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NONLINEAR DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES: FILIPPOV SOLUTIONS, NONSMOOTH STABILITY AND DISSIPATIVITY THEORY, AND OPTIMAL DISCONTINUOUS FEEDBACK CONTROL

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This paper is dedicated to the memory of V. Lakshmikantham: a true scholar and a friend.

ABSTRACT. In this paper, we develop stability, dissipativity, and optimality notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. In addition, we extend classical dissipativity theory to address the problem of dissipative discontinuous dynamical systems. These results are then used to derive extended Kalman-Yakubovich-Popov conditions for characterizing necessary and sufficient conditions for dissipativity of discontinuous systems using Clarke gradients and locally Lipschitz continuous storage functions. In addition, feedback interconnection stability results for discontinuous systems are developed thereby providing a generalization of the small gain and positivity theorems to systems with discontinuous vector fields. Moreover, we consider a discontinuous control problem involving a notion of optimality that is directly related to a specified nonsmooth Lyapunov function to obtain a characterization of optimal discontinuous feedback controllers. Furthermore, using the newly developed dissipativity notions we develop a return difference inequality to provide connections between dissipativity and optimality of nonlinear discontinuous controllers for Filippov dynamical systems. Specifically, using the extended Kalman-Yakubovich-Popov conditions we show that our discontinuous feedback control law satisfies a return difference inequality if and only if the controller is dissipative with respect to a quadratic supply rate.

Key words and phrases: Discontinuous systems, differential inclusions, Filippov solutions, stability theory, nonsmooth Lyapunov functions, semistability, finite-time stability, dissipativity theory, Kalman-Yakubovich-Popov conditions, Clarke generalized gradients, set-valued Lie derivatives, nonlinear control, optimal control, inverse optimality, gain, sector, and disk margins.

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1. INTRODUCTION

Numerous engineering applications give rise to discontinuous dynamical systems. Specifically, in impact mechanics the motion of a dynamical system is subject to velocity jumps and force discontinuities leading to nonsmooth dynamical systems [1,2]. In mechanical systems subject to unilateral constraints on system positions [3], discontinuities occur naturally through system-environment interactions. Alternatively, control of networks and control over networks with dynamic topologies also give rise to discontinuous systems [4]. Specifically, link failures or creations in network systems result in switchings of the communication topology leading to dynamical systems with discontinuous right-hand sides. In addition, open-loop and feedback controllers also give rise to discontinuous dynamical systems. In particular, bang-bang controllers discontinuously switch between maximum and minimum control input values to generate minimum-time system trajectories [5], whereas sliding mode controllers [6, 7] use discontinuous feedback control for system stabilization. In switched systems [8,9], switching algorithms are used to select an appropriate plant (or controller) from a given finite parameterized family of plants (or controllers) giving rise to discontinuous systems.

In the case where the vector field defining the dynamical system is a discontinuous function of the state, system stability can be analyzed using nonsmooth Lyapunov theory involving concepts such as weak and strong stability notions, differential inclusions, and generalized gradients of locally Lipschitz continuous functions and proximal subdifferentials of lower semicontinuous functions [10]. The consideration of nonsmooth Lyapunov functions for proving stability of discontinuous systems is an important extension to classical stability theory since, as shown in [11], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory.

In many applications of discontinuous dynamical systems such as mechanical systems having rigid-body modes, isospectral matrix dynamical systems, and consensus protocols for dynamical networks, the system dynamics give rise to a continuum of equilibria. Under such dynamics, the limiting system state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [12, 13], and not asymptotic stability, is the relevant notion of stability. Semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium.

To address the stability analysis of discontinuous dynamical systems having a continuum of equilibria, in this paper we extend the theory of semistability to discontinuous time-invariant dynamical systems. In particular, we develop sufficient conditions to guarantee weak and strong invariance of Filippov solutions. Moreover, we present Lyapunov-based tests for semistability of autonomous differential inclusions. In addition, we develop sufficient conditions for finite-time semistability of autonomous discontinuous dynamical systems.

Many physical and engineering systems are *open systems*, that is, the system behaviour is described by an evolution law that involves the system state and the system input with, possibly, an output equation wherein past trajectories together with the knowledge of any inputs define future trajectories (uniquely or nonuniquely) and the system output depends on the instantaneous (present) values of the system state. Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of open dynamical systems based on generalized system energy considerations. In particular, dissipativity theory exploits the notion that numerous physical dynamical systems have certain input-output and state properties related to conservation, dissipation, and transport of mass and energy. Such conservation laws are prevalent in dynamical systems, in general, and feedback control systems, in particular. The dissipation hypothesis on dynamical systems results in a fundamental constraint on the system dynamical behavior, wherein the stored energy of a dissipative dynamical system is at most equal to sum of the initial energy stored in the system and the total externally supplied energy to the system. Thus, the energy that can be extracted from the system through its input-output ports is less than or equal to the initial energy stored in the system, and hence, there can be no internal creation of energy; only conservation or dissipation of energy is possible.

The key foundation in developing dissipativity theory for nonlinear dynamical systems with continuously differentiable flows was presented by Willems [14,15] in his seminal two-part paper on dissipative dynamical systems. In particular, Willems [14] introduced the definition of dissipativity for general nonlinear dynamical systems in terms of a *dissipation inequality* involving a generalized system power input, or *supply rate*, and a generalized energy function, or *storage function*. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system during this time interval. The set of all possible system storage functions is convex and every system storage function is bounded from below by the *available system storage* and bounded from above by the *required energy supply*.

In light of the fact that energy notions involving conservation, dissipation, and transport also arise naturally for discontinuous systems, it seems natural that dissipativity theory can play a key role in the analysis and control design of discontinuous dynamical systems. Specifically, as in the analysis of continuous dynamical systems with continuously differentiable flows, dissipativity theory for discontinuous dynamical systems can involve conditions on system parameters that render an input, state, and output system dissipative. In addition, robust stability for discontinuous dynamical systems can be analyzed by viewing a discontinuous dynamical system as an interconnection of discontinuous dissipative dynamical subsystems. Alternatively, discontinuous dissipativity theory can be used to design discontinuous feedback controllers that add dissipation and guarantee stability robustness allowing discontinuous stabilization to be understood in physical terms. As for dynamical systems with continuously differentiable flows [16], dissipativity theory can play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections, and optimality for discontinuous dynamical systems.

Even though passivity notions for the specific problem of the control of mechanical systems with discontinuous friction-type nonlinearities are considered in [17–19] using input-to-state stability notions and set-valued nonlinearity extensions of the circle and Popov criterion, the general problem of dissipativity theory in the sense of Willems [14, 15] for discontinuous dynamical systems and its connections to nonlinear discontinuous feedback regulator theory and inverse optimal control have not been addressed in the literature. It is important to note, however, that the problem of stabilization for discontinuous systems with nonsmooth control Lyapunov functions has been extensively addressed in the literature; see [20–25] and the references therein. However, with the exception of [26, 27] that address the specific problem of $\mathcal{L}_2$-gain stabilizability, these results do not explore the underlying connections between steady-state viscosity supersolutions of the Hamilton-Jacobi-Bellman equation and nonsmooth closed-loop Lyapunov functions for guaranteeing both stability and optimality for discontinuous dynamical systems. In addition, gain, sector, and disk margin guarantees are not provided in the aforementioned references by exploiting connections between dissipativity theory, discontinuous nonlinear regulator theory, and an inverse optimal control problem.

In this paper, we develop Lyapunov-based tests for Lyapunov stability, semistability, finite-time stability, finite-time semistability, and asymptotic stability for nonlinear dynamical systems with discontinuous right-hand sides. Specifically, we develop new Lyapunov-based results for semistability that do not make assumptions of sign definiteness on the Lyapunov functions. Instead, our results extend the results of [13] to discontinuous systems and use nontangency notions between the discontinuous vector field and weakly invariant or weakly negatively invariant subsets of the level or sublevel sets of the Lyapunov function. It is important to note that our stability results are different from the results in the literature [28, 29] since the Lipschitz conditions in [28, 29] are not valid for the autonomous differential inclusions considered in the paper. Moreover, using an extended notion of control Lyapunov functions [21] we develop a universal feedback controller for discontinuous dynamical systems based on the existance of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients and set-valued Lie derivatives.

Next, we extend the results of [30] to develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. Moreover, we develop extended Kalman-Yakubovich-Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions. In addition, using the concepts of dissipativity for discontinuous dynamical systems with appropriate storage functions and supply rates, we construct nonsmooth Lyapunov functions for discontinuous feedback systems by appropriately combining the storage functions for the forward and feedback subsystems. General stability criteria are given for Lyapunov, asymptotic, and exponential stability as well as finite-time stability for feedback interconnections of discontinuous dynamical systems. In the case where the supply rate involves the net system power or weighted input-output energy, these results provide extensions of the positivity and small gain theorems to discontinuous dynamical systems.

Finally, we consider a notion of optimality that is directly related to a given nonsmooth Lyapunov function. Specifically, an optimal control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal discontinuous feedback controller. In addition, we develop sufficient conditions for gain, sector, and disk margin guarantees for Filippov nonlinear dynamical systems controlled by optimal and inverse optimal discontinuous regulators. Furthermore, we develop a counterpart to the classical return difference inequality for continuous-time systems with continuously differentiable flows [31,32] for Filippov dynamical systems and provide connections between dissipativity and optimality for discontinuous nonlinear controllers. In particular, we show an equivalence between dissipativity and optimality of discontinuous controllers holds for Filippov dynamical systems. Specifically, we show that an optimal nonlinear controller $\phi(x)$ satisfying a return difference condition is equivalent to the fact that the Filippov dynamical system with input u and output $y = -\phi(x)$ is dissipative with respect to a supply rate of the form $[u + y]^{\mathrm{T}}[u + y] - u^{\mathrm{T}}u$.

2. NOTATION AND MATHEMATICAL PRELIMINARIES

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denotes transpose. We write $\partial\mathcal{S}$ and $\overline{\mathcal{S}}$ to denote the boundary and the closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm on $\mathbb{R}^n$, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the *open ball centered* at α with *radius* ε, $\mathrm{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $T > t_0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$. Finally, the notions of openness, convergence, continuity, and compactness that we use throughout the paper refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

In this paper, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $f : \mathcal{D} \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [33] with respect to x, that is, f is bounded on a bounded neighborhood of every point x, excluding sets of measure zero, and admits an equilibrium point at $x_{\mathrm{e}} \in \mathcal{D}$; that is, $f(x_{\mathrm{e}}) = 0$.

An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a *Filippov solution* [33] of (1) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[f](x(t)), \quad \text{a.e.} \quad t \in [t_0, \tau], \tag{2}$$

where the *Filippov set-valued map* $\mathcal{K}[f] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{K}[f](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{f(\mathcal{B}_\delta(x)\backslash\mathcal{S}\}, \quad x \in \mathbb{R}^n, \tag{3}$$

$2^{\mathbb{R}^n}$ denotes the collection of all subsets of $\mathbb{R}^n$, $\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\mathrm{co}}$" denotes convex closure, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.[1] Note that since f is locally essentially bounded, $\mathcal{K}[f](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values. Thus, Filippov solutions are limits of solutions to $\mathcal{G}$ with f averaged over progressively smaller neighborhoods around the solution point, and hence, allow solutions to be defined at points where f itself is not defined. Hence, the tangent vector to a Filippov solution, when it exists, lies in the convex closure of the limiting values of the system vector field $f(\cdot)$ in progressively smaller neighborhoods around the solution point. Dynamical systems of the form given by (1) are called *differential inclusions* in the literature [34] and, for every state $x \in \mathbb{R}^n$, they specify a *set* of possible evolutions of $\mathcal{G}$ rather than a single one.

[1] Alternatively, we can consider Krasovskii solutions of (1) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[f](x)$ is replaced with $\mathcal{K}[f](x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\{f(\mathcal{B}_\delta(x))\}$ and where f is assumed to be locally bounded.

Since the Filippov set-valued map given by (3) is upper semicontinuous with nonempty, convex, and compact values, and $\mathcal{K}[f](\cdot)$ is also locally bounded [33, p. 85], it follows that Filippov solutions to (1) exist [33, Thm. 1, p. 77]. Recall that the Filippov solution $t \mapsto x(t)$ to (1) is a *right maximal solution* if it cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal Filippov solutions to (1) exist on $[t_0, \infty)$, and hence, we assume that (1) is *forward complete*. Recall that (1) is forward complete if and only if the Filippov solutions to (1) are uniformly globally sliding time stable [35, Lem 1, p. 182]. An *equilibrium point* of (1) is a point $x_{\mathrm{e}} \in \mathbb{R}^n$ such that $0 \in \mathcal{K}[f](x_{\mathrm{e}})$. It is easy to see that x_{e} is an equilibrium point of (1) if and only if the constant function $x(\cdot) = x_{\mathrm{e}}$ is a Filippov solution of (1). We denote the set of equilibrium points of (1) by $\mathcal{E}$. Since the set-valued map $\mathcal{K}[f](\cdot)$ is upper semicontinuous, it follows that $\mathcal{E}$ is closed.

To develop stability properties for discontinuous dynamical systems given by (1), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [24].

Definition 2.1 ([24], [25])**.** Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(\cdot)$ at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$V^{o}(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h}. \tag{4}$$

The *Clarke generalized gradient* $\partial V : \mathbb{R}^n \to 2^{\mathbb{R}^{1\times n}}$ of $V(\cdot)$ at x is the set

$$\partial V(x) \triangleq \mathrm{co}\left\{ \lim_{i\to\infty} \nabla V(x_i) : x_i \to x,\ x_i \notin \mathcal{N} \cup \mathcal{S} \right\}, \tag{5}$$

where co denotes the convex hull, ∇ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero of points where ∇V does not exist, $\mathcal{S}$ is any subset of $\mathbb{R}^n$ of measure zero, and the increasing unbounded sequence $\{x_i\}_{i \in \overline{\mathbb{Z}}_+} \subset \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$.

Note that (4) always exists. Furthermore, note that it follows from Definition 2.1 that the generalized gradient of V at x consists of all convex combinations of all the possible limits of the gradient at neighboring points where V is differentiable. In addition, note that since $V(\cdot)$ is Lipschitz continuous, it follows from Rademacher's theorem [36, Thm 6, p. 281] that the gradient $\nabla V(\cdot)$ of $V(\cdot)$ exists almost everywhere, and hence, $\nabla V(\cdot)$ is bounded. Specifically, for every $x \in \mathbb{R}^n$, every $\varepsilon > 0$, and every Lipschitz constant L for V on $\overline{\mathcal{B}}_\varepsilon(x)$, $\partial V(x) \subseteq \overline{\mathcal{B}}_L(0)$. Thus, since for every $x \in \mathbb{R}^n$, $\partial V(x)$ is convex, closed, and bounded, it follows that $\partial V(x)$ is compact.

In order to state the main results of this paper, we need some additional notation and definitions. Given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the *set-valued Lie derivative* $\mathcal{L}_f V : \mathbb{R}^n \to 2^{\mathbb{R}}$ of V with respect to f at x [25, 37] is defined as

$$\begin{aligned}\mathcal{L}_f V(x) &\triangleq \Big\{a \in \mathbb{R}: \text{ there exists } v \in \mathcal{K}[f](x) \text{ such that } p^{\mathrm{T}}v = a \text{ for all } p^{\mathrm{T}} \in \partial V(x)\Big\} \\ &\subseteq \bigcap_{p^{\mathrm{T}} \in \partial V(x)} p^{\mathrm{T}}\mathcal{K}[f](x). \end{aligned} \tag{6}$$

If $\mathcal{K}[f](x)$ is convex with compact values, then $\mathcal{L}_f V(x)$, $x \in \mathbb{R}^n$, is a closed and bounded, possibly empty, interval in $\mathbb{R}$. If $V(\cdot)$ is continuously differentiable at x, then $\mathcal{L}_f V(x) = \{\nabla V(x) \cdot v : v \in \mathcal{K}[f](x)\}$. In the case where $\mathcal{L}_f V(x)$ is nonempty, we use the notion $\max \mathcal{L}_f V(x)$ (resp., $\min \mathcal{L}_f V(x)$) to denote the largest (resp., smallest) element of $\mathcal{L}_f V(x)$. Furthermore, we adopt the convention $\max \varnothing = -\infty$. Finally, recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ [24, Def. 2.3.4] if, for all $v \in \mathbb{R}^n$, the right directional derivative $V'_+(x, v) \triangleq \lim_{h\to 0^+} \frac{1}{h}[V(x + hv) - V(x)]$ exists and $V'_+(x, v) = V^o(x, v)$. V is called *regular* on $\mathbb{R}^n$ if it is regular at every $x \in \mathbb{R}^n$.

