation (3). Furthermore, for a given semistable system define the $\mathcal{H}_2$ norm of $G(s)\left[ \begin{array}{c} A \\ C \end{array} \right] x_0$ by

$$
\| G \|_2 \triangleq \left[ \int_0^\infty \| G(t) \|_F^2 \, dt \right]^{1/2} = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \| G(j\omega) \|_F^2 \, d\omega \right]^{1/2}. \tag{9}
$$

The following proposition presents necessary and sufficient conditions for well-posedness of the $\mathcal{H}_2$ norm of a semistable system.

Proposition 2.2. Consider the linear dynamical system (8) with output (3) and assume $A$ is semistable. Then the following statements are equivalent:

(i) For every $x_0 \in \mathbb{R}^n$, $\| G \|_2 < \infty$.
(ii) $\int_0^\infty e^{At} R e^{At} \, dt < \infty$, where $R \triangleq C^T C$.
(iii) $\mathcal{N}(A) \subset \mathcal{N}(C)$. 

Proof. The equivalence of (i) and (ii) follows from the fact
\[ \| G \|_2^2 = x_0^T \int_0^\infty e^{AT} R e^{At} \, dt \, x_0. \] (10)

To show (ii) implies (iii) note that since \( A \) is Hurwitz or there exists an invertible matrix \( S \in \mathbb{R}^{n \times n} \) such that \( A = S J_0^{-1} S^{-1} \), where \( J \in \mathbb{R}^{r \times r}, \quad r = \text{rank } A \), and \( J \) is Hurwitz. Now, if \( A \) is Hurwitz, then (iii) holds trivially since \( \mathcal{N}(A) = \{0\} \subset \mathcal{N}(C) \).

Alternatively, if \( A \) is not Hurwitz, then
\[ \mathcal{N}(A) = \{x \in \mathbb{R}^n : x = S[0_{1 \times r}, y]^T, y \in \mathbb{R}^{n-r}\}. \] (11)

Now,
\[ \int_0^\infty e^{AT} R e^{At} \, dt = S^{-T} \int_0^\infty e^{jT} \hat{R} e^{jt} \, dt S = S^{-T} \int_0^\infty \begin{bmatrix} e^{jT} \hat{R}_1 e^{jt} & e^{jT} \hat{R}_{12} \end{bmatrix} \, dt S, \] (12)

where
\[ \hat{j} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{R} = S^T RS = \begin{bmatrix} \hat{R}_1 & \hat{R}_{12} \\ \hat{R}_{12}^T & \hat{R}_2 \end{bmatrix}. \] (13)

Next, it follows from Eq. (12) that
\[ \int_0^\infty e^{AT} R e^{At} \, dt < \infty \] (14)
if and only if \( \hat{R}_2 = 0 \) or, equivalently,
\[ [0_{1 \times r}, y]^T \hat{R}[0_{1 \times r}, y]^T = 0, \quad y \in \mathbb{R}^{n-r}, \] (15)
which is further equivalent to \( x^T R x = 0, \quad x \in \mathcal{N}(A) \). Hence, \( \mathcal{N}(A) \subset \mathcal{N}(C) \).

Finally, the proof of (iii) implies (ii) is immediate by reversing the steps of the proof given above. \( \square \)

Theorem 2.1. Consider the linear dynamical system (8). Suppose there exist an \( n \times n \) matrix \( P \geq 0 \) and an \( m \times n \) matrix \( C \) such that \( (A, C) \) is semiobservable and
\[ 0 = A^T P + PA + R, \] (16)
where \( R \triangleq C^T C \). Then Eq. (8) is semistable with respect to \( \mathbb{R}^n \). Furthermore, \( \| G(s) \|_2^2 = (x_0 - x_\varepsilon)^T P(x_0 - x_\varepsilon) \), where \( x_\varepsilon \triangleq x_0 - AA^# x_0 \).