3. NONSMOOTH STABILITY THEORY FOR DISCONTINUOUS DIFFERENTIAL EQUATIONS

In this section, we study the stability of discontinuous systems. For stating the main stability theorems we assume that all right maximal Filippov solutions to (1) exist on $[0, \infty)$. We say that a set $\mathcal{M}$ is *weakly positively invariant* (resp., *strongly positively invariant*) with respect to (1) if, for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a right maximal solution (resp., all right maximal solutions) of (1) [25,38]. The set $\mathcal{M} \subseteq \mathbb{R}^q$ is *weakly negatively invariant* if, for every $x \in \mathcal{N}$ and $t \geq 0$, there exist $z \in \mathcal{N}$ and a Filippov solution $\psi(\cdot)$ to (1) with $\psi(0) = z$ such that $\psi(t) = x$ and $\psi(\tau) \in \mathcal{N}$ for all $\tau \in [0, t]$. Finally, the set $\mathcal{M} \subseteq \mathbb{R}^q$ is *weakly invariant* if $\mathcal{M}$ is weakly positively invariant as well as weakly negatively invariant.

The next definition introduces the notion of Lyapunov stability, semistability, and asymptotic stability for discontinuous dynamical systems. The adjective "weak" is used in reference to a stability property when the stability property is satisfied by at least one Filippov solution starting from every initial condition in $\mathcal{D}$, whereas "strong" is used when the stability property is satisfied by all Filippov solutions starting from every initial condition in $\mathcal{D}$. In this section, however, we provide strong stability theorems for (1) and, hence, we omit the adjective "strong" in the statement of our results.

Definition 3.1. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (1). An equilibrium point $x_e \in \mathcal{D}$ of (1) is *Lyapunov stable* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every initial condition $x_0 \in \mathcal{B}_\delta(x_e)$ and every Filippov solution $x(t)$ with the initial condition $x(0) = x_0$, $x(t) \in \mathcal{B}_\varepsilon(x_e)$ for all $t \geq 0$. An equilibrium point $x_e \in \mathcal{D}$ of (1) is *semistable* if x_e is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing x_e such that, for all initial conditions in

$\mathcal{D}_0$, the Filippov solutions of (1) converge to a Lyapunov stable equilibrium point. An equilibrium point $x_{\rm e} \in \mathcal{D}$ of (1) is *asymptotically stable* if $x_{\rm e}$ is Lyapunvov stable and there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then the Filippov solutions of (1) converge to $x_{\rm e}$. An equilibrium point $x_{\rm e} \in \mathcal{D}$ of (1) is *exponentially stable* if there exits positive constants α, β, and δ such that if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then every Filipov solution to (1) satisfies $\|x(t)\| \leq \|x_0\|e^{-\beta}$, $t \geq 0$. The system (1) is *semistable* (resp., *asymptotically* stable) with respect to $\mathcal{D}$ if every Filippov solution with initial condition in $\mathcal{D}$ converges to a Lyapunov stable equilibrium (resp., the Lyapunov stable equilibrium $x_{\rm e}$). Finally, (1) is said to be *globally semistable* (resp., *globally asymptotically stable*, *globally exponentially stable*) if (1) is semistable (resp., asymptotically stable, exponentially stable) with respect to $\mathbb{R}^n$.

Next, we introduce the definition of finite-time semistability and finite-time stability of (1).

Definition 3.2. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (1). An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (1) is said to be *finite-time-semistable* (resp., *finite-time stable*) if there exist an open neighborhood $\mathcal{U} \subseteq \mathcal{D}$ of $x_{\rm e}$ and a function $T : \mathcal{U}\backslash\mathcal{E} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) For every $x \in \mathcal{U}\backslash\mathcal{E}$ and every Filippov solution $\psi(t)$ of (1) with $\psi(0) = x$, $\psi(t) \in \mathcal{U}\backslash\mathcal{E}$ for all $t \in [0, T(x))$, and $\lim_{t\to T(x)} \psi(t)$ exists (resp., $\lim_{t\to T(x)} \psi(t) = x_{\rm e}$) and is contained in $\mathcal{U} \cap \mathcal{E}$.

ii) $x_{\rm e}$ is semistable (resp., Lyapunov stable and $\mathcal{U} \cap \mathcal{E} = \{x_{\rm e}\}$).

An equilibrium point $x_{\rm e} \in \mathcal{E}$ of (1) is said to be *globally finite-time-semistable* (resp., *globally finite-time stable*) if it is finite-time-semistable (resp., finite-time stable) with $\mathcal{D} = \mathcal{U} = \mathbb{R}^n$. The system (1) is said to be *finite-time-semistable* if every equilibrium point in $\mathcal{E}$ is finite-time-semistable. Finally, (1) is said to be *globally finite-time-semistable* if every equilibrium point in $\mathcal{E}$ is globally finite-time-semistable.

Given an absolutely continuous curve $\gamma : [0, \infty) \to \mathbb{R}^n$, *the positive limit set of* γ is the set $\Omega(\gamma)$ of points $y \in \mathbb{R}^n$ for which there exists an increasing divergent sequence $\{t_i\}_{i=1}^\infty$ satisfying $\lim_{i\to\infty} \gamma(t_i) = y$. We denote the positive limit set of a Filippov solution $\psi(\cdot)$ of (1) by $\Omega(\psi)$. The positive limit set of a bounded Filippov solution of (1) is nonempty and weakly invariant with respect to (1) [33, Lem. 4, p. 130].

Next, we state sufficient conditions for stability of discontinuous dynamical systems. Here, we state the stability theorems for only the local case; the global stability theorems are similar except for the additional assumption of properness on the Lyapunov function and nonrestricting the domain of analysis.

Theorem 3.1 ([25, 39]). *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (1). *Let x_{e} be an equilibrium point of $\mathcal{G}$ and let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open and connected set with $x_{\mathrm{e}} \in \mathcal{D}$. If $V : \mathcal{D} \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous, and regular function such that* $\max \mathcal{L}_f V(x) \leq 0$ *(resp.,* $\max \mathcal{L}_f V(x) < 0$, *$x \neq x_{\mathrm{e}}$) for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$, then x_{e} is Lyapunov (resp., asymptotically) stable. Finally, if there exists scalars $\alpha, \beta, \gamma > 0$, and $p \geq 1$ such that $V : \mathcal{D} \to \mathbb{R}$ satisfies $\alpha \|x - x_{\mathrm{e}}\|^p \leq V(x) \leq \|x - x_{\mathrm{e}}\|^p$ and* $\max \mathcal{L}_f V(x) \leq -\gamma \|x - x_{\mathrm{e}}\|^p$ *for almost all $x \in \mathcal{D}$, $x \neq x_{\mathrm{e}}$, such that $\mathcal{L}_f V(x) \neq \varnothing$, then x_{e} is exponentially stable.*

The next result presents an extenson of the Krasovskii-LaSalle invariant set theorem to discontinuous dynamical systems.

Theorem 3.2 ([25, 39]). *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (1). *Let x_{e} be an equilibrium point of $\mathcal{G}$, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *such that $x_{\mathrm{e}} \in \mathcal{D}$, and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that, for every $x \in \mathcal{D}$ and every Filippov solution $\psi(\cdot)$ satisfying $\psi(t_0) = x$, there exists a compact subset $\mathcal{D}_{\mathrm{c}}$ of $\mathcal{D}$ containing $\psi(t)$ for all $t \geq 0$. Furthermore, assume that* $\max \mathcal{L}_f V(x) \leq 0$ *for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Finally, define $\mathcal{R} \triangleq \{x \in \mathcal{D} : 0 \in \mathcal{L}_f V(x)\}$ and let $\mathcal{M}$ be the largest weakly positively invariant subset of $\overline{\mathcal{R}} \cap \mathcal{D}$. If $x(t_0) \in \mathcal{D}_{\mathrm{c}}$, then $x(t) \to \mathcal{M}$ as $t \to \infty$. If, alternatively, $\mathcal{R}$ contains no invariant set other than $\{x_{\mathrm{e}}\}$, then the Filippov solution $x(t) \equiv x_{\mathrm{e}}$ of $\mathcal{G}$ is asymptotically stable for all $x_0 \in \mathcal{D}_{\mathrm{c}}$.*

Next, we develop Lyapunov-based semistability and finite-time semistability theory for discontinuous dynamical systems of the form given by (1). The following proposition is needed.

Proposition 3.1. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *and let $\psi(\cdot)$ be a Filippov solution of* (2) *with $\psi(0) \in \mathcal{D}$. If $z \in \Omega(\psi) \cap \mathcal{D}$ is a Lyapunov stable equilibrium point, then $z = \lim_{t\to\infty} \psi(t)$ and $\Omega(\psi) = \{z\}$.*

Proof. Suppose $z \in \Omega(\psi) \cap \mathcal{D}$ is Lyapunov stable and let $\varepsilon > 0$. Since z is Lyapunov stable, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $y \in \mathcal{B}_\delta(z)$ and every Filippov solution $\eta(\cdot)$ of (2) satisfying $\eta(0) = y$, $\eta(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq 0$. Now, since $z \in \Omega(\psi)$, it follows that there exists a divergent sequence $\{t_i\}_{i=1}^\infty$ in $[0, \infty)$ such that $\lim_{i\to\infty} \psi(t_i) = z$, and hence, there exists $k \geq 1$ such that $\psi(t_k) \in \mathcal{B}_\delta(z)$. It now follows from our construction of δ that $\psi(t) \in \mathcal{B}_\varepsilon(z)$ for all $t \geq t_k$. Since ε was chosen arbitrarily, it follows that $z = \lim_{t\to\infty} \psi(t)$. Thus, $\lim_{n\to\infty} \psi(t_n) = z$ for every divergent sequence $\{t_n\}_{n=1}^\infty$, and hence, $\Omega(\psi) = \{z\}$. □

Next, we present sufficient conditions for semistability of (1).

Theorem 3.3. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that, for every $x \in \mathcal{D}$ and every Filippov solution $\psi(\cdot)$ satisfying $\psi(0) = x$, there exists a compact subset of $\mathcal{D}$ containing $\psi(t)$ for all $t \geq 0$. Furthermore, assume that* $\max \mathcal{L}_f V(x) \leq 0$ *for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Finally, define*

$$\mathcal{R} \triangleq \{x \in \mathcal{D} : 0 \in \mathcal{L}_f V(x)\}. \tag{7}$$

If every point in the largest weakly positively invariant subset $\mathcal{M}$ of $\overline{\mathcal{R}} \cap \mathcal{D}$ is a Lyapunov stable equilibrium point, then (1) *is semistable with respect to $\mathcal{D}$.*

Proof. Let $x \in \mathcal{D}$, $\psi(\cdot)$ be a Filippov solution to (1) with $\psi(0) = x$, and $\Omega(\psi)$ be the positive limit set of ψ. First, we show that $\Omega(\psi) \subseteq \overline{\mathcal{R}}$. Since either $\max \mathcal{L}_f V(x) \leq 0$ or $\mathcal{L}_f V(x) = \varnothing$ for almost all $x \in \mathcal{D}$, it follows from Lemma 1 of [25] that $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t))$ exists and is contained in $\mathcal{L}_f V(\psi(t))$ for almost every $t \geq 0$. Now, by assumption, $V(\psi(t)) - V(\psi(\tau)) = \int_\tau^t \frac{\mathrm{d}}{\mathrm{d}t} V(\psi(s))\mathrm{d}s \leq 0$, $t \geq \tau$, and hence, $V(\psi(t)) \leq V(\psi(\tau))$, $t \geq \tau$, which implies that $V(\psi(t))$ is a nonincreasing function of time.

The continuity of V and the boundedness of ψ imply that $V(\psi(\cdot))$ is bounded. Hence, $\gamma_x \triangleq \lim_{t\to\infty} V(\psi(t))$ exists. Next, consider $p \in \Omega(\psi)$. There exists an increasing unbounded sequence $\{t_n\}_{n=1}^\infty$ in $[0, \infty)$ such that $\psi(t_n) \to p$ as $n \to \infty$. Since V is continuous on $\mathcal{D}$, it follows that $V(p) = V(\lim_{n\to\infty} \psi(t_n)) = \lim_{n\to\infty} V(\psi(t_n)) = \gamma_x$, and hence, $V(p) = \gamma_x$ for $p \in \Omega(\psi)$. In other words, $\Omega(\psi)$ is contained in a level set of V.

Let $y \in \Omega(\psi)$. Since $\Omega(\psi)$ is weakly positively invariant, there exists a Filippov solution $\hat{\psi}(\cdot)$ of (1) such that $\hat{\psi}(0) = y$ and $\hat{\psi}(t) \in \Omega(\psi)$ for all $t \geq 0$. Since $V(\Omega(\psi)) = \{V(y)\}$, $\frac{\mathrm{d}}{\mathrm{d}t} V(\hat{\psi}(t)) = 0$, and hence, it follows from Lemma 1 of [25] that $0 \in \mathcal{L}_f V(\hat{\psi}(t))$, that is, $\hat{\psi}(t) \in \mathcal{R}$ for almost all $t \in [0, \hat{t}]$. In particular, $y \in \mathcal{R}$. Since $y \in \Omega(\psi)$ was chosen arbitrarily, it follows that $\Omega(\psi) \subseteq \overline{\mathcal{R}}$.

Next, since $\Omega(\psi)$ is weakly positively invariant, it follows that $\Omega(\psi) \subseteq \mathcal{M}$. Moreover, since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point of (1), it follows from Proposition 3.1 that $\Omega(\psi)$ contains a single point and $\lim_{t\to\infty} \psi(t)$ is a Lyapunov stable equilibrium. Now, since $x \in \mathcal{D}$ was chosen arbitrarily, it follows from Definition 3.1 that (1) is semistable with respect to $\mathcal{D}$. $\square$

The following corollary to Theorem 3.3 provides sufficient conditions for finite-time semistability of (1).

Corollary 3.1. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that* $\max \mathcal{L}_f V(x) < 0$ *for almost all $x \in \mathcal{D} \backslash \mathcal{E}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. If every equilibrium in $\mathcal{D}$ is Lyapunov stable, then every equilibrium in $\mathcal{D}$ is semistable. If, in addition,*

$\max \mathcal{L}_f V(x) \le -\varepsilon < 0$ for almost every $x \in \mathcal{D} \backslash \mathcal{E}$ such that $\mathcal{L}_f V(x) \neq \varnothing$, then (1) *is finite-time semistable.*

Proof. To prove the first statement, suppose every equilibrium in $\mathcal{D}$, that is, every point in $\mathcal{E} \cap \mathcal{D}$ is Lyapunov stable. By Lyapunov stability, there exists an open set $\mathcal{D}'$ containing $\mathcal{E} \cap \mathcal{D}$ such that $\mathcal{D}'$ is strongly positively invariant with respect to (1) and every Filippov solution having initial condition in $\mathcal{D}'$ is bounded. Let $\mathcal{M}$ denote the largest weakly positively invariant subset of the set $\mathcal{R}' \triangleq \{x \in \mathcal{D}' : 0 \in \mathcal{L}_f V(x)\}$. Note that $0 \in \mathcal{L}_f V(x)$ for every $x \in \mathcal{E}$. Since $\mathcal{E} \cap \mathcal{D}$ is weakly positively invariant and contained in $\mathcal{D}'$, it follows that $\mathcal{E} \cap \mathcal{D} \subseteq \mathcal{M}$. Since either $\max \mathcal{L}_f V(x) < 0$ or $\mathcal{L}_f V(x) = \varnothing$ for almost all $x \in \mathcal{D} \backslash \mathcal{E}$, it follows that $\mathcal{R}' \subseteq \mathcal{E}$. Hence, it follows that $\mathcal{M} = \mathcal{E} \cap \mathcal{D}$. Theorem 3.3 now implies that (1) is semistable with respect to $\mathcal{D}'$. Since $\mathcal{E} \cap \mathcal{D} = \mathcal{E} \cap \mathcal{D}'$, it follows that every equilibrium in $\mathcal{D}$ is semistable.

If, in addition, $\max \mathcal{L}_f V(x) \le -\varepsilon < 0$ for almost every $x \in \mathcal{D} \backslash \mathcal{E}$ such that $\mathcal{L}_f V(x) \neq \varnothing$, then it follows from Proposition 2.8 of [37] that every Filippov solution originating in $\mathcal{D}'$ reaches $\mathcal{R}'$ in finite time. Thus, it follows from Definition 3.2 that (1) is finite-time-semistable. □

Example 3.1. Consider the nonlinear switched dynamical system on $\mathcal{D} = \mathbb{R}^2$ given by

$$\dot{x}_1(t) = f_{\sigma(t)}(x_2(t)) - g_{\sigma(t)}(x_1(t)), \quad x_1(0) = x_{10}, \quad t \ge 0, \quad \sigma(t) \in \mathcal{S}, \tag{8}$$

$$\dot{x}_2(t) = g_{\sigma(t)}(x_1(t)) - f_{\sigma(t)}(x_2(t)), \quad x_2(0) = x_{20}, \tag{9}$$

where $x_1, x_2 \in \mathbb{R}$, $\sigma : [0, \infty) \to \mathcal{S}$ is a piecewise constant switching signal, $\mathcal{S}$ is a finite index set, for every $\sigma \in \mathcal{S}$, $f_\sigma(\cdot)$ and $g_\sigma(\cdot)$ are Lipschitz continuous, $f_\sigma(x_2) - g_\sigma(x_1) = 0$ if and only if $x_1 = x_2$, and $(x_1 - x_2)(f_\sigma(x_2) - g_\sigma(x_1)) \le 0$, $x_1, x_2 \in \mathbb{R}$. Note that $f^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$. To show that (8) and (9) is semistable, consider the Lyapunov function candidate $V(x_1 - \alpha, x_2 - \alpha) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that

$$\begin{aligned} \dot{V}(x_1 - \alpha, x_2 - \alpha) &= (x_1 - \alpha)[f_\sigma(x_2) - g_\sigma(x_1)] + (x_2 - \alpha)[g_\sigma(x_1) - f_\sigma(x_2)] \\ &= x_1[f_\sigma(x_2) - g_\sigma(x_1)] + x_2[g_\sigma(x_1) - f_\sigma(x_2)] \\ &= (x_1 - x_2)[f_\sigma(x_2) - g_\sigma(x_1)] \\ &\le 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{10}$$

which, by Theorem 3.1, implies that $x_1 = x_2 = \alpha$ is Lyapunov stable for all $\alpha \in \mathbb{R}$.

Next, we rewrite (8) and (9) in the form of the differential inclusion (2) where $x \triangleq [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ and $f(x) \triangleq [f_\sigma(x_2) - g_\sigma(x_1), g_\sigma(x_1) - f_\sigma(x_2)]^{\mathrm{T}}$. Let v_x be an arbitrary element of $\mathcal{K}[f](x)$ and note that the Clarke upper generalized derivative of $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ along a vector $v_x \in \mathcal{K}[f](x)$ is given by $V^o(x, v_x) = x^{\mathrm{T}} v_x$. Furthermore, note that the set $\mathcal{D}_{\mathrm{c}} \triangleq \{x \in \mathbb{R}^2 : V(x) \le c\}$, where $c > 0$, is a compact set. Next, consider $\max V^o(x,v_x) \triangleq \max_{v_x\in\mathcal{K}[f]}\{x^{\mathrm{T}}v_x\}$. It follows from Theorem 1 of [40] and (10) that $x^{\mathrm{T}}\mathcal{K}[f](x) = \mathcal{K}[x^{\mathrm{T}}f](x) = \mathcal{K}\left[(x_1-x_2)(f_\sigma(x_2)-g_\sigma(x_1))\right](x)$, and hence, by definition of $\mathcal{K}[f](x)$, it follows that $\max V^o(x,v_x) = \max\overline{\mathrm{co}}\{(x_1-x_2)(f_\sigma(x_2)-g_\sigma(x_1))\}$. Note that since, by (10), $(x_1-x_2)(f_\sigma(x_2)-g_\sigma(x_1)) \le 0$, $x\in\mathbb{R}^2$, it follows that $\max V^o(x,v_x)$ cannot be positive, and hence, the largest value that $\max V^o(x,v_x)$ can achieve is zero.

Finally, let $\mathcal{R} \triangleq \{(x_1,x_2)\in\mathbb{R}^2 : (x_1-x_2)(f_\sigma(x_2)-g_\sigma(x_1)) = 0\} = \{(x_1,x_2)\in\mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha\in\mathbb{R}\}$. Since $\mathcal{R}$ consists of equilibrium points, it follows that $\mathcal{M} = \mathcal{R}$. Note that $\max\mathcal{L}_f V(x) \le \max V^o(x,v_x)$ for every $x\in\mathbb{R}^2$ [25]. Hence, it follows from Theorem 3.3 that $x_1 = x_2 = \alpha$ is semistable for all $\alpha\in\mathbb{R}$. △

Example 3.2. Consider the discontinuous dynamical system on $\mathcal{D} = \mathbb{R}^2$ given by

$$\dot{x}_1(t) = \mathrm{sign}(x_2(t)-x_1(t)), \quad x_1(0) = x_{10}, \quad t\ge 0, \tag{11}$$

$$\dot{x}_2(t) = \mathrm{sign}(x_1(t)-x_2(t)), \quad x_2(0) = x_{20}, \tag{12}$$

where $x_1,x_2\in\mathbb{R}$, $\mathrm{sign}(x) \triangleq x/|x|$ for $x\ne 0$, and $\mathrm{sign}(0) \triangleq 0$. Let $f(x_1,x_2) \triangleq [\mathrm{sign}(x_2-x_1), \mathrm{sign}(x_1-x_2)]^{\mathrm{T}}$. Consider $V(x_1,x_2) = \frac{1}{2}(x_1-\alpha)^2 + \frac{1}{2}(x_2-\alpha)^2$, where $\alpha\in\mathbb{R}$. Since $V(x_1,x_2)$ is differentiable at $x = (x_1,x_2)$, it follows that $\mathcal{L}_f V(x_1,x_2) = [x_1-\alpha, x_2-\alpha]\mathcal{K}[f](x_1,x_2)$. Now, it follows from Theorem 1 of [40] that

$$\begin{aligned}
[x_1-\alpha, x_2-\alpha]\mathcal{K}[f](x) &= \mathcal{K}[[x_1-\alpha, x_2-\alpha]f](x)\\
&= \mathcal{K}[-(x_1-x_2)\mathrm{sign}(x_1-x_2)](x)\\
&= -(x_1-x_2)\mathcal{K}[\mathrm{sign}(x_1-x_2)](x)\\
&= -(x_1-x_2)\mathrm{SGN}(x_1-x_2)\\
&= -|x_1-x_2|, \quad (x_1,x_2)\in\mathbb{R}^2,
\end{aligned} \tag{13}$$

where $\mathrm{SGN}(\cdot)$ is defined by [11, 40]

$$\mathrm{SGN}(x) \triangleq \begin{cases} -1, & x<0,\\ [-1,1], & x=0,\\ 1, & x>0. \end{cases} \tag{14}$$

Hence, $\max\mathcal{L}_f V(x_1,x_2) \le 0$ for almost all $(x_1,x_2)\in\mathbb{R}^2$. Now, it follows from Theorem 3.1 that $(x_1,x_2) = (\alpha,\alpha)$ is Lyapunov stable. Finally, note that $0\in\mathcal{L}_f V(x_1,x_2)$ if and only if $x_1 = x_2$, and hence, $\mathcal{R} = \{(x_1,x_2)\in\mathbb{R}^2 : x_1 = x_2\}$. Since $\mathcal{R}$ is weakly positively invariant and every point in $\mathcal{R}$ is a Lyapunov stable equilibrium, it follows from Theorem 3.3 that (11) and (12) is semistable.

Finally, we show that (11) and (12) is finite-time-semistable. To see this, consider the nonnegative function $U(x_1,x_2) = |x_1-x_2|$. Note that

$$\partial U(x_1,x_2) = \begin{cases} \{\mathrm{sign}(x_1-x_2)\}\times\{\mathrm{sign}(x_2-x_1)\}, & x_1\ne x_2,\\ [-1,1]\times[-1,1], & x_1 = x_2. \end{cases} \tag{15}$$

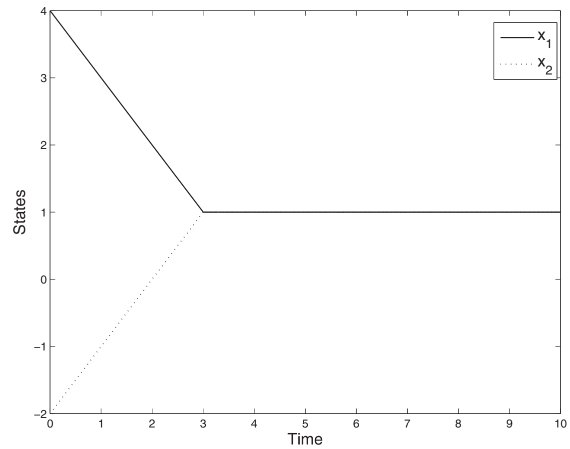

FIGURE 1. State trajectories versus time for Example 3.2

Hence, it follows that

$$\mathcal{L}_f U(x_1, x_2) = \begin{cases} \{-2\}, & x_1 \neq x_2, \\ \{0\}, & x_1 = x_2, \end{cases} \tag{16}$$

which implies that $\max \mathcal{L}_f U(x_1, x_2) = -2 < 0$ for almost all $(x_1, x_2) \in \mathbb{R}^2 \backslash \mathcal{R}$. Now, it follows from Corollary 3.1 that (11) and (12) is globally finite-time-semistable. Figure 1 shows the solutions of (11) and (12) for $x_{10} = 4$ and $x_{20} = -2$. $\triangle$

Note that Theorem 3.3 and Corollary 3.1 require verifying Lyapunov stability for concluding semistability and finite-time semistability, respectively. However, finding the corresponding Lyapunov function can be a difficult task. To overcome this drawback, we extend the nontangency-based approach of [13] to discontinuous dynamical systems in order to guarantee semistability and finite-time semistability by testing a condition on the vector field f which avoids proving Lyapunov stability. Before stating our result, we introduce some notation and definitions as well as extended versions of some results from [13].

A set $\mathcal{E} \subseteq \mathbb{R}^n$ is *connected* if and only if every pair of open sets $\mathcal{U}_i \subseteq \mathbb{R}^n$, $i = 1, 2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \varnothing$, $i = 1, 2$, has a nonempty intersection. A *connected component* of the set $\mathcal{E} \subseteq \mathbb{R}^n$ is a connected subset of $\mathcal{E}$ that is not properly contained in any connected subset of $\mathcal{E}$. Given a set $\mathcal{E} \subseteq \mathbb{R}^n$, let $\operatorname{coco} \mathcal{E}$ denote the convex cone generated by $\mathcal{E}$.

Definition 3.3. Given $x \in \mathbb{R}^n$, the *direction cone* $\mathcal{F}_x$ of f at x is the intersection of closed convex cones of the form $\overline{\bigcap_{\mu(\mathcal{S})=0} \operatorname{coco}\{f(\mathcal{U} \backslash \mathcal{S})\}}$, where $\mathcal{U} \subseteq \mathbb{R}^n$ is an open neighborhood of x. Let $\mathcal{E} \subseteq \mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is *tangent* to $\mathcal{E}$ at $z \in \mathcal{E}$ if there exist a sequence $\{z_i\}_{i=1}^{\infty}$ in $\mathcal{E}$ converging to z and a sequence $\{h_i\}_{i=1}^{\infty}$ of positive real numbers converging to zero such that $\lim_{i\to\infty} \frac{1}{h_i}(z_i - z) = v$. The *tangent cone* to $\mathcal{E}$ at z is the closed cone $T_z\mathcal{E}$ of all vectors tangent to $\mathcal{E}$ at z. Finally, the vector field f is *nontangent* to the set $\mathcal{E}$ at the point $z \in \mathcal{E}$ if $T_z\mathcal{E} \cap \mathcal{F}_z \subseteq \{0\}$.

Definition 3.4. Given a point $x \in \mathbb{R}^n$ and a bounded open neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of x, the *restricted prolongation* of x with respect to $\mathcal{U}$ is the set $\mathcal{R}_x^{\mathcal{U}} \subseteq \overline{\mathcal{U}}$ of all subsequential limits of sequences of the form $\{\psi_i(t_i)\}_{i=1}^{\infty}$, where $\{t_i\}_{i=1}^{\infty}$ is a sequence in $[0,\infty)$, $\psi_i(\cdot)$ is a Filippov solution to (1) with $\psi_i(0) = x_i$, $i = 1, 2, \ldots$, and $\{x_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{U}$ converging to x such that the set $\{z \in \mathbb{R}^n : z = \psi_i(t), t \in [0, t_i]\}$ is contained in $\overline{\mathcal{U}}$ for every $i = 1, 2, \ldots$.

Proposition 3.2. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1). *Furthermore, let $x \in \mathcal{D}$ and let $\mathcal{U} \subseteq \mathcal{D}$ be a bounded open neighborhood of x. Then $\mathcal{R}_x^{\mathcal{U}}$ is connected. Moreover, if x is an equilibrium point of* (1)*, then $\mathcal{R}_x^{\mathcal{U}}$ is weakly negatively invariant.*

Proof. The proof of connectedness is similar to the proof of the first part of Proposition 6.1 of [13] and, hence, is omitted. To prove weak negative invariance, suppose $x \in \mathcal{D}$ is an equilibrium point of (1), and consider $z \in \mathcal{R}_x^{\mathcal{U}}$. Then there exist a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0,\infty)$, a sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{D}$ converging to x, and a sequence $\{\psi_i(\cdot)\}_{i=1}^{\infty}$ of Filippov solutions of (1) such that $\lim_{i\to\infty} \psi_i(t_i) = z$ and, for every i, $\psi_i(0) = x_i$ and $\psi_i(h) \in \overline{\mathcal{U}}$ for every $h \in [0, t_i]$.

Now, let $t \geq 0$. First, assume $z = x$. Then $\psi \equiv x$ is a Filippov solution of (1) such that $\psi(0) = x$, $\psi(t) = z$ and $\psi(\tau) \in \mathcal{R}_x^{\mathcal{U}}$ for all $\tau \in [0, t]$. Next consider the case $z \neq x$. First, suppose that the sequence $\{t_i\}_{i=1}^{\infty}$ has a subsequence $\{t_{i_k}\}_{k=1}^{\infty}$ in $[0, t]$. By choosing a subsequence if necessary, we may assume that the subsequence $\{t_{i_k}\}_{k=1}^{\infty}$ converges to T. Necessarily, $T \leq t$. By Lemma 1 in [33, p. 87], a subsequence of the sequence $\{\psi_{i_k}\}_{k=1}^{\infty}$ converges uniformly on compact subsets of $(0, T)$ to a Filippov solution ψ of (1). Moreover, the solution ψ satisfies $\psi(0) = x$ and $\psi(T) = z$. For each $s \in [0, T]$, $\psi(s)$ is a subsequential limit of the sequence $\{\psi_{i_k}(s)\}_{k=1}^{\infty}$, and hence, contained in $\mathcal{R}_x^{\mathcal{U}}$. It is now easy to verify that the function $\beta : [0, t] \to \mathcal{D}$ defined by

$$\begin{aligned} \beta(s) &= x, & 0 \leq s \leq t - T, \\ &= \psi(s - t + T), & t - T < s \leq t, \end{aligned}$$

is a Filippov solution of (1) satisfying $\beta(0) = x$, $\beta(t) = z$, and $\beta(s) \in \mathcal{R}_x^{\mathcal{U}}$ for all $s \in [0, t]$.

Next, suppose that the sequence $\{t_i\}_{i=1}^{\infty}$ has no subsequence in $[0, t]$. Then there exists $N > 0$ such that $t_i > t$ for all $i \geq N$. For each i, define $\beta_i : [0, t] \to \mathcal{D}$ by $\beta(s) = \psi_{i+N}(t_{i+N} - t + s)$. Clearly, each β_i is a Filippov solution of (1). Moreover, the sequence $\{\beta_i(t)\}_{i=1}^{\infty}$ converges to z. Let $y \in \mathcal{D}$ be a subsequential limit of the bounded sequence $\{\beta_i(0)\}_{i=1}^{\infty}$. By definition, $y \in \mathcal{R}_x^{\mathcal{U}}$. By Lemma 1 in [33, p. 87], a subsequence of $\{\beta_i\}_{i=1}^{\infty}$ converges uniformly on compact subsets of $(0, t)$ to a Filippov solution β of (1). Moreover, we may choose the subsequence such that $\beta(0) = y$ and $\beta(t) = z$. Finally, for each $s \in [0, t]$, $\beta(s)$ is a subsequential limit of the sequence $\{\beta_i(s)\}_{i=1}^{\infty}$, and hence, in $\mathcal{R}_x^{\mathcal{U}}$. We have thus shown that there exists a Filippov solution β defined on $[0,t]$ such that $\beta(s) \in \mathcal{R}_x^{\mathcal{U}}$ for all $s \in [0,t]$ and $\beta(t) = z$. Since $t \geq 0$ and $z \in \mathcal{R}_x^{\mathcal{U}}$ were chosen to be arbitrary, it follows that $\mathcal{R}_x^{\mathcal{U}}$ is weakly negatively invariant. $\square$

The following two lemmas and proposition extend related results from [13], and are needed for the main result of this section.