Proof. The first part of the result is a direct consequence of Proposition 4.1 of [16]. Now, since \( A \) is semistable, it follows from (ix) of Proposition 11.7.2 of [15] that \( \lim_{t \to \infty} e^{At} = I_q - AA^# \). Next, noting that \( Ax_\varepsilon = 0 \), Eq. (8) can be equivalently written as
\[ \dot{x}(t) = A(x(t) - x_\varepsilon), \quad x(0) = x_0, \quad t \geq 0. \] (17)

Hence,
\[ \int_0^t (x(s) - x_\varepsilon)^T R(x(s) - x_\varepsilon) \, ds = -(x(t) - x_\varepsilon)^T P(x(t) - x_\varepsilon) + (x_0 - x_\varepsilon)^T P(x_0 - x_\varepsilon). \] (18)
Now, it follows from the semiobservability of \((A, C)\) that \(Rx_c = 0\). Hence, letting \(t \to \infty\) and noting that \(x(t) \to x_c\) as \(t \to \infty\) it follows from Eq. (18) that
\[
\int_0^\infty x^T(t)Rx(t) \, dt = (x_0-x_c)^T P(x_0-x_c). \tag{19}
\]

Finally, defining the free response of Eq. (8) by \(z(t) \triangleq Cx(t) = Ce^{At}x_0\), \(t \geq 0\), and noting that \(R = C^T C\), it follows from Parseval’s theorem that
\[
(x_0-x_c)^T P(x_0-x_c) = \int_0^\infty z^T(t)z(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 \, d\omega. \tag{20}
\]
This completes the proof. \(\square\)

**Example 2.1.** Consider the linear system (8) where \(A\) is given by
\[
A = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \tag{21}
\]
and note that \(\mathcal{N}(A) = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 = x_2 = z, z \in \mathbb{R}\}\). Let \(C = [2, -1]\). It is easy to verify that Eq. (16) holds with
\[
P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{22}
\]
Furthermore, note that \(\mathcal{N}(C) = \mathcal{N}(A)\), and hence, the pair \((A, C)\) is semiobservable. Now, it follows from **Theorem 2.1** that Eq. (8) is semistable. \(\square\)

Next, we give a necessary and sufficient condition for characterizing semistability using the Lyapunov equation (16). Before we state this result, the following lemmas are needed.

**Lemma 2.1.** Consider the linear dynamical system (8). If Eq. (8) is semistable, then, for every \(n \times n\) nonnegative definite matrix \(R\),
\[
\int_0^\infty (x(t)-x_c)^T R(x(t)-x_c) \, dt < \infty, \tag{23}
\]
where \(x_c = (I_n - AA^H)x_0\).

**Proof.** Since \(A\) is semistable, it follows from the Jordan decomposition that there exists an invertible matrix \(S \in \mathbb{C}^{n \times n}\) such that \(A = S J_0 S^{-1}\), where \(J \in \mathbb{C}^{r \times r}\), \(r = \text{rank } A\), and \(J\) is asymptotically stable. Let \(z(t) \triangleq S^{-1}x(t)\) and \(z_c \triangleq S^{-1}x_c\), \(t \geq 0\). Then Eq. (8) becomes
\[
\dot{z}(t) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} z(t), \quad z(0) = S^{-1}x_0, \quad t \geq 0, \tag{24}
\]
which implies that \(\lim_{t \to \infty} z_i(t) = 0, \ i = 1, \ldots, r\), and \(z_j(t) = z_j(0), \ j = r+1, \ldots, n\), that is, \(z_c = [0, \ldots, 0, z_{r+1}(0), \ldots, z_n(0)]^T\). Now,
\[
\int_0^\infty (x(t)-x_c)^T R(x(t)-x_c) \, dt = \int_0^\infty (z(t)-z_c)^* S^* RS(z(t)-z_c) \, dt \\
= \int_0^\infty \dot{z}^*(t) S^* RS \dot{z}(t) \, dt, \quad \tag{25}
\]
where \( \hat{z}(t) \triangleq [z_1(t), \ldots, z_r(t), 0, \ldots, 0]^T \). Since
\[
\hat{z}(t) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \hat{z}(t)
\]
and \( J \) is asymptotically stable, it follows that
\[
\int_0^\infty \hat{z}^*(t) S^* R S \hat{z}(t) \, dt < \infty,
\]
which proves the result. \( \square \)

**Lemma 2.2.** Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times m} \). If \( A \) and \( B \) are semistable, then \( A \oplus B \) is semistable.