Lemma 3.1. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that $V(x) \geq 0$, for all $x \in \mathcal{D}$, $V(z) = 0$ for all $z \in \mathcal{E}$, and $\max \mathcal{L}_f V(x) \leq 0$ for almost every $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. For every $z \in \mathcal{E}$, let $\mathcal{N}_z$ denote the largest weakly negatively invariant connected subset of $\overline{\mathcal{R}} \cap \mathcal{D}$ containing z, where $\mathcal{R}$ is given by (7). Then, for every $x \in \mathcal{E}$ and every bounded open neighborhood $\mathcal{V} \subset \mathcal{D}$ of x, $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{N}_x$.*

Proof. Let $x \in \mathcal{E}$ and let $\mathcal{V} \subset \mathcal{D}$ be a bounded open neighborhood of x. Consider $z \in \mathcal{R}_x^{\mathcal{V}}$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{V}$ converging to x and let $\{t_i\}_{i=1}^{\infty}$ be a sequence in $[0, \infty)$ such that the sequence $\{\psi_i(t_i)\}_{i=1}^{\infty}$ converges to z and, for every i, $\psi_i(\tau) \in \overline{\mathcal{V}} \subset \mathcal{D}$ for every $\tau \in [0, t_i]$, where $\psi_i(\cdot)$ is a Filippov solution to (1) with $\psi_i(0) = x_i$. Since either $\max \mathcal{L}_f V(y) \leq 0$ or $\mathcal{L}_f V(y) = \varnothing$ for almost every $y \in \mathcal{D}$, it follows from Lemma 1 of [25] that $\frac{\mathrm{d}}{\mathrm{d}t} V(\psi(t))$ exists and is contained in $\mathcal{L}_f V(\psi(t))$ for almost all $t \in [0, \tau]$, where $\psi(\cdot)$ is a Filippov solution to (1) with $\psi(0) = y$. Now, by assumption, $V(\psi(\tau)) - V(y) = \int_0^{\tau} \frac{\mathrm{d}}{\mathrm{d}t} V(\psi(s))\mathrm{d}s \leq 0$, $\tau \geq 0$, and hence, $V(\psi(\tau)) \leq V(y)$ for $y \in \mathcal{D}$ and $\tau \geq 0$.

Next, note that $V(z) = \lim_{i\to\infty} V(\psi_i(t_i)) \leq \lim_{i\to\infty} V(x_i) = V(x)$, and hence, $V(z) \leq V(x)$. Since $V(z) \geq 0$ and $V(x) = 0$ by assumption, it follows that $V(z) = V(x) = 0$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq V^{-1}(0) \cap \overline{\mathcal{V}} \subset V^{-1}(0)$. By Proposition 3.2, $\mathcal{R}_x^{\mathcal{V}}$ is weakly negatively invariant and connected, and $x \in \mathcal{R}_x^{\mathcal{V}}$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{M}_x$, where $\mathcal{M}_x$ denotes the largest, weakly, negatively invariant connected subset of $V^{-1}(0)$ containing x.

Finally, we show that $\mathcal{M}_x \subseteq \mathcal{N}_x$. Let $z \in \mathcal{M}_x$ and let $t > 0$. By weak negative invariance, there exists $w \in \mathcal{M}_x$ and a Filippov solution $\psi(\cdot)$ to (1) satisfying $\psi(0) = w$ such that $\psi(t) = z$ and $\psi(\tau) \in \mathcal{M}_x \subseteq V^{-1}(0)$ for all $\tau \in [0,t]$. Thus, $V(\psi(\tau)) = V(x) = 0$ for every $\tau \in [0,t]$, and hence, by Lemma 1 of [25], $0 \in \mathcal{L}_f V(\psi(\tau))$ for almost every $\tau \in [0,t]$, that is, $\psi(\tau) \in \mathcal{R}$ for almost every $\tau \in [0,t]$. It immediately follows that $z \in \overline{\mathcal{R}}$, and hence, $\mathcal{M}_x \subseteq \overline{\mathcal{R}}$. Since $\mathcal{M}_x$ is weakly negatively invariant, connected, contains x, and is contained in $\mathcal{U}$, it follows that $\mathcal{M}_x \subseteq \mathcal{N}_x$. Hence, $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{M}_x \subseteq \mathcal{N}_x$. $\square$

Lemma 3.2. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1). *Furthermore, let $x \in \mathcal{D}$ and let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{D}$ converging to x. Let $\mathcal{I}_i \subseteq [0, \infty)$, $i = 1, 2, \ldots$, be intervals containing 0, and let $\mathcal{B} \subseteq \mathcal{D}$ be the set of all* *subsequential limits contained in $\mathcal{D}$ of sequences of the form $\{\psi_i(\tau_i)\}_{i=1}^{\infty}$, where, for each i, $\tau_i \in \mathcal{I}_i$ and $\psi_i : \mathcal{I}_i \to \mathcal{D}$ is a Filippov solution of (1) satisfying $\psi_i(0) = x_i$. Then $\mathcal{B} = \{x\}$ if and only if f is nontangent to $\mathcal{B}$ at x.*

Proof. First, we note that $x \in \mathcal{B}$ since $x = \lim_{i\to\infty} \psi_i(0)$. Necessity now follows by noting that if $\mathcal{B} = \{x\}$, then $T_x\mathcal{B} = \{0\}$ and, hence, $T_x\mathcal{B} \cap \mathcal{F}_x \subseteq \{0\}$.

To prove sufficiency, suppose $z_0 \in \mathcal{B}$, $z_0 \neq x$. Let $\{\mathcal{U}_k\}_{k=1}^{\infty}$ be a nested sequence of bounded open neighborhoods of x in $\mathcal{D}$ such that $\overline{\mathcal{U}_{k+1}} \subset \mathcal{U}_k$ and $x_k \in \mathcal{U}_k$ for every $k = 1, 2, \ldots$, $\bigcap_k \mathcal{U}_k = \{x\}$ and $z_0 \notin \mathcal{U}_1$. Since $z_0 \in \mathcal{B}$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $\tau_i \in \mathcal{I}_i$ for every i, and $\lim_{i\to\infty} \psi_i(\tau_i) = z_0 \notin \mathcal{U}_1$. The continuity of Filippov solutions implies that, for every k, there exists a sequence $\{h_j^k\}_{j=k}^{\infty}$ in $[0, \infty)$ such that, for every $j \geq k$, $h_j^k \in \mathcal{I}_j$, $h_j^k \leq \tau_j$, $\psi_j(\tau) \in \mathcal{U}_k$ for every $\tau \in [0, h_j^k)$, and $\psi_j(h_j^k) \in \partial\mathcal{U}_k$. For each k, let $z_k \in \partial\mathcal{U}_k$ be a subsequential limit of the bounded sequence $\{\psi_j(h_j^k)\}_{j=k}^{\infty}$. Then, for every k, it follows that $z_k \in \mathcal{B}$, $z_k \neq x$ and $\lim_{k\to\infty} z_k = x$. Now, consider a subsequential limit v of the bounded sequence $\{\|z_k - x\|^{-1}(z_k - x)\}$. Clearly, $v \in T_x\mathcal{B}$. Also $\|v\| = 1$ so that $v \neq 0$. We claim that $v \in \mathcal{F}_x$.

Let $\mathcal{V} \subseteq \mathcal{D}$ be an open neighborhood of x and consider $\varepsilon > 0$. By construction, there exists k such that $\|v - \|z_k - x\|^{-1}(z_k - x)\| < \varepsilon/3$. Moreover, since $\bigcap_i \mathcal{U}_i = \{x\}$, we can assume that $\mathcal{U}_k \subseteq \mathcal{V}$. Since z_k belongs to the boundary of an open neighborhood of x, $\delta \triangleq \|z_k - x\| > 0$. Since $z_k = \lim_{i\to\infty} \psi_i(h_i^k)$ and $x = \lim_{i\to\infty} x_i$, there exists i such that $x_i \in \mathcal{V}$, $\|x - x_i\| < \varepsilon\delta/3$ and $\|z_k - \psi_i(h_i^k)\| < \varepsilon\delta/3$. Let $\mathcal{S} \subset \mathcal{D}$ be a zero measure set. Then, $\mathcal{K}[f](\psi_i(\tau)) \subseteq \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$ for all $\tau \in [0, h_i^k]$, so that $\dot{\psi}_i(\tau) \in \mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$ for almost every $\tau \in [0, h_i^k]$. Therefore, it follows from Theorem I.6.13 of [41, p. 145] that $w \triangleq \psi_i(h_i^k) - x_i = \int_0^{h_i^k} \dot{\psi}_i(\tau)d\tau$ is contained in the convex cone generated by $\mathrm{co}\{f(\mathcal{V}\backslash\mathcal{S})\}$. Since $\mathcal{S}$ was chosen to be an arbitrary zero-measure set, it follows that $w \in \bigcap_{\mu(\mathcal{S})=0} \mathrm{coco}\{f(\mathcal{V}\backslash\mathcal{S})\}$.

Now,

$$\begin{aligned}
\big\|v - \delta^{-1}w\big\| &= \big\|v - \delta^{-1}(z_k - x) - \delta^{-1}(\psi(h_i^k, x_i) - z_k) - \delta^{-1}(x - x_i)\big\| \\
&\leq \big\|v - \|z_k - x\|^{-1}(z_k - x)\big\| + \delta^{-1}\|\psi(h_i^k, x_i) - z_k\| + \delta^{-1}\|x - x_i\| \\
&< \varepsilon.
\end{aligned}$$

We have thus shown that, for every $\varepsilon > 0$ there exists $w \in \bigcap_{\mu(\mathcal{S})=0} \mathrm{coco}\{f(\mathcal{V}\backslash\mathcal{S})\}$ and $\delta > 0$ such that $w \neq 0$ and $\|v - \delta^{-1}w\| < \varepsilon$. It follows that v is contained in the closed convex cone $\overline{\bigcap_{\mu(\mathcal{S})=0} \mathrm{coco}\{f(\mathcal{V}\backslash\mathcal{S})\}}$. Since $\mathcal{V}$ was chosen to be an arbitrary open neighborhood of x, it follows that v is contained in $\mathcal{F}_x$. Thus, if $\mathcal{B} \neq \{x\}$, then there exists $v \in \mathbb{R}^n$ such that $v \neq 0$ and $v \in T_x\mathcal{B} \cap \mathcal{F}_x$, that is, f is not nontangent to $\mathcal{B}$ at x. Sufficiency now follows. □

Proposition 3.3. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (1). Furthermore, let $x \in \mathcal{D}$ and let $\mathcal{U} \subseteq \mathcal{D}$ be a bounded open neighborhood of x.*

If the vector field f of (1) *is nontangent to $\mathcal{R}_x^{\mathcal{U}}$ at x, then the point x is a Lyapunov stable equilibrium of* (1)*.*

Proof. Since f is nontangent to $\mathcal{R}_x^{\mathcal{U}}$ at x, by definition, it follows that $T_x\mathcal{R}_x^{\mathcal{U}} \cap \mathcal{F}_x \subseteq \{0\}$. Let $z \in \mathcal{R}_x^{\mathcal{U}}$. Then there exist a sequence $\{x_i\}_{i=1}^{\infty}$ converging to x, a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0,\infty)$, and a sequence $\{\psi_i\}_{i=1}^{\infty}$ of Filippov solutions of (2) such that $\psi_i(0) = x_i$ and $\psi([0,t_i]) \subseteq \overline{\mathcal{U}}$ for every $i = 1, 2, \ldots$, and $\lim_{i\to\infty} \psi_i(t_i) = z$.

First, suppose that the sequence $\{t_i\}_{i=1}^{\infty}$ converges to 0. Then it follows from Theorem 11 of [42] that there exists a Filippov solution $\hat{\psi}(\cdot)$ to (1) with $\hat{\psi}(0) = x$ such that $\lim_{i\to\infty} \psi_i(t_i) = \hat{\psi}(0) = x$. Next, suppose the sequence $\{t_i\}_{i=1}^{\infty}$ does not converge to 0. Then there exists a subsequence $\{t_{i_k}\}_{k=1}^{\infty}$ of the sequence $\{t_i\}_{i=1}^{\infty}$ such that $\liminf_{k\to\infty} t_{i_k} > 0$. Let $\mathcal{I}_k \triangleq [0, t_{i_k}]$ for each k and let $\mathcal{B} \subseteq \overline{\mathcal{U}}$ denote the set of all subsequential limits of sequences of the form $\{\psi_{i_k}(\tau_k)\}_{k=1}^{\infty}$, where $\tau_k \in \mathcal{I}_k$ for every k. By construction, $z \in \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{R}_x^{\mathcal{U}}$. Hence, $T_x\mathcal{B} \cap \mathcal{F}_x \subseteq T_x\mathcal{R}_x^{\mathcal{U}} \cap \mathcal{F}_x \subseteq \{0\}$, that is, f is nontangent to $\mathcal{B}$ at x. Now, it follows from Lemma 3.2 that $\mathcal{B} = \{x\}$. Hence, $z = x$. Since $z \in \mathcal{R}_x^{\mathcal{U}}$ is arbitrary, it follows that $\mathcal{R}_x^{\mathcal{U}} = \{x\}$.

Suppose, *ad absurdum*, that x is not a Lyapunov stable equilibrium. Then there exist a bounded open neighborhood $\mathcal{V} \subseteq \mathcal{U}$ of x, a sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{V}$ converging to x, a sequence $\{\psi_i\}_{i=1}^{\infty}$ of Filippov solutions to (2), and a sequence $\{t_i\}_{i=1}^{\infty}$ in $[0,\infty)$ such that $\psi_i(x_i) = x_i$ and $\psi_i(t_i) \in \partial\mathcal{V}$ for every i. Without loss of generality, we can assume that the sequence $\{t_i\}_{i=1}^{\infty}$ is chosen such that, for every i, $\psi_i(h) \in \mathcal{V}$ for all $h \in [0, t_i)$. Now, every subsequential limit of the bounded sequence $\{\psi_i(t_i)\}_{i=1}^{\infty}$ is distinct from x by construction and is contained in $\mathcal{R}_x^{\mathcal{U}}$ by definition, which implies that $\mathcal{R}_x^{\mathcal{U}} \backslash \{x\} \neq \varnothing$. This contradicts our earlier conclusion that $\mathcal{R}_x^{\mathcal{U}} = \{x\}$. Hence, x is Lyapunov stable. □

The following theorem gives sufficient conditions for semistability using nontangency of the vector field f.

Theorem 3.4. *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to* (1) *and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that $V(x) \geq 0$ for all $x \in \mathcal{D}$, $V(z) = 0$ for all $z \in \mathcal{E} \cap \mathcal{D}$, and $\max \mathcal{L}_f V(x) \leq 0$ for almost every $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Furthermore, for every $z \in \mathcal{E}$, let $\mathcal{N}_z$ denote the largest weakly negatively invariant connected subset of $\overline{\mathcal{R}} \cap \mathcal{D}$ containing z, where $\mathcal{R}$ is given by (7). If f is nontangent to $\mathcal{N}_z$ at every $z \in \mathcal{E}$, then every equilibrium in $\mathcal{D}$ is semistable.*

Proof. Let $\mathcal{V} \subset \mathcal{D}$ be a bounded open neighborhood of $x \in \mathcal{E} \cap \mathcal{D}$. Since f is nontangent to $\mathcal{N}_x$ at the point $x \in \mathcal{E} \cap \mathcal{V}$, it follows that $T_x\mathcal{N}_x \cap \mathcal{F}_x \subseteq \{0\}$. Next, we show that f is nontangent to $\mathcal{R}_x^{\mathcal{V}}$ at the point x. It follows from Lemma 3.1 that $\mathcal{R}_x^{\mathcal{V}} \subseteq \mathcal{N}_x$. Hence, $T_x\mathcal{R}_x^{\mathcal{V}} \cap \mathcal{F}_x \subseteq T_x\mathcal{N}_x \cap \mathcal{F}_x \subseteq \{0\}$, that is, $T_x\mathcal{R}_x^{\mathcal{V}} \cap \mathcal{F}_x \subseteq \{0\}$. By definition, f is nontangent to $\mathcal{R}_x^{\mathcal{V}}$ at the point x. Now, it follows from Proposition 3.3 that x is a Lyapunov stable equilibrium. Since $x \in \mathcal{E} \cap \mathcal{D}$ was chosen arbitrarily, it follows that every equilibrium of (1) in $\mathcal{D}$ is Lyapunov stable.