**Proof.** Let \( \lambda \in \text{spec}(A) \) and \( \mu \in \text{spec}(B) \). Since \( A \) and \( B \) are both semistable, it follows that \( \Re \lambda < 0 \) or \( \lambda = 0 \) and \( \text{am}_A(0) = \text{gm}_A(0) \), and \( \Re \mu < 0 \) or \( \mu = 0 \) and \( \text{am}_B(0) = \text{gm}_B(0) \), where \( \text{am}_X(\lambda) \) and \( \text{gm}_X(\lambda) \) denote algebraic multiplicity of \( \lambda \in \text{spec}(X) \) and geometric multiplicity of \( \lambda \in \text{spec}(X) \), respectively. Now, it follows from the fact that \( \lambda + \mu \in \text{spec}(A \oplus B) \), that \( \text{spec}(A \oplus B) \subset \{ z \in \mathbb{C} : \Re z < 0 \} \cup \{ 0 \} \). Next, it follows from Fact 7.5.2 of [15] that \( \text{gm}_A(0) \text{gm}_B(0) \leq \text{gm}_{A \oplus B}(0) \leq \text{am}_A(0) \text{am}_B(0) \). Since \( \text{am}_A(0) = \text{gm}_A(0) \) and \( \text{am}_B(0) = \text{gm}_B(0) \), it follows that \( \text{gm}_{A \oplus B}(0) = \text{am}_{A \oplus B}(0) \), and hence, \( A \oplus B \) is semistable. \( \square \)

**Lemma 2.3.** Let \( x \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \), and assume \( A \) is semistable. Then \( \int_0^\infty e^{At} x \, dt \) exists if and only if \( x \in \mathcal{R}(A) \). In this case, \( \int_0^\infty e^{At} x \, dt = -A^# x \).

**Proof.** The proof is similar to the proofs of (vii) and (viii) of Lemma 2.2 of [17] and, hence, is omitted. \( \square \)

**Lemma 2.4** (Bhat and Bernstein [16]). Let \( A \in \mathbb{R}^{n \times n} \). If there exists an \( n \times n \) matrix \( P \geq 0 \) and an \( m \times n \) matrix \( C \) such that \((A, C)\) is semiobservable and Eq. (16) holds, then (i) \( \mathcal{N}(P) \subset \mathcal{N}(A) \subset \mathcal{N}(R) \) and (ii) \( \mathcal{N}(A) \cap \mathcal{R}(A) = \{ 0 \} \).

**Theorem 2.2.** Consider the linear dynamical system (8). Then Eq. (8) is semistable if and only if for every semiobservable pair \((A, C)\) there exists an \( n \times n \) matrix \( P \geq 0 \) such that Eq. (16) holds. Furthermore, if \((A, C)\) is semiobservable and \( P \) satisfies Eq. (16), then
\[
P = \int_0^\infty e^{At} Re^{At} \, dt + P_0
\]
for some \( P_0 = P_0^T \in \mathbb{R}^{n \times n} \) satisfying
\[
0 = A^T P_0 + P_0 A
\]
and
\[
P_0 \geq -\int_0^\infty e^{At} Re^{At} \, dt.
\]
In addition, \( \min_{P \in \mathcal{D}} \| P \|_F \) has a unique solution \( P \) given by
\[
P = \int_0^\infty e^{At} Re^{At} \, dt,
\]
where \( P \) denotes the set of all \( P \) satisfying Eq. (16). Finally, Eq. (8) is semistable if and only if for every semiobservable pair \((A, C)\) there exists an \( n \times n \) matrix \( P > 0 \) such that Eq. (16) holds.

**Proof.** Sufficiency for the first implication follows from Theorem 2.1. To show necessity, assume Eq. (8) is semistable. Then, \( \lim_{t \to \infty} x(t) = x_c \), where \( x_c = (I_n - AA^\#)x_0 \). For a semiobservable pair \((A, C)\), let

\[
P = \int_0^{\infty} (AA^\#)^T e^{At} Re^{At} AA^\# \, dt.
\]

Then, for \( x_0 \in \mathbb{R}^n \),

\[
x_0^T P x_0 = \int_0^{\infty} x_0^T (AA^\#)^T e^{At} Re^{At} AA^\# x_0 \, dt
\]

\[
= \int_0^{\infty} (x_0 - x_c)^T e^{At} Re^{At} (x_0 - x_c) \, dt
\]

\[
= \int_0^{\infty} (x(t) - x_c)^T R (x(t) - x_c) \, dt,
\]

where we used the fact that \( x(t) - x_c = e^{At}(x_0 - x_c) \). It follows from Lemma 2.1 that \( P \) is well defined. Since \( x_c \in \mathcal{N}(A) \), it follows from Eq. (5) that \( Rx_c = 0 \), and hence,

\[
x_0^T P x_0 = \int_0^{\infty} x^T(t) Rx(t) \, dt = \int_0^{\infty} x_0^T e^{At} Re^{At} x_0 \, dt,
\]

which implies that

\[
P = \int_0^{\infty} e^{At} Re^{At} \, dt.
\]

Now, Eq. (16) is immediate using the fact that \( Rx_c = 0 \).

Next, since \( A \) is semistable, it follows from the above result that there exists an \( n \times n \) nonnegative-definite matrix \( P \) such that Eq. (16) holds or, equivalently, \((A \oplus A)^T \text{vec } P = -\text{vec } R \). Hence, \( \text{vec } R \in \mathcal{R}((A \oplus A)^T) \) and \( \mathcal{P} = \{ P \in \mathbb{R}^{n \times n} : P = -\text{vec}^{-1}(((A \oplus A)^T)^\# \text{vec } R) + \text{vec}^{-1}(z) \} \) for some \( z \in \mathcal{N}((A \oplus A)^T) \). Next, it follows from Lemma 2.2 that \( A \oplus A \) is semistable, and hence, by Lemma 2.3,

\[
\text{vec}^{-1}(((A \oplus A)^T)^\# \text{vec } R) = -\int_0^{\infty} \text{vec}^{-1}(e^{(A \oplus A)^T} \text{vec } R) \, dt
\]

\[
= -\int_0^{\infty} \text{vec}^{-1}(e^{At} \otimes e^{At}) \text{vec } R \, dt
\]

\[
= -\int_0^{\infty} e^{At} Re^{At} \, dt,
\]

where in Eq. (36) we used the facts that \((X \otimes Y)^T = X^T \otimes Y^T \), \( e^{X \oplus Y} = e^X \otimes e^Y \), and \( \text{vec}(XYZ) = (Z^T \otimes X)\text{vec } Y \) [15, Chapter 7]. Hence,

\[
P = \int_0^{\infty} e^{At} Re^{At} \, dt + \text{vec}^{-1}(z),
\]

where \( \text{vec}^{-1}(z) \) satisfies \( \text{vec}^{-1}(z) = (\text{vec}^{-1}(z))^T \), \( A^T \text{vec}^{-1}(z) + \text{vec}^{-1}(z)A = 0 \), and \( \text{vec}^{-1}(z) \geq -\int_0^{\infty} e^{At} Re^{At} \, dt \). If \( P \) is such that \( \min_{P \in \mathcal{P}} \| P \|_F \) holds, then it follows that \( P \) is the unique
solution of a least squares minimization problem and is given by
\[ P = -\text{vec}^{-1}((A \oplus A)^T)^{\#} \text{vec} \, R = \int_0^\infty e^{tA}Re^{A^T} \, dt. \] (38)

Finally, suppose \((A, C)\) is semiobservable. Then it follows from the first part of the theorem that there exists an \(n \times n\) matrix \(P \geq 0\) such that Eq. (16) holds. Since, by Lemma 2.4, \(\mathcal{N}(A) \cap \mathcal{R}(A) = \emptyset\), it follows from Lemma 4.14 of [7] that \(A\) is group invertible. Thus, let \(L \triangleq I_n - AA^\#\) and note that \(L^2 = L\). Hence, \(L\) is the unique \(n \times n\) matrix satisfying \(\mathcal{N}(L) = \mathcal{R}(A), \mathcal{R}(L) = \mathcal{N}(A),\) and \(Lx = x\) for all \(x \in \mathcal{N}(A)\). Now, define
\[ \hat{P} \triangleq P + L^TL. \] (39)
Next, we show that \(\hat{P}\) is positive definite. Consider the function \(V(x) = x^T\hat{P}x, x \in \mathbb{R}^n\). If \(V(x) = 0\) for some \(x \in \mathbb{R}^n\), then \(Px = 0\) and \(Lx = 0\). It follows from (i) of Lemma 2.4 that \(x \in \mathcal{N}(A),\) and \(Lx = 0\) implies that \(x \in \mathcal{R}(A)\). Now, it follows from (ii) of Lemma 2.4 that \(x = 0\). Hence, \(\hat{P}\) is positive definite. Next, since \(LA = A - AA^\#A = 0\), it follows that
\[ A^T\hat{P} + \hat{P}A + R = A^TP + PA + R + A^TL^TL + L^TLA = (LA)^TL + L^TLA = 0. \] (40)