By Lyapunov stability of x, it follows that there exists a strongly positively invariant neighborhood $\mathcal{U} \subset \mathcal{V}$ of x that is open and bounded, and such that $\overline{\mathcal{U}} \subset \mathcal{V}$. Consider $z \in \mathcal{U}$, and let $\psi(\cdot)$ be a Filippov solution of (1) with $\psi(0) = z$. Then $\psi(\cdot)$ is bounded in $\mathcal{D}$. Hence, it follows from [33, p. 129] and Theorem 3 of [25] that $\Omega(\psi) \subseteq \overline{\mathcal{U}}$ is nonempty and contained in $\overline{\mathcal{R}}$.

Let $w \in \Omega(\psi)$. The invariance and connectedness of $\Omega(\psi)$ implies that $\Omega(\psi) \subseteq \mathcal{N}_w$. Hence, $T_w\Omega(\psi) \cap \mathcal{F}_w \subseteq T_w\mathcal{N}_w \cap \mathcal{F}_w \subseteq \{0\}$. Now, it follows from Lemma 3.2 (see the proof of Proposition 5.2 of [13]) that $\lim_{t\to\infty} \psi(t)$ exists. Since $z \in \mathcal{U}$ was chosen arbitrarily, it follows that every Filippov solution in $\mathcal{U}$ converges to a limit. The strong invariance of $\mathcal{U}$ implies that the limit of every Filippov solution in $\mathcal{U}$ is contained in $\overline{\mathcal{U}}$. Since every equilibrium in $\overline{\mathcal{U}} \subset \mathcal{V}$ is Lyapunov stable, it follows from Theorem 24 that x is semistable. Finally, since $x \in \mathcal{E} \cap \mathcal{D}$ was chosen arbitrarily, it follows that every equilibrium in $\mathcal{D}$ is semistable. □

Example 3.3. Consider the discontinuous dynamical system on $\mathcal{D} = \mathbb{R}^4$ given by

$$\dot{x}_1(t) = \operatorname{sign}(x_3(t) - x_4(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{17}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_4(t) - x_3(t)), \quad x_2(0) = x_{20}, \tag{18}$$

$$\dot{x}_3(t) = \operatorname{sign}(x_4(t) - x_3(t)) + \operatorname{sign}(x_2(t) - x_1(t)), \quad x_3(0) = x_{30}, \tag{19}$$

$$\dot{x}_4(t) = \operatorname{sign}(x_3(t) - x_4(t)) + \operatorname{sign}(x_1(t) - x_2(t)), \quad x_4(0) = x_{40}, \tag{20}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ denote the vector field of (17)–(20) and $x \triangleq [x_1, x_2, x_3, x_4] \in \mathbb{R}^4$. Consider the function $V(x) = |x_1 - x_2| + |x_3 - x_4|$. Note that

$$\partial V(x) = \begin{cases} \{\operatorname{sign}(x_1 - x_2)\} \times \{\operatorname{sign}(x_2 - x_1)\} & \\ \times\{\operatorname{sign}(x_3 - x_4)\} \times \{\operatorname{sign}(x_4 - x_3)\}, & x_1 \neq x_2,\ x_3 \neq x_4, \\ [-1,1] \times [-1,1] \times \{\operatorname{sign}(x_3 - x_4)\} \times \{\operatorname{sign}(x_4 - x_3)\}, & x_1 = x_2,\ x_3 \neq x_4, \\ \{\operatorname{sign}(x_1 - x_2)\} \times \{\operatorname{sign}(x_2 - x_1)\} \times [-1,1] \times [-1,1], & x_1 \neq x_2,\ x_3 = x_4, \\ \overline{\mathrm{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\}, & x_1 = x_2,\ x_3 = x_4. \end{cases}$$

Hence,

$$\mathcal{L}_f V(x) = \begin{cases} \{-2\}, & x_1 \neq x_2,\ x_3 \neq x_4, \\ \varnothing, & x_1 = x_2,\ x_3 \neq x_4, \\ \varnothing, & x_1 \neq x_2,\ x_3 = x_4, \\ \{0\}, & x_1 = x_2,\ x_3 = x_4, \end{cases} \tag{21}$$

which implies that $\max \mathcal{L}_f V(x) \leq 0$ for almost every $x \in \mathbb{R}^4$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Consequently, $\mathcal{R} = \{x \in \mathbb{R}^4 : x_1 = x_2, x_3 = x_4\}$. Let $\mathcal{N}$ denote the largest weakly,

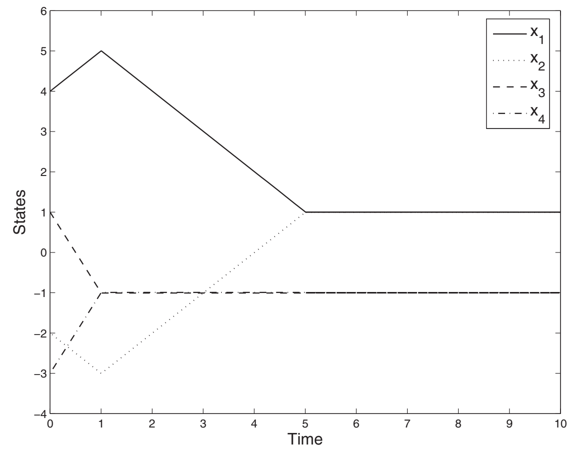

FIGURE 2. State trajectories versus time for Example 3.3

negatively invariant subset contained in $\mathcal{R}$. On $\mathcal{N}$, it follows from (17)–(20) that $\dot{x}_1 = \dot{x}_2 = 0$ and $\dot{x}_3 = \dot{x}_4 = 0$. Hence, $\mathcal{N} = \{x \in \mathbb{R}^4 : x_1 = x_2 = a, x_3 = x_4 = b\}$, $a, b \in \mathbb{R}$, which implies that $\mathcal{N}$ is the set of equilibrium points.

Next, we show that f for (17)–(20) is nontangent to $\mathcal{N}$ at the point $z \in \mathcal{N}$. To see this, note that the tangent cone $T_z\mathcal{N}$ to the equilibrium set $\mathcal{N}$ is orthogonal to the vectors $\mathbf{u}_1 \triangleq [1, -1, 0, 0]^{\mathrm{T}}$ and $\mathbf{u}_2 \triangleq [0, 0, 1, -1]^{\mathrm{T}}$. On the other hand, since $f(z) \in \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ for all $z \in \mathbb{R}^4$, it follows that $f(\mathcal{V}) \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ for every subset $\mathcal{V} \subseteq \mathbb{R}^4$. Consequently, the direction cone $\mathcal{F}_z$ of f at $z \in \mathcal{N}$ relative to $\mathbb{R}^4$ satisfies $\mathcal{F}_z \subseteq \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Hence, $T_z\mathcal{N} \cap \mathcal{F}_z = \{0\}$, which implies that the vector field f is nontangent to the set of equilibria $\mathcal{N}$ at the point $z \in \mathcal{N}$. Note that for every $z \in \mathcal{N}$, the set $\mathcal{N}_z$ required by Theorem 3.4 is contained in $\mathcal{N}$. Since nontangency to $\mathcal{N}$ implies nontangency to $\mathcal{N}_z$ at the point $z \in \mathcal{N}$, it follows from Theorem 3.4 that every equilibrium point of (17)–(20) in $\mathbb{R}^4$ is semistable.

Finally, note that either $\max \mathcal{L}_f V(x) \leq -2 < 0$ or $\mathcal{L}_f V(x) = \varnothing$ for almost all $x \in \mathbb{R}^4 \backslash \mathcal{R}$, and hence, it follows from Corollary 3.1 that (17)–(20) is globally finite-time-semistable. Figure 2 shows the solutions of (17)–(20) for $x_{10} = 4$, $x_{20} = -2$, $x_{30} = 1$, and $x_{40} = -3$. $\triangle$

4. UNIVERSAL FEEDBACK CONTROLLERS FOR DISCONTINUOUS SYSTEMS

The consideration of nonsmooth Lyapunov functions for proving stability of feedback discontinuous systems is an important extension to classical stability theory since, as shown in [11], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory. For dynamical systems with continuously differentiable flows, the concept of smooth control Lyapunov functions was developed by Artstein [21] to show the existence of a feedback stabilizing controller. A constructive feedback control law based on smooth control Lyapunov functions was given in [43]. Even though a stabilizing continuous feedback controller guarantees the existence of a smooth control Lyapunov function, many systems that possess smooth control Lyapunov functions do not necessarily admit a continuous stabilizing feedback controller [21, 44]. However, as shown in [44], the existence of a control Lyapunov function allows for the design of a stabilizing feedback controller that admits Filippov and Krasovskii closed-loop system solutions. In this and the next section, we use the results of [44, 45] to develop a constructive universal feedback control law for discontinuous dynamical systems based on the existence of a nonsmooth control Lyapunov function defined in the sense of generalized Clarke gradients [24] and set-valued Lie derivatives [25].

Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) \quad = \quad F(x(t), u(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{22}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [33] with respect to x, continuous with respect to u, and admits an equilibrium point at $x_{\mathrm{e}} \in \mathcal{D}$ for some $u_{\mathrm{e}} \in U$; that is, $F(x_{\mathrm{e}}, u_{\mathrm{e}}) = 0$. The control $u(\cdot)$ in (22) is restricted to the class of *admissible* controls consisting of measurable and locally essentially bounded functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$. For each value $u \in U$, we define the function F_u by $F_u(x) = F(x, u)$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(x_{\mathrm{e}}) = u_{\mathrm{e}}$ is called a *control law*. If $u(t) = \phi(x(t))$, where ϕ is a control law and $x(t)$ satisfies (22), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, the *closed-loop system* is given by

$$\dot{x}(t) = F(x(t), \phi(x(t))), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0. \tag{23}$$

Analogous to the open-loop case, we define the function F_ϕ by $F_\phi(x) = F(x, \phi(x))$. Note that an arc $x(\cdot)$ (i.e., an absolutely continuous function from $[t_0, t]$ to $\mathcal{D}$) satisfies (22) for an admissible control $u(t) \in U$ if and only if [33, p. 152]

$$\dot{x}(t) \in \mathcal{F}(x(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{24}$$

where $\mathcal{F}(x) \triangleq F(x, U)$, that is, $\mathcal{F}(x) \triangleq \{F(x, u) : u \in U\}$.

Here $\mathcal{F} : \mathcal{D} \to 2^{\mathbb{R}^n}$ is a *set-valued map* that assigns sets to points. The set $\mathcal{F}(x)$ captures all of the directions in $\mathbb{R}^n$ that can be generated at x with inputs $u = u(t) \in U$. The inputs $u(\cdot)$ can be selected as either $u : [t_0, \infty) \to U$ or $u : \mathcal{D} \to U$. We assume that $\mathcal{F}(x)$ is an upper semicontinuous, nonempty, convex, and compact set for all $x \in \mathbb{R}^n$. That is, for every $x \in \mathcal{D}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\|z - x\| \leq \delta$, $\mathcal{F}(z) \subseteq \mathcal{F}(x) + \overline{\mathcal{B}}_\varepsilon(0)$. This assumption is mainly used to guarantee the existence of Filippov solutions to (23) [33].

An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a Filippov solution [33] of (23) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies

$$\dot{x}(t) \in \mathcal{K}[F_\phi](x(t)), \quad \text{a.e.} \quad t \in [t_0, \tau], \tag{25}$$

where the Filippov set-valued map $\mathcal{K}[F_\phi] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{K}[F_\phi](x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{F_\phi(\mathcal{B}_\delta(x)\backslash\mathcal{S})\}, \quad x \in \mathcal{D}, \tag{26}$$

$\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\mathrm{co}}$" denotes convex closure, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.[2] Note that since F is locally essentially bounded, $\mathcal{K}[F_\phi](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values.

5. NONSMOOTH CONTROL LYAPUNOV FUNCTIONS

In this section, we consider a feedback control problem and introduce the notion of *control Lyapunov functions* for discontinuous dynamical systems. Furthermore, using the concept of control Lyapunov functions we provide necessary and sufficient conditions for stabilization of discontinuous nonlinear dynamical systems. To address the problem of control Lyapunov functions for discontinuous dynamical systems, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set and let $U \subseteq \mathbb{R}^m$, where $0 \in \mathcal{D}$ and $0 \in U$. Next, consider the controlled nonlinear discontinuous dynamical system (22), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$ for almost all $t \geq 0$ and the constraint set U is given. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop dynamical system is given by (23).

The following definitions are required for stating the main result of this section.

Definition 5.1. Let $\phi : \mathcal{D} \to U$ be a measurable mapping on $\mathcal{D}\backslash\{0\}$ with $\phi(0) = 0$. Then (22) is *feedback asymptotically stabilizable* if the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous nonlinear dynamical system (23) is asymptotically stable.

Definition 5.2. Consider the controlled discontinuous nonlinear dynamical system given by (22). A locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ satisfying

$$\inf_{u \in U} [\max \mathcal{L}_{F_u} V(x)] < 0, \quad \text{a.e.} \quad x \in \mathcal{D} \setminus \{0\}, \tag{27}$$

is called a *control Lyapunov function*.

[2] Alternatively, we can consider Krasovskii solutions of (23) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[F_\phi](x)$ is replaced with $\mathcal{K}[F_\phi](x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\{F_\phi(\mathcal{B}_\delta(x))\}$ and where F_ϕ is assumed to be locally bounded.

Note that if (27) holds, then there exists a feedback control law $\phi : \mathcal{D} \to U$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0$, $x \in \mathcal{D}$, $x \neq 0$, and hence, Theorem 3.1 with $f(x) = F_\phi(x) = F(x, \phi(x))$ implies that if there exists a control Lyapunov function for the discontinuous nonlinear dynamical system (22), then there exists a feedback control law $\phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the closed-loop system (23) is strongly asymptotically stable. Conversely, if there exists a feedback control law $u = \phi(x)$ such that the zero Filippov solution $x(t) \equiv 0$ of the discontinuous nonlinear dynamical system (22) is strongly asymptotically stable, then, since $\mathcal{L}_{F_\phi} V(x) \subseteq \{p^{\mathrm{T}} v : p^{\mathrm{T}} \in \partial V(x) \text{ and } v \in \mathcal{K}[F_\phi](x)\}$, it follows from Theorem 2.7 of [44] that there exists a locally Lipschitz continuous, regular, and positive-definite function $V : \mathcal{D} \to \mathbb{R}$ such that $\max \mathcal{L}_{F_\phi} V(x) < 0$ for all nonzero $x \in \mathcal{D}$ or, equivalently, there exists a control Lyapunov function for the discontinous nonlinear dynamical system (22). Hence, a given discontinuous dynamical system of the form (22) is strongly feedback asymptotically stabilizable if and only if there exists a control Lyapunov function satisfying (27). Finally, in the case where $\mathcal{D} = \mathbb{R}^n$ and $U = \mathbb{R}^m$ the zero Filippov solution $x(t) \equiv 0$ to (22) is globally strongly asymptotically stabilizable if and only if $V(x) \to \infty$ as $||x|| \to \infty$.

Next, we consider the special case of discontinuous nonlinear systems affine in the control, and we construct state feedback controllers that globally asymptotically stabilize the zero Filippov solution of the discontinuous nonlinear dynamical system under the assumption that the system has a radially unbounded control Lyapunov function. Specifically, we consider discontinuous nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{28}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. We assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. Note that (28) is a special case of (22) with $F(x, u) = f(x) + G(x)u$. We use the notation $f + Gu$ to denote the function $F_u(x) = f(x) + G(x)u$ for each $u \in \mathbb{R}^m$.

Note that (28) includes piecewise continuous dynamical systems as well as switched dynamical systems as special cases. For example, if $f(\cdot)$ and $G(\cdot)$ are piecewise continuous, then (28) can be represented as a differential inclusion involving Filippov set-valued maps of piecewise-continuous vector fields given by $\mathcal{K}[f](x) = \overline{\mathrm{co}}\{\lim_{i\to\infty} f(x_i) : x_i \to x, x_i \notin \mathcal{S}_f\}$, where $\mathcal{S}_f$ has measure zero and denotes the set of points where f is discontinuous [40], and similarly for $G(\cdot)$. Here, we assume that $\mathcal{K}[f](\cdot)$ has at least one equilibrium point so that, without loss of generality, $0 \in \mathcal{K}[f](0)$.