Conversely, if there exists \(P > 0\) such that Eq. (16) holds, consider the function \(U(x) = x^TPx, x \in \mathbb{R}^n\). Then \(\dot{U}(x) = -x^TRx \leq 0\) and \(\dot{U}^{-1}(0) = \mathcal{N}(R)\). To obtain the largest invariant set \(\mathcal{M}\) contained in \(\mathcal{N}(R)\), consider a solution \(x(t)\) of Eq. (8) such that \(Cx(t) = 0\) for all \(t \geq 0\). On \(\mathcal{M}\), it follows that \(C(d^{k-1}/dt^{k-1})x(t) = 0\) for all \(t \geq 0\) and \(k = 1, \ldots, n\), and hence, \(CA^{k-1}x(t) = 0\) for all \(t \geq 0\) and \(k = 1, \ldots, n\). Now, it follows from Eq. (5) that \(x(t) \in \mathcal{N}(A)\) for all \(t \geq 0\). Thus, \(\mathcal{M} \subseteq \mathcal{N}(A)\). Since \(\mathcal{N}(A)\) consists of equilibrium points, it follows that \(\mathcal{M} = \mathcal{N}(A)\). For \(x_c \in \mathcal{N}(A)\), Lyapunov stability of \(x_c\) now follows by considering the Lyapunov function \(U(x-x_c)\). \(\square\)

**Example 2.2.** Consider the two-agent network consensus problem given by the linear system (8) where \(A\) is given by [18]
\[ A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \] (41)
Note that \(\mathcal{N}(A) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = x, x \in \mathbb{R}\}\). Let \(C = [1, -1]\). It is easy to verify that Eq. (16) holds with
\[ P = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \] (42)
Furthermore, note that \(\mathcal{N}(C) = \mathcal{N}(A)\), and hence, the pair \((A, C)\) is semiobservable. Now, it follows from Theorem 2.2 that Eq. (8) is semistable. \(\square\)

Next, we show that the unique solution \(P\) given by Eq. (16) and satisfying \(\min_{P \in \mathcal{P}} \|P\|_F\) can be characterized by a linear matrix inequality minimization problem.

**Theorem 2.3.** Consider the linear dynamical system (8) with output (3). Assume \(A\) is semistable and \((A, C)\) is semiobservable. Let \(P_{\min}\) be the solution to the linear matrix inequality minimization problem
\[ \min \{\text{tr} \; PV : P \geq 0 \text{ and } A^TP + PA + R \leq 0\}, \] (43)
where $V \in \mathbb{R}^{n \times n}$, $V \geq 0$. Then

$$\text{tr } P_{\min} V = \text{tr } \int_0^\infty e^{A^T t} R e^{A t} \, dt \, V.$$  \hfill (44)

**Proof.** Let $\hat{P} = \int_0^\infty e^{A^T t} R e^{A t} \, dt$ and let $P \geq 0$ be such that

$$A^T P + PA + R \leq 0.$$  \hfill (45)

(Note that $A^T \hat{P} + \hat{P} A + R = 0$, which implies that a $P \geq 0$ satisfying Eq. (45) exists.) Now, let $W \in \mathbb{R}^{n \times n}$, $W \geq 0$, be such that

$$0 = A^T P + PA + R + W.$$  \hfill (46)

Next, since $(A, C)$ is semiobservable, it follows that if $x_c \in \mathcal{N}(A)$, then $R x_c = 0$, and hence, it follows from Eq. (46) that $W x_c = 0$. Now, using identical arguments as in the proof of Theorem 2.2 it follows that

$$P = \int_0^\infty e^{A^T t}(R + W)e^{A t} \, dt \geq \int_0^\infty e^{A^T t} Re^{A t} \, dt = \hat{P}.$$  \hfill (47)

Finally, since $\hat{P}$ is an element of the feasible set of the optimization problem (43),

$$\text{tr } P_{\min} V = \text{tr } \hat{P} V.$$  \hfill \Box

Finally, we provide a dual result to Theorem 2.3 which is necessary for developing feedback controllers guaranteeing closed-loop semistability.

**Theorem 2.4.** Consider the linear dynamical system (8) with output (3). Assume $A$ is semistable and let $V \in \mathbb{R}^{n \times n}$, $V \geq 0$, be such that $(A, V)$ is semicontrollable. Let $Q_{\min}$ be the solution to the LMI minimization problem

$$\min \{ \text{tr } Q R : Q \geq 0 \text{ and } A Q + Q A^T + V \leq 0 \}.$$  \hfill (48)

Then

$$\text{tr } Q_{\min} R = \text{tr } \int_0^\infty e^{A^T t} R e^{A t} \, dt \, V = \text{tr } P_{\min} V,$$  \hfill (49)

where $P_{\min}$ is the solution to the LMI minimization problem given by Eq. (43).

**Proof.** The proof is a direct consequence of Theorem 2.3 by noting that $(A, V)$ is semicontrollable if and only if $(A^T, V)$ is semiobservable. Now, replacing $A$ with $A^T$ and $R$ with $V$ in Theorem 2.3 it follows that

$$\text{tr } Q_{\min} R = \text{tr } \int_0^\infty e^{A t} V e^{A^T t} \, dt \, R = \text{tr } \int_0^\infty e^{A^T t} Re^{A t} \, dt \, V = \text{tr } P_{\min} V.$$  \hfill (50)

This completes the proof. \hfill \Box

**3. Optimal semistable stabilization**

In this section, we consider the problem of optimal state feedback control for semistable stabilization of linear dynamical systems. Specifically, we consider the controlled linear system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0,$$  \hfill (51)
where \( x(t) \in \mathbb{R}^n \), \( t \geq 0 \), is the state vector, \( u(t) \in \mathbb{R}^m \), \( t \geq 0 \), is the control input, \( A \in \mathbb{R}^{n \times n} \), and \( B \in \mathbb{R}^{n \times m} \), with the state feedback controller \( u(t) = Kx(t) \), where \( K \in \mathbb{R}^{m \times n} \), such that the closed-loop system given by

\[
\dot{x}(t) = (A + BK)x(t), \quad x(0) = x_0, \quad t \geq 0
\]

is semistable and the performance criterion

\[
J(K) \triangleq \int_0^\infty [(x(t) - x_c)^T R_1 (x(t) - x_c) + (u(t) - u_c)^T R_2 (u(t) - u_c)] \, dt
\]

is minimized, where \( R_1 \triangleq E_1^T E_1 \), \( R_2 \triangleq E_2^T E_2 > 0 \), \( R_{12} \triangleq E_1^T E_2 = 0 \), \( u_c = Kx_c \), and \( x_c = \lim_{t \to \infty} x(t) \).

The difficulty in addressing the optimal semistable control problem is that the classical optimal control theory cannot be applied to this optimal control problem. More specifically, Examples 2.1–2.3 in [19] show that the optimal semistable control problem proposed here may have a unique solution, infinitely many solutions, and no solution at all, whereas the classical \( H_2 \) optimal control problem involves a unique optimal solution. Hence, it is necessary to develop a new optimal control framework to solve the optimal semistable control problem.

Note that it follows from Lemma 2.1 that if the closed-loop system is semistable, then \( J(K) \) is well defined. To develop necessary conditions for the optimal semistable control problem, we assume that \((A, B)\) is semicontrollable, \((A, E_1)\) is semiobservable, and \( x_c \in \mathcal{N}(K) \). In this case, it follows from Proposition 2.1 that \((A + BK, R_1 + K^T R_2 K)\) is semiobservable with respect to \( K \), and hence, \((R_1 + K^T R_2 K)x_c = 0\). Thus,

\[
J(K) = \int_0^\infty x_0^T e^{\hat{A}^T t} (R_1 + K^T R_2 K) e^{\hat{A} t} x_0 \, dt
\]