Next, define

$$\mathcal{L}_G V(x) \triangleq \{q \in \mathbb{R}^{1\times m} : \text{ there exists } v \in \mathfrak{G}(x) \text{ such that } p^{\mathrm{T}} v = q \text{ for all } p^{\mathrm{T}} \in \partial V(x)\},$$

where $\mathfrak{G}(x) \triangleq \bigcap_{\delta>0}\bigcap_{\mu(\mathcal{S})=0}\overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x)\backslash\mathcal{S})\}$, $x\in\mathbb{R}^n$, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Finally, we assume that the set $\mathcal{L}_G V(x)$ is single-valued[3] for almost all $x\in\mathbb{R}^n$, and that $\mathcal{L}_G V(x)\neq\varnothing$ at all other points x.

Theorem 5.1. *Consider the controlled discontinuous nonlinear dynamical system given by* (28). *Then a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function* $V:\mathbb{R}^n\to\mathbb{R}$ *is a control Lyapunov function for* (28) *if and only if*

$$\max\mathcal{L}_f V(x)<0, \quad \textit{a.e.} \quad x\in\mathcal{R}, \tag{29}$$

where $\mathcal{R}\triangleq\{x\in\mathbb{R}^n\setminus\{0\}:\mathcal{L}_G V(x)=0\}$.

Proof. Sufficiency is a direct consequence of the definition of a control Lyapunov function and the sum rule for computing the generalized gradient of locally Lipschitz continuous functions [40]. Specifically, for systems of the form (28), note that $\mathcal{L}_{f+Gu}V(x)\subseteq\mathcal{L}_f V(x)+\mathcal{L}_G V(x)u$ for almost all x and all u, and hence, $\inf_{u\in U}[\max\mathcal{L}_f V(x)+\mathcal{L}_G V(x)u]=-\infty$ when $x\notin\mathcal{R}$ and $x\neq 0$, whereas $\inf_{u\in U}[\max\mathcal{L}_f V(x)+\mathcal{L}_G V(x)u]<0$ for almost all $x\in\mathcal{R}$. Hence, (29) implies (27) with $F_u(x)=f(x)+G(x)u$.

To prove necessity suppose, *ad absurdum*, that $V(\cdot)$ is a control Lyapunov function and (29) does not hold. In this case, there exists a set $\mathcal{M}\subseteq\mathcal{R}$ of positive measure such that $\max\mathcal{L}_f V(x)\geq 0$ for all $x\in\mathcal{M}$. Let $x\in\mathcal{M}$ and let $\alpha\in\mathcal{L}_f V(x)\cap[0,\infty)$. From the definition of a control Lyapunov function, x is such that there exists u such that $\max\mathcal{L}_{f+Gu}V(x)<0$ and, by the sum rule for generalized gradients, the inclusion $\mathcal{L}_f V(x)\subseteq\mathcal{L}_{f+Gu}V(x)+\mathcal{L}_{-Gu}V(x)$ is satisfied (since the sum rule holds for almost all x). Now, since $x\in\mathcal{M}$, we have $\mathcal{L}_{-Gu}V(x)=-\mathcal{L}_{Gu}V(x)\subseteq-\mathcal{L}_G V(x)u\subseteq\{0\}$. Hence, there exists a nonnegative $\alpha\in\mathcal{L}_{f+Gu}V(x)$, which is a contradiction. This proves the theorem. □

It follows from Theorem 5.1 that the zero Filippov solution $x(t)\equiv 0$ of a discontinuous nonlinear affine system of the form (28) is globally strongly feedback asymptotically stabilizable if and only if there exists a locally Lipschitz continuous, regular, positive-definite, and radially unbounded function $V:\mathbb{R}^n\to\mathbb{R}$ satisfying (29). Hence, Theorem 5.1 provides necessary and sufficient conditions for discontinuous nonlinear system stabilization.

[3]The assumption that $\mathcal{L}_G V(x)$ is single-valued is necessary. Specifically, as will be seen later in the paper, the requirement that there exists $\overline{z}\in\mathcal{L}_G V(x)$ such that, for all $u\in\mathbb{R}^m$, $\max[\mathcal{L}_G V(x)u]=\overline{z}u$ holds if and only if $\mathcal{L}_G V(x)$ is a singleton. To see this, let $q,r\in\mathcal{L}_G V(x)$, with $q\neq r$, and assume, *ad absurdum*, that the required $\overline{z}$ exists. Then, either $q-\overline{z}\neq 0$ or $r-\overline{z}\neq 0$. Assume $q-\overline{z}\neq 0$ and let $u^{\mathrm{T}}=q-\overline{z}$. Then, $qu-\overline{z}u=(q-\overline{z})u=(q-\overline{z})(q-\overline{z})^{\mathrm{T}}=\|q-\overline{z}\|_2^2>0$. Hence, $qu>\overline{z}u$, which leads to a contradiction.

Next, using Theorem 5.1 we *construct* an explicit feedback control law that is a function of the control Lyapunov function $V(\cdot)$. Specifically, consider the feedback control law given by

$$\phi(x) = \begin{cases} -\left(c_0 + \frac{\alpha(x)+\sqrt{\alpha^2(x)+(\beta^{\mathrm{T}}(x)\beta(x))^2}}{\beta^{\mathrm{T}}(x)\beta(x)}\right)\beta(x), & \beta(x) \neq 0, \\ 0, & \beta(x) = 0, \end{cases} \tag{30}$$

where $\alpha(x) \triangleq \max \mathcal{L}_f V(x)$, $\beta(x) \triangleq (\mathcal{L}_G V(x))^{\mathrm{T}}$, and $c_0 \geq 0$ is a constant. In this case, the control Lyapunov function $V(\cdot)$ of (28) is a Lyapunov function for the closed-loop system (28) with $u = \phi(x)$, where $\phi(x)$ is given by (30). To see this, recall that using the sum rule for computing the generalized gradient of locally Lipschitz continuous functions [40] it follows that $\mathcal{L}_{f+Gu}V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_{Gu}V(x)$ for almost all $x \in \mathbb{R}^n$. Now, Theorem 5.1 gives

$$\begin{aligned} \max \mathcal{L}_{F_\phi} V(x) &= \max \mathcal{L}_{f+G\phi} \qquad (31)\\ &\leq \max\,[\mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)] \\ &= \max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x) \\ &= \alpha(x) + \beta^T(x)\phi(x) \\ &= \begin{cases} -c_0\beta^{\mathrm{T}}(x)\beta(x) - \sqrt{\alpha^2(x) + (\beta^{\mathrm{T}}(x)\beta(x))^2}, & \beta(x) \neq 0, \\ \alpha(x), & \beta(x) = 0, \end{cases} \\ &< 0, \quad x \in \mathbb{R}^n, \quad \text{a.e.} \quad x \neq 0, \qquad (32) \end{aligned}$$

which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (28), and hence, by Theorem 3.1, guaranteeing global strong asymptotic stability with $u = \phi(x)$ given by (30).

Example 5.1. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction given by [25]

$$\dot{x}_1(t) = -\operatorname{sign}(x_2(t)) - \frac{1}{2}\operatorname{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{33}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \tag{34}$$

where $\operatorname{sign}(\sigma) \triangleq \frac{\sigma}{|\sigma|}$, $\sigma \neq 0$, and $\operatorname{sign}(0) \triangleq 0$. Next, consider the locally Lipschitz continuous function $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & x_1 \neq 0, x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & x_1 \neq 0, x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & x_2 \neq 0, x_1 = 0, \\ \overline{\operatorname{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence, with $f(x) = [-\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_1), \operatorname{sign}(x_1)]^{\mathrm{T}}$ and $G = [0, 1]^{\mathrm{T}}$,

$$\mathcal{L}_f V(x) = \begin{cases} \{-\frac{1}{2}\}, & x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & x_1 \neq 0, x_2 = 0, \\ \varnothing, & x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

and

$$\mathcal{L}_G V(x) = \begin{cases} \{\operatorname{sign}(x_2)\}, & x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & x_1 \neq 0, x_2 = 0, \\ \{\operatorname{sign}(x_2)\}, & x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \setminus \{0\} : \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 5.1 that $V(x) = |x_1| + |x_2|$ is a control Lyapunov function for (33) and (34).

Next, note that it follows from (30) that

$$\begin{aligned} \phi(x) &= \begin{cases} -\left(c_0 + \dfrac{-\frac{1}{2} + \sqrt{\frac{1}{4} + \operatorname{sign}^4(x_2)}}{\operatorname{sign}^2(x_2)}\right)\operatorname{sign}(x_2), & \operatorname{sign}(x_2) \neq 0, \\ 0, & \operatorname{sign}(x_2) = 0, \end{cases} \\ &= \begin{cases} -\left(c_0 + \dfrac{\sqrt{5} - 1}{2}\right)\operatorname{sign}(x_2), & \operatorname{sign}(x_2) \neq 0, \\ 0, & \operatorname{sign}(x_2) = 0, \end{cases} \end{aligned} \tag{35}$$

where $c_0 \geq 0$, and hence, since $\mathcal{L}_{f+G\phi} V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\phi(x)$ for almost all x,

$$\max \mathcal{L}_{f+G\phi} V(x) \leq -\left(c_0 + \frac{\sqrt{5}}{2}\right) < 0, \quad \operatorname{sign}(x_2) \neq 0.$$

Now, it follows from Theorem 3.1 that (35) is a globally strongly stabilizing feedback controller. In fact, by Corollary 3.1, (35) is a globally finite-time stabilizing controller. $\triangle$

Example 5.2. Consider the controlled dynamical system $\mathcal{G}$ given by (28), where $x = [x_1, x_2]^{\mathrm{T}}$, $u = [u_1, u_2]^{\mathrm{T}}$,

$$f(x) = \begin{bmatrix} |x_1|(-x_1 + |x_2|) \\ x_2(-x_1 - |x_2|) \end{bmatrix}, \quad G(x) = \begin{bmatrix} |x_1| & 0 \\ 0 & x_2 \end{bmatrix}.$$

Next, consider the locally Lipschitz continuous function $V(x) = 2|x_1|+2|x_2|$ and note that

$$\partial V(x) = \begin{cases} \{2\,\mathrm{sign}(x_1)\} \times \{2\,\mathrm{sign}(x_2)\}, & x_1 \neq 0, x_2 \neq 0, \\ \{2\,\mathrm{sign}(x_1)\} \times [-2, 2], & x_1 \neq 0, x_2 = 0, \\ [-2, 2] \times \{2\,\mathrm{sign}(x_2)\}, & x_2 \neq 0, x_1 = 0, \\ \overline{\mathrm{co}}\{(2, 2), (-2, 2), (-2, -2), (2, -2)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V(x) \;=\; \begin{cases} \{-2x_1^2 - 2x_2^2\}, & x_1 \neq 0, x_2 \neq 0, \\ \{-2x_1^2\}, & x_1 \neq 0, x_2 = 0, \\ \{-2x_2^2\}, & x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

and

$$\mathcal{L}_G V(x) \;=\; \begin{cases} \{(2x_1, 2|x_2|)\}, & x_1 \neq 0, x_2 \neq 0, \\ \{(2x_1, 0)\}, & x_1 \neq 0, x_2 = 0, \\ \{(0, 2|x_2|)\}, & x_2 \neq 0, x_1 = 0, \\ \{(0, 0)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Now, since $\max \mathcal{L}_f V(x) < 0$ for all $x \in \mathcal{R}$, where $\mathcal{R} = \{x \in \mathbb{R}^2 \setminus \{0\} : \mathcal{L}_G V(x) = 0\}$, it follows from Theorem 5.1 that $V(x) = 2|x_1| + 2|x_2|$ is a control Lyapunov function.

Setting $\alpha(x) = \max \mathcal{L}_f V(x)$ and $\beta(x) = (\mathcal{L}_G V(x))^{\mathrm{T}}$, it follows that $\beta(x)\beta^T(x) = 4(x_1^2 + x_2^2)$ and $\alpha^2(x) + (\beta^T(x)\beta(x))^2 = 4(x_1^2 + x_2^2)^2 + 16(x_1^4 + x_2^4 + 2x_1^2x_2^2) = 20(x_1^4 + x_2^4) + 40x_1^2x_2^2 = 20(x_1^2 + x_2^2)^2$, and hence, (30) gives

$$\phi(x) = \begin{cases} -\left(c_0 + (\sqrt{5} - 1)\right) \begin{bmatrix} x_1 \\ |x_2| \end{bmatrix}, & (x_1, x_2) \neq (0, 0), \\ 0, & (x_1, x_2) = (0, 0), \end{cases} \tag{36}$$

where $c_0 \geq 0$. Thus, $\max \mathcal{L}_{f+G\phi} V(x) \leq -|x|^2$ for all $x \neq 0$. Now, it follows from Theorem 3.1 that (36) is a globally strongly stabilizing feedback controller. △

6. DISSIPATIVITY THEORY FOR DISCONTINUOUS SYSTEMS

In this section, we extend the notion of classical dissipativity [14,15] of dynamical systems with continuously differentiable flows to discontinuous systems. Specifically, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = F(x(t), u(t)), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{37}$$

$$y(t) = H(x(t)u(t)), \tag{38}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded [33] with respect to x, continuous with respect to u, admits an equilibrium point at $x_{\mathrm{e}} \in \mathcal{D}$ for some $u_{\mathrm{e}} \in U$;

that is, $F(x_{\rm e}, u_{\rm e}) = 0$, and $H : \mathcal{D} \times U \to \mathbb{R}^l$. The following definition is needed for the main results of this section.

Definition 6.1. *i*) The discontinuous dynamical system $\mathcal{G}$ given by (37) and (38) is *weakly* (resp., *strongly*) *dissipative with respect to the* (locally Lebesgue integrable) *supply rate* $s : U \times Y \to \mathbb{R}$ if there exists a locally Lipschitz continuous, regular, and nonnegative definite *storage function* $V_{\rm s} : \mathcal{D} \to \mathbb{R}$, such that $V_{\rm s}(0) = 0$ and the dissipation inequality

$$V_{\rm s}(x(t)) \leq V_{\rm s}(x(t_0)) + \int_{t_0}^{t} s(u(\sigma), y(\sigma)){\rm d}\sigma, \quad t \geq t_0, \tag{39}$$

is satisfied for at least one (resp., every) Filippov solution $x(t)$, $t \geq t_0$, of $\mathcal{G}$ with $u(t) \in U$.

ii) The discontinuous dynamical system $\mathcal{G}$ given by (37) and (38) is *weakly* (resp., *strongly*) *exponentially dissipative with respect to the* (locally Lebesgue integrable) *sypply rate* $s : U \times Y \to \mathbb{R}$ if there exist a locally Lipschitz continuous, regular, and nonnegative storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ such that $V_{\rm s}(0) = 0$ and the dissipation inequality

$$e^{\varepsilon t} V_{\rm s}(x(t)) \leq e^{\varepsilon t_0} V_{\rm s}(x(t_0)) + \int_{t_0}^{t} e^{\varepsilon \sigma} s(u(\sigma), y(\sigma)){\rm d}\sigma, \quad t \geq t_0, \tag{40}$$

is satisfied for one (resp., every) Filippov solution $x(t)$, $t \geq 0$, of $\mathcal{G}$ with $u(t) \in U$.

iii) The discontinuous dynamical system $\mathcal{G}$ given by (37) and (38) is *strictly weakly* (resp., *strongly*) *dissipative with respect to the* (locally Lebesgue integrable) *supply rate* $s : U \times Y \to \mathbb{R}$ if there exist a locally Lipschitz continuous, regular, and nonnegative storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ and a scalar $\varepsilon > 0$ such that $V_{\rm s}(0) = 0$ and the dissipation inequality

$$V_{\rm s}(x(t)) \leq V_{\rm s}(x(t_0)) + \int_{t_0}^{t} [s(u(\sigma), y(\sigma)) - \varepsilon]{\rm d}\sigma, \quad t \geq t_0, \tag{41}$$

is satisfied for at least one (resp., every) Filippov solution $x(t)$, $t \geq t_0$, of $\mathcal{G}$ with $u(t) \in U$.

Since $V_{\rm s}(\cdot)$ is assumed to be locally Lipschitz continuous and regular, an equivalent statement for the dissipativeness of $\mathcal{G}$ involving supply rates $s(u, y)$ is

$$\dot{V}_{\rm s}(x(t)) \leq s(u(t), y(t)), \text{ a.e.} \quad t \geq 0, \tag{42}$$

or, equivalently, $\dot{V}_{\rm s}(x) \leq s(u, y)$, where

$$\dot{V}_{\rm s}(x) = \frac{\rm d}{{\rm d}t} V_{\rm s}(\psi(t, x, u))\bigg|_{t=0} \triangleq \limsup_{h \to 0^+} \frac{V_{\rm s}(\psi(h, x, u)) - V_{\rm s}(x)}{h}, \tag{43}$$

for every $x \in \mathbb{R}^n$, denotes the upper right directional Dini derivative of $V_{\rm s}(x)$ along the Filippov state trajectories $\psi(t, x, u)$ of (37) through $x \in \mathcal{D}$ with $u(t) \in U$ at

$t = 0$. Alternatively, an equivalent statement for exponentional dissipativeness and strict dissipativeness of $\mathcal{G}$ involving the supply rate $s(u, y)$ is, respectively,

$$\dot{V}_{\rm s}(x(t)) + \varepsilon V_{\rm s}(x(t)) \le s(u(t), y(t)), \text{ a.e. } \quad t \ge 0, \tag{44}$$

and

$$\dot{V}_{\rm s}(x(t)) \le s(u(t), y(t)) - \varepsilon, \text{ a.e. } \quad t \ge 0. \tag{45}$$

The following lemma is necessary for the next proposition. For the statement of this lemma we require some additional notation. Specifically, given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, define the set-valued Lie derivative $\mathcal{L}_{F(x,u)}V : \mathbb{R}^n \times U \to 2^{\mathbb{R}}$ of V with respect to F at x and u by

$$\mathcal{L}_{F(\cdot,u)}V(x) \triangleq \Big\{ a \in \mathbb{R} : \text{ there exists } v \in \mathcal{K}[F(\cdot,u)](x) \\ \text{such that } p^{\rm T} v = a \text{ for all } p^{\rm T} \in \partial V(x) \Big\},$$

where $\mathcal{K}[F(\cdot, u)](x)$ denotes the Filippov set-valued map of $F(x, u)$ over x for each admissible input $u(t) \in U$. That is, $F(\cdot, u)$ is averaged over progressively smaller neighborhoods around $x \in \mathbb{R}^n$ with $u \in U$. Analogously, for fixed $t > 0$, $x \in \mathbb{R}^n$, and a measurable and locally essentially bounded $u : \mathbb{R} \to U$, define the set-valued Lie derivative $\mathcal{L}_{F(x,u(\cdot))}V : \mathbb{R}^n \times U \to 2^{\mathbb{R}}$ by

$$\mathcal{L}_{F(\cdot,u(\cdot))}V(x) \triangleq \Big\{ a \in \mathbb{R} : \text{ there exists } v \in \mathcal{K}[F(\cdot,u(t))](x) \\ \text{such that } p^{\rm T} v = a \text{ for all } p^{\rm T} \in \partial V(x) \Big\},$$

that is, we fix $u(\cdot) \in \mathcal{L}_\infty(\mathbb{R}, U)$ and apply the Filippov construction over x. Note that if $\psi(\cdot)$ is a Filippov solution to (37) with $u(\cdot) = \overline{u}(\cdot)$, then $\mathcal{L}_{F(\cdot,\overline{u}(\cdot))}V(\psi(t)) \subseteq \mathcal{L}_{F(\cdot,u)}V(\psi(t))$. In addition, note that $\mathcal{L}_{F(\cdot,u)}V(x)$ is a closed and bounded, possibly empty, interval in $\mathbb{R}$.

Lemma 6.1. *Let $x : [t_0, t] \to \mathbb{R}^n$ be a Filippov solution of (37) corresponding to the input $u(\cdot)$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and regular. Then $\frac{\rm d}{{\rm d}\sigma}V(x(\sigma))$ exists for almost all $\sigma \in [t_0, t]$ and $\frac{\rm d}{{\rm d}\sigma}V(x(\sigma)) \in \mathcal{L}_{F(\cdot,u(\cdot))}V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$.*

Proof. The proof is similar to the proof of Lemma 1 of [25] and, hence, is omitted. □

Proposition 6.1. *Consider the discontinuous dynamical system $\mathcal{G}$ given by (37) and (38), and let $V : \mathcal{D} \to \mathbb{R}$ be a locally Lipschitz continuous and regular function such that $V(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$. Assume there exist a Lebesgue measurable function $s : U \times Y \to \mathbb{R}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that*

$$\max \mathcal{L}_{F(\cdot,u)}V(x) \le -\varepsilon V(x) + s(u, y), \quad \text{a.e.} \quad u \in U. \tag{46}$$

Then $\mathcal{G}$ is strongly exponentially dissipative (resp., strongly dissipative) with respect to the supply rate $s(u, y)$.

Proof. It suffices to show that if (46) holds, then (39) holds on the interval $[t_0, t]$. To see this, let $x : [t_0, t] \to \mathbb{R}^n$ be a Filippov solution of (24) with initial condition $x(0) = x_0$. Now, since by Lemma 6.1 $\dot{V}(x(\sigma)) \leq \max \mathcal{L}_{\mathcal{F}(\cdot,u(\cdot))} V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$, it follows from (46) that $\dot{V}(x(\sigma)) \leq -\varepsilon V(x(\sigma)) + s(u(\sigma), y(\sigma))$ for almost all $\sigma \in [t_0, t]$, and hence,

$$e^{\varepsilon\sigma}\left[\dot{V}(x(\sigma)) + \varepsilon V(x(\sigma))\right] \leq e^{\varepsilon\sigma} s(u(\sigma), y(\sigma)), \quad \text{a.e.} \quad \sigma \in [t_0, t]. \tag{47}$$

Now, integrating (47), where the integral is a Lebesgue integral, it follows that (39) holds with $\varepsilon > 0$ (resp., $\varepsilon = 0$). □

Example 6.1. Consider the controlled discontinuous dynamical system $\mathcal{G}$ representing a mass sliding on a horizontal surface subject to a Coulomb frictional force. During sliding, the Coulomb frictional model states that the magnitude of the friction force is independent of the magnitude of the system velocity and is equal to the normal contact force times the coefficient of kinetic friction. The application of this model to a sliding mass on a horizontal frictional surface gives

$$\dot{x}(t) = -\operatorname{sign}(x(t)) + u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{48}$$

$$y(t) = x(t). \tag{49}$$

Equation (48) can be rewritten in the form of a differential inclusion

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) + u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{50}$$

where the Filippov set-valued map $\mathcal{K}[f] : \mathbb{R} \to 2^{\mathbb{R}}$ is given by

$$\mathcal{K}[f](x) = \begin{cases} -1, & x > 0, \\ [-1, 1], & x = 0, \\ 1, & x < 0. \end{cases} \tag{51}$$

Let $V_{\mathrm{s}_1}(x) = x^2$. Since

$$\begin{aligned} \dot{V}_{\mathrm{s}}(x) &\in \mathcal{L}_{F(\cdot,u)} V_{s1}(x) \\ &= \partial V_{\mathrm{s}}(x)(\mathcal{K}[f](x) + u) \\ &= 2x\mathcal{K}[f](x) + 2xu \\ &= -|x| + 2uy \\ &\leq 2uy, \end{aligned} \tag{52}$$

it follows that $\max \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}_1}(x) \leq 2uy$ for all Filippov solutions, which, by Proposition 6.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $2uy$.

Next, let $V_{\mathrm{s}}(x) = |x|$. Since

$$\dot{V}_{\mathrm{s}}(x) \in \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}}(x) = \begin{cases} -1 + \operatorname{sign}(x)u, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$= -1 + u\,\mathrm{sign}(y), \quad x \neq 0, \tag{53}$$

it follows that $\max \mathcal{L}_{F(\cdot,u)} V_{\mathrm{s}_2}(x) \leq u\,\mathrm{sign}(y)$ for almost all $x \in \mathbb{R}$ and all Filippov solutions, which, by Proposition 6.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $u\,\mathrm{sign}(y)$. △

Next, we show that dissipativeness of discontinuous nonlinear affine dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad \text{a.e.} \quad t \geq t_0, \tag{54}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{55}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n \times m}$, $h : \mathcal{D} \to Y$, and $J : \mathcal{D} \to \mathbb{R}^{l \times m}$, can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$. Here, we assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are Lebesgue measurable and locally essentially bounded.

For the remainder of this section, we consider the special case of dissipative systems with quadratic supply rates [15], [16]. Specifically, set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l \times m}$ be given, and assume $s(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $s(\kappa(y), y) < 0$, $y \neq 0$, so that, as shown by Theorem 3.2 of [46], all storage functions for $\mathcal{G}$ are positive definite. Next, define

$$\mathcal{L}_G V_{\mathrm{s}}(x) \triangleq \{q \in \mathbb{R}^{1 \times m} : \text{ there exists } v \in \mathfrak{G}(x) \\ \text{such that } p^{\mathrm{T}} v = q \text{ for all } p^{\mathrm{T}} \in \partial V_{\mathrm{s}}(x)\},$$

where $\mathfrak{G}(x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S}) = 0} \overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x)) \backslash \mathcal{S}\}$, $x \in \mathbb{R}^n$, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Finally, we assume that the set $\mathcal{L}_G V_{\mathrm{s}}(x)$ is single-valued[4] for almost all $x \in \mathbb{R}^n$ modulo $\mathcal{L}_G V_{\mathrm{s}}(x) \neq \varnothing$. The following definition is necessary for the statement of the next result.

Definition 6.2 ([46]). The nonlinear dynamical system $\mathcal{G}$ given by (37) and (38) is *weakly (*resp.*, strongly) completely reachable* if for every $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ there exists a finite time $t_{\mathrm{i}} < t_0$ and an admissible input $u(t)$ defined on $[t_{\mathrm{i}}, t_0]$ such that at least one (resp., every) Filippov solution $x(t)$, $t \geq t_{\mathrm{i}}$, of $\mathcal{G}$ can be driven from $x(t_{\mathrm{i}}) = 0$ to

[4]The assumption that $\mathcal{L}_G V_{\mathrm{s}}(x)$ is single-valued is necessary for obtaining Kalman-Yakubovich-Popov conditions for (54) and (55) with Lebesgue measurable and locally essentially bounded system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$, and with locally Lipschitz continuous storage functions $V_{\mathrm{s}}(\cdot)$. Specifically, as will be seen in the proof of Theorem 6.1, the requirement that there exists $\overline{z} \in \mathcal{L}_G V_{\mathrm{s}}(x)$ (resp., $\underline{z} \in \mathcal{L}_G V_{\mathrm{s}}(x)$) such that, for all $u \in \mathbb{R}^m$, $\max[\mathcal{L}_G V_{\mathrm{s}}(x)u] = \overline{z}u$ (resp., $\min[\mathcal{L}_G V_{\mathrm{s}}(x)u] = \underline{z}u$) used in the proof of Theorem 6.1 holds if and only if $\mathcal{L}_G V_{\mathrm{s}}(x)$ is a singleton. This fact is shown in Footnote 3 for $\overline{z} \in \mathcal{L}_G V_{\mathrm{s}}(x)$. A similar construction shows the result for $\underline{z} \in \mathcal{L}_G V_{\mathrm{s}}(x)$.

$x(t_0) = x_0$. The nonlinear dynamical system $\mathcal{G}$ given by (1) and (2) is *weakly (*resp., *strongly) zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$ for at least one (resp., every) Filippov solution of $\mathcal{G}$.

The following theorem gives necessary and sufficient Kalman-Yakubovich-Popov conditions for dynamical systems with Lebesgue measurable and locally essentially bounded system functions.

Theorem 6.1. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$\begin{aligned} 0 &= \min \mathcal{L}_f V_{\rm s}(x) + \varepsilon V_{\rm s}(x) - h^{\rm T}(x)Qh(x) + \ell^{\rm T}(x)\ell(x), && (56)\\ 0 &= \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) - h^{\rm T}(x)(QJ(x) + S) + \ell^{\rm T}(x)\mathcal{W}(x), && (57)\\ 0 &= R + S^{\rm T}J(x) + J^{\rm T}(x)S + J^{\rm T}(x)QJ(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x), && (58)\end{aligned}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\rm T}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\rm s}(x) - \min \mathcal{L}_f V_{\rm s}(x), \quad u \in \mathbb{R}^m, \qquad (59)$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate $s(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$. Conversely, if $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate $s(u, y)$, then there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\rm s}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (56)–(58) *hold.*

Proof. First, suppose that there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ such that $V_{\rm s}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (56)–(59) are satisfied. Then, for every admissible input $u(t) \in \mathbb{R}^m$, $t \geq 0$, it follows from (56)–(59) that

$$\begin{aligned} &\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t)){\rm d}t\\ &= \int_{t_1}^{t_2} e^{\varepsilon t}\big[y^{\rm T}(t)Qy(t) + 2y^{\rm T}(t)Su(t) + u^{\rm T}(t)Ru(t)\big]{\rm d}t\\ &= \int_{t_1}^{t_2} e^{\varepsilon t}\big[h^{\rm T}(x(t))Qh(x(t)) + 2h^{\rm T}(x(t))(S + QJ(x(t)))u(t)\\ &\qquad +u^{\rm T}(t)(J^{\rm T}(x(t))QJ(x(t)) + S^{\rm T}J(x(t)) + J^{\rm T}(x(t))S + R)u(t)\big]{\rm d}t\\ &= \int_{t_1}^{t_2} e^{\varepsilon t}\big[\min \mathcal{L}_f V_{\rm s}(x(t)) + \varepsilon V_{\rm s}(x(t)) + \mathcal{L}_G V_{\rm s}(x(t))u(t) + \ell^{\rm T}(x(t))\ell(x(t))\\ &\qquad +2\ell^{\rm T}(x(t))\mathcal{W}(x(t))u(t) + u^{\rm T}(t)\mathcal{W}^{\rm T}(x(t))\mathcal{W}(x(t))u(t)\big]{\rm d}t \end{aligned}$$

$$\begin{aligned}
&= \int_{t_1}^{t_2} e^{\varepsilon t}[\min \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t)) \\
&\qquad + [\ell(x(t)) + \mathcal{W}(x(t))u(t)]^{\mathrm{T}}[\ell(x(t)) + \mathcal{W}(x(t))u(t)]\big]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t))]\mathrm{d}t, \qquad (60)
\end{aligned}$$

where $x(t)$, $t \geq 0$, satisfies (54).

Next, using the sum rule for computing the generalized gradient of a locally Lipschitz continuous function [40] it follows that

$$\mathcal{L}_{f+Gu} V_{\mathrm{s}}(x) \subseteq \mathcal{L}_f V_{\mathrm{s}}(x) + \mathcal{L}_{Gu} V_{\mathrm{s}}(x)$$

for almost all $x \in \mathbb{R}^n$. Now, it follows from Lemma 6.1 that $\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{s}}(x(t)) \in \mathcal{L}_{f+Gu} V_{\mathrm{s}}(x(t)) \subseteq \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_{Gu} V_{\mathrm{s}}(x(t))$ for almost all $t \geq 0$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{s}}(x(t)) &\leq \max \mathcal{L}_{f+Gu} V_{\mathrm{s}}(x(t)) \\
&\leq \max\left[\mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t)\right] \\
&= \max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t), \quad \text{a.e.} \quad t \geq 0, \quad u(t) \in U. \qquad (61)
\end{aligned}$$

Next, note that

$$e^{\varepsilon t} V_{\mathrm{s}}(x(t)) = e^{\varepsilon t_0} V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}(e^{\varepsilon\sigma} V_{\mathrm{s}}(x(\sigma)))\mathrm{d}\sigma, \qquad (62)$$

where the integral in (62) is the Lebesgue integral.

Using (61) and (62), it follows from (60) that

$$\begin{aligned}
\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t &\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(e^{\varepsilon t} V_{\mathrm{s}}(x(t)))\mathrm{d}t \\
&= e^{\varepsilon t_2} V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1} V_{\mathrm{s}}(x(t_1)), \quad \text{a.e.} \quad t \geq 0, \quad u(t) \in U.
\end{aligned}$$

The assertion now follows from Definition 6.1.