\[
= \text{tr} \int_0^\infty e^{\hat{A}^T t} (R_1 + K^T R_2 K) e^{\hat{A} t} x_0 x_0^T \, dt = \text{tr} P_{LS} V,
\]

where we assume that the initial state \( x_0 \) is a random variable such that \( \mathbb{E}[x_0] = 0 \) and \( \mathbb{E}[x_0 x_0^T] = V \), \( \hat{A} \triangleq A + BK \), and \( P_{LS} \triangleq \int_0^\infty e^{\hat{A}^T t} (R_1 + K^T R_2 K) e^{\hat{A} t} \, dt \) denotes the least squares solution to

\[
0 = \hat{A}^T P + P \hat{A} + \hat{R},
\]

where \( \hat{R} \triangleq R_1 + K^T R_2 K \). Unlike the standard \( H_2 \) optimal control problem, \( P_{LS} \geq 0 \) is not a unique solution to Eq. (55).

The following theorem presents an LMI solution to the \( H_2 \) optimal semistable control problem.

**Theorem 3.1.** Consider the linear dynamical system (51) and assume \((A, E_1)\) is semiobservable and \((A, V)\) is semicontrollable. Let \( Q \in \mathbb{R}^{n \times n} \) and \( X \in \mathbb{R}^{m \times n} \) be the solution to the LMI minimization problem

\[
\min_{Q \in \mathbb{R}^{n \times n}, X \in \mathbb{R}^{m \times n}, W \in \mathbb{R}^{p \times p}} \text{tr } W,
\]

subject to

\[
\begin{bmatrix}
Q & (E_1 Q + E_2 X)^T \\
E_1 Q + E_2 X & W
\end{bmatrix} > 0,
\]

where
Then, $K = XQ^{-1}$ is a semistabilizing controller for Eq. (51), that is, $A + BK$ is semistable. Furthermore, $K$ minimizes the $H_2$ performance criterion $J(K)$ given by Eq. (53).

**Proof.** Since $K = XQ^{-1}$ it follows from Eq. (58) that

$$\begin{align*}
(A + BK)Q + Q(A + BK)^T + V &\leq 0,
\end{align*}$$

which, since $(A, V)$ is semicontrollable, implies that $A + BK$ is semistable. Next, note that Eq. (57) holds if and only if

$$W > (E_1Q + E_2X)Q^{-1}(E_1Q + E_2X)^T,$$

which implies that the minimization problem (56)–(58) is equivalent to

$$\min \operatorname{tr}(E_1Q + E_2X)Q^{-1}(E_1Q + E_2X)^T,$$

subject to

$$AQ + BX + QA^T + X^TB^T + V \leq 0,$$

$$Q > 0.$$  \hspace{1cm} (62)

$$Q \geq 0,$$  \hspace{1cm} (63)

Hence, noting that (61)–(63) is equivalent to

$$\min \operatorname{tr} \tilde{Q} \tilde{R},$$

subject to

$$\tilde{AQ} + \tilde{Q}^T \tilde{A}^T + V \leq 0,$$

$$Q > 0,$$

the result follows as a direct consequence of Theorems 2.4 and 2.2. \qed

**Remark 3.1.** If the dimension of the system is very large, then LMIs cannot be implemented in finite time. In this case, instead of seeking a globally optimal solution, we can solve for an approximate suboptimal solution. For the LMI problem developed by Theorem 3.1, a suboptimal solution to this problem can be obtained by using a two-stage optimization process. Specifically, by fixing $Q$ one can design the controller $K$. Then, with $K$ fixed, $Q$ can be obtained. This process continues until convergence or an acceptable controller is found.

Finally, we note that the framework developed in this paper is restricted to time-invariant dynamical systems. However, part of the results developed in the paper can be applied to switched systems. For details, see [20].

4. Conclusion

In this paper, we extended $H_2$ theory to include semistable systems. Using this framework along with linear matrix inequalities we developed an $H_2$ optimal semistable stabilization framework for linear dynamical systems. Future extensions will concentrate on the development of mixed-norm $H_2/H_{\infty}$, $H_2/L_1$, and $H_{\infty}/L_1$ semistable stabilization problems.
References