Conversely, suppose that $\mathcal{G}$ is weakly exponentially dissipative with respect to the supply rate map $\{s(u, y)\}$. Now, it follows from Theorem 3.1 of [46] that the smallest available storage map $V_{\mathrm{as}}(x)$ of $\mathcal{G}$ is finite for all $x \in \mathbb{R}^n$, $V_{\mathrm{as}}(0) = 0$, and

$$e^{\varepsilon t_2} V_{\mathrm{as}}(x(t_2)) \leq e^{\varepsilon t_1} V_{\mathrm{as}}(x(t_1)) + \int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t \qquad (63)$$

for almost all $t_2 \geq t_1$ and $u(\cdot) \in \mathcal{U}$. Dividing (63) by $t_2 - t_1$ and letting $t_2 \to t_1$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\mathrm{as}}(x(t)) + \varepsilon V_{\mathrm{as}}(x(t)) \leq s(u(t), y(t)), \ \text{a.e. } t \geq 0, \qquad (64)$$

where $x(t)$, $t \geq 0$, is a solution satisfying (54) and $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x(t)) = \limsup_{h\to 0^+}[V_{\mathrm{as}}(x(t+h)) - V_{\mathrm{as}}(x(t))]/h$. Now, with $t = 0$, it follows from (64) that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x_0) + \varepsilon V_{\mathrm{as}}(x_0) \leq s(u, y(0)), \qquad u \in \mathbb{R}^m.$$

Next, let $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be such that

$$d(x,u) \triangleq -\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) - \varepsilon V_{\mathrm{as}}(x) + s(u,y). \tag{65}$$

Now, it follows from (64) that $d(x,u) \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Since $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \in \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_{Gu} V_{\mathrm{as}}(x)$ for almost all $x \in \mathbb{R}^n$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \geq \min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u, \quad \text{a.e.} \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{66}$$

and hence, it follows from (65) and (66) that

$$\begin{aligned} -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) &\geq d(x,u) \geq 0, \\ \text{a.e.} \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m&. \end{aligned} \tag{67}$$

Since the left-hand side of (67) is quadratic in u, there exist functions $\ell : \mathbb{R}^n \to \mathbb{R}^p$ and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that

$$\begin{aligned} &[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \\ &\quad = -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) \\ &\quad = -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + [h(x) + J(x)u]^{\mathrm{T}} \\ &\qquad \times Q[h(x) + J(x)u] + 2[h(x) + J(x)u]^{\mathrm{T}} Su + u^{\mathrm{T}} Ru. \end{aligned}$$

Now, equating coefficients of equal powers yields (56)–(58) with $V_{\mathrm{s}}(x) = V_{\mathrm{as}}(x)$ and with the positive definiteness of $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, following from Theorem 3.2 of [46].

Finally, the proof for the weakly dissipative case follows by using an identical construction with $\varepsilon = 0$. □

Remark 6.1. Note that if $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is invertible for all $x \in \mathbb{R}^n$, then inequality (59) can be equivalently written as

$$\begin{aligned} [\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)]^{\mathrm{T}}[\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)] \\ \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{68}$$

which is free of $u \in \mathbb{R}^m$. This follows from the fact that (59) holds if and only if

$$\min_u [\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \tag{69}$$

holds. A similar expression to (68) involving generalized inverses also holds in the case where $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is singular for some $x \in \mathbb{R}^n$.

The following result gives sufficient conditions for weak dissipativity and weak exponential dissipativity of $\mathcal{G}$ based on $\max \mathcal{L}_f V_{\mathrm{s}}(\cdot)$.

Theorem 6.2. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \max \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{70}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{71}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{72}$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate $s(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$.

Proof. Suppose that there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (70)–(72) are satisfied. Then, for every admissible input $u(t) \in \mathbb{R}^m$, it follows from (70)–(72) and (61) that

$$\begin{aligned}
\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t &= \int_{t_1}^{t_2} e^{\varepsilon t}\big[y^{\mathrm{T}}(t)Qy(t) + 2y^{\mathrm{T}}(t)Su(t) + u^{\mathrm{T}}(t)Ru(t)\big]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\big[h^{\mathrm{T}}(x(t))Qh(x(t)) + 2h^{\mathrm{T}}(x(t))(S + QJ(x(t)))u(t) \\
&\quad + u^{\mathrm{T}}(t)(J^{\mathrm{T}}(x(t))QJ(x(t)) + S^{\mathrm{T}}J(x(t)) \\
&\quad + J^{\mathrm{T}}(x(t))S + R)u(t)\big]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t)) \\
&\quad + [\ell(x(t)) + \mathcal{W}(x(t))u(t)]^{\mathrm{T}}[\ell(x(t)) + \mathcal{W}(x(t))u(t)]\big]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t))]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\
&= e^{\varepsilon t_2}V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1}V_{\mathrm{s}}(x(t_1)), \quad \text{a.e.} \quad t \geq 0,
\end{aligned}$$

where $x(t)$, $t \geq t_0$, is a solution satisfying (54). The result is now immediate from Definition 6.1. The proof for the weak dissipative case follows an identical construction by setting $\varepsilon = 0$. □

Next, we provide several definitions of nonlinear discontinuous dynamical systems which are dissipative or exponentially dissipative with respect to supply rates of a specific form.

Definition 6.3. A discontinuous dynamical system $\mathcal{G}$ of the form (37) and (38) with $m = l$ is *weakly* (resp., *strongly*) *passive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the supply rate $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 6.4. A discontinuous dynamical system $\mathcal{G}$ of the form (37) and (38) is *weakly* (resp., *strongly*) *nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the supply rate $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

Definition 6.5. A discontinuous dynamical system $\mathcal{G}$ of the form (37) and (38) with $m = l$ is *weakly* (resp., *strongly*) *exponentially passive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the supply rate $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 6.6. A discontinuous dynamical system $\mathcal{G}$ of the form (37) and (38) is *weakly* (resp., *strongly*) *exponentially nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the supply rate $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

The following results present the nonlinear versions of the Kalman-Yakubovich-Popov *strict positive real lemma* (resp., *positive real lemma*) and *strict bounded real lemma* (resp., *bounded real lemma*) for weakly exponentially passive (resp., weakly passive) and weakly exponentially nonexpansive (resp., weakly nonexpansive) discontinuous systems, respectively.

Corollary 6.1. *Let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \min \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{73}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{74}$$

$$0 = J(x) + J^{\mathrm{T}}(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{75}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad u \in \mathbb{R}^m, \tag{76}$$

then $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive). Conversely, if $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive), then there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (73)–(75) hold.

Proof. The result is a direct consequence of Theorem 6.1 with $l = m$, $Q = 0$, $S = I_m$, and $R = 0$. Specifically, with $\kappa(y) = -y$ it follows that $s(\kappa(y), y) = -2y^{\mathrm{T}}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 6.1 are satisfied. □

Example 6.2. Consider the harmonic oscillator $\mathcal{G}$ with Coulomb friction given by [11]

$$m\ddot{x}(t) + b\,\mathrm{sign}(\dot{x}(t)) + kx(t) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \text{a.e.} \quad t \geq 0, \tag{77}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{78}$$

or, equivalently,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_1(t) - \frac{b}{m}\,\mathrm{sign}(x_2(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \tag{79}$$

$$\text{a.e.} \quad t \geq 0,$$

$$y(t) = \frac{1}{2}x_2(t), \tag{80}$$

where $m, b, k > 0$. Next, consider the continuously differentiable storage function $V_{\mathrm{s}}(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$ and note that, for almost all $x \in \mathbb{R}^2$, $\mathcal{L}_f V_{\mathrm{s}}(x) = \{-b|x_2|\}$ and $\mathcal{L}_G V_{\mathrm{s}}(x) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_{\mathrm{s}}(x) = \max \mathcal{L}_f V_{\mathrm{s}}(x) = -b|x_2|$. Now, with $\ell(x) = \pm\sqrt{b|x_2|}$ and $\mathcal{W}(x) = 0$, (73)–(76) are satisfied. Hence, it follows from Corollary 6.1 that $\mathcal{G}$ is weakly passive. △

Example 6.3. Consider a controlled smooth oscillator with nonsmooth friction and uncertain coefficients given in [25] represented by the differential inclusion $\mathcal{G}$ given by

$$\begin{aligned} \dot{x}(t) &\in \mathcal{K}[f](x(t)) + Gu(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, && (81) \\ y(t) &= \frac{1}{2}x_2(t), && (82) \end{aligned}$$

where $G = [0, 1]^{\mathrm{T}}$ and $\mathcal{K}[f] : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ is given by

$$\mathcal{K}[f](x) \triangleq \begin{cases} [-2x_2 - 1, -x_2 - 1] \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-x_2 - \mathrm{sign}(x_1)\} \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 \backslash (\{(0, x_2) : x_2 \in \mathbb{R}\}, \\ & \quad \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}, \\ [-2x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0, x_1 = 0, \\ [-x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 < 0, x_1 = 0, \\ [-1, 1] \times \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Next, consider the continuously differentiable storage function $V_{\mathrm{s}}(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and note that for almost all $x \in \mathbb{R}^2$,

$$\begin{aligned} \mathcal{L}_f V_{\mathrm{s}}(x) &= \begin{cases} \{[-1, 0]x_1x_2 - x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-|x_1|\}, & (x_1, x_2) \in \mathbb{R}^2 \backslash (\{(0, x_2) : x_2 \in \mathbb{R}\}, \\ & \quad \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}, \\ \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases} \\ \mathcal{L}_G V_{\mathrm{s}}(x) &= \{x_2\}, \end{aligned}$$

which implies that $\max \mathcal{L}_f V_{\rm s}(x) = 0$ and $\min \mathcal{L}_f V_{\rm s}(x) = -|x_1|$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (73)–(76) that

$$
\begin{aligned}
0 &= -|x_1| + \ell^2(x), &(83)\\
0 &= \frac{1}{2}x_2 - \frac{1}{2}x_2 + \ell(x)\mathcal{W}(x), &(84)\\
0 &= \mathcal{W}^2(x), &(85)\\
|x_1| &\leq [\ell(x) + \mathcal{W}(x)u]^2, \quad u \in \mathbb{R}. &(86)
\end{aligned}
$$

Hence, with $\ell(x) = \pm\sqrt{|x_1|}$ and $\mathcal{W}(x) = 0$, it follows from Corollary 6.1 that $\mathcal{G}$ is weakly passive. △

Example 6.4. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction and nonsmooth output given by ([25])

$$
\begin{aligned}
\dot{x}(t) &= f(x(t)) + Gu(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, &(87)\\
y(t) &= \frac{1}{2}\,\text{sign}(x_2(t)), &(88)
\end{aligned}
$$

where $f(x) = [-\text{sign}(x_2) - \frac{1}{2}\text{sign}(x_1), \text{sign}(x_1)]^{\rm T}$ and $G = [0, 1]^{\rm T}$. Next, consider the locally Lipschitz continuous storage function $V_{\rm s}(x) = |x_1| + |x_2|$ and note that

$$
\partial V_{\rm s}(x_1, x_2) = \begin{cases}
\{\text{sign}(x_1)\} \times \{\text{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0,\\
\{\text{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0,\\
[-1, 1] \times \{\text{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0,\\
\overline{\text{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0).
\end{cases}
$$

Hence,

$$
\begin{aligned}
\mathcal{L}_f V_{\rm s}(x_1, x_2) &= \begin{cases}
\{-\frac{1}{2}\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0,\\
\varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0,\\
\varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0,\\
\{0\}, & (x_1, x_2) = (0, 0),
\end{cases}\\
\mathcal{L}_G V_{\rm s}(x_1, x_2) &= \begin{cases}
\{\text{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0,\\
\varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0,\\
\{\text{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0,\\
\{0\}, & (x_1, x_2) = (0, 0),
\end{cases}
\end{aligned}
$$

which implies that $\max \mathcal{L}_f V_{\rm s}(x) = 0$, $\min \mathcal{L}_f V_{\rm s}(x) = -\frac{1}{2}$, and $\mathcal{L}_G V_{\rm s}(x) = \{\text{sign}(x_2)\}$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (73)–(76) that

$$
\begin{aligned}
0 &= -\frac{1}{2} + \ell^2(x), &(89)\\
0 &= \frac{1}{2}\,\text{sign}(x_2) - \frac{1}{2}\,\text{sign}(x_2) + \ell(x)\mathcal{W}(x), &(90)
\end{aligned}
$$

$$0 = \mathcal{W}^2(x), \tag{91}$$

$$\frac{1}{2} \leq [\ell(x) + \mathcal{W}(x)u]^2, \quad u \in \mathbb{R}. \tag{92}$$

Hence, with $\ell(x) = \pm\sqrt{\frac{1}{2}}$ and $\mathcal{W}(x) = 0$, it follows from Corollary 6.1 that $\mathcal{G}$ is weakly passive. △

Corollary 6.2. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \min \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) + h^{\mathrm{T}}(x)h(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{93}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) + h^{\mathrm{T}}(x)J(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{94}$$

$$0 = \gamma^2 I_m - J^{\mathrm{T}}(x)J(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{95}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad u \in \mathbb{R}^m, \tag{96}$$

where $\gamma > 0$, then $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive). Conversely, if $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive), then there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (93)–(95) *hold.*

Proof. The result is a direct consequence of Theorem 6.1 with $Q = -I_l$, $S = 0$, and $R = \gamma^2 I_m$. Specifically, with $\kappa(y) = -\frac{1}{2\gamma}y$ it follows that $s(\kappa(y), y) = -\frac{3}{4}y^{\mathrm{T}}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 6.1 are satisfied. □

Example 6.5. Consider the controlled dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{97}$$

$$y(t) = x(t), \tag{98}$$

where $x(t) = [x_1(t), x_2(t)]^{\mathrm{T}}$, $u(t) = [u_1(t), u_2(t)]^{\mathrm{T}}$,

$$f(x) = \begin{bmatrix} |x_1|(-x_1 + |x_2|) \\ x_2(-x_1 - |x_2|) \end{bmatrix}, \quad G(x) = \begin{bmatrix} |x_1| & 0 \\ 0 & x_2 \end{bmatrix}.$$

Next, consider the locally Lipschitz continuous storage function $V_{\mathrm{s}}(x) = 2|x_1| + 2|x_2|$ and note that

$$\partial V_{\mathrm{s}}(x_1, x_2) = \begin{cases} \{2\,\mathrm{sign}(x_1)\} \times \{2\,\mathrm{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{2\,\mathrm{sign}(x_1)\} \times [-2, 2], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ [-2, 2] \times \{2\,\mathrm{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \overline{\mathrm{co}}\{(2, 2), (-2, 2), (-2, -2), (2, -2)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V_{\mathrm{s}}(x_1,x_2) = \begin{cases} \{-2x_1^2-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0,\\ \{-2x_1^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2= 0,\\ \{-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1= 0,\\ \{0\}, & (x_1,x_2)=(0,0), \end{cases}$$

$$\mathcal{L}_G V_{\mathrm{s}}(x_1,x_2) = \begin{cases} \{[2x_1, 2|x_2|]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0,\\ \{[2x_1, 0]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2= 0,\\ \{[0, 2|x_2|]\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1= 0,\\ \{[0,0]\}, & (x_1,x_2)=(0,0), \end{cases}$$

which implies that $\min \mathcal{L}_f V_{\mathrm{s}}(x) = \max \mathcal{L}_f V_{\mathrm{s}}(x) = -2x_1^2 - 2x_2^2$ and $\mathcal{L}_G V_{\mathrm{s}}(x) = \{[2x_1, 2|x_2|]\}$ for almost all $x\in\mathbb{R}^2$. Now, it follows from (93)–(96) that

$$0 = -2x_1^2 - 2x_2^2 + x_1^2 + x_2^2 + \ell^{\mathrm{T}}(x)\ell(x), \tag{99}$$

$$0 = \frac{1}{2}[2x_1, 2|x_2|] + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{100}$$

$$0 = \gamma^2 I_2 - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{101}$$

$$0 \leq [\ell(x)+\mathcal{W}(x)u]^{\mathrm{T}}[\ell(x)+\mathcal{W}(x)u], \quad u\in\mathbb{R}^2. \tag{102}$$

Hence, with $\gamma = 1$, $\ell(x) = -[x_1, |x_2|]^{\mathrm{T}}$, and $\mathcal{W}(x) = I_2$, it follows from Corollary 6.2 that $\mathcal{G}$ is weakly nonexpansive. △

In light of Definition 6.3 the following result is immediate.

Proposition 6.2. *Consider the discontinuous dynamical system $\mathcal{G}$ given by* (37) *and* (38)*. Then the following statements hold:*

i) If $\mathcal{G}$ is strongly passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_{\mathrm{s}}(\cdot)$, then the zero solution $x(t)\equiv 0$ of the undisturbed ($u(t)\equiv 0$) system $\mathcal{G}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}$ is strongly exponentially passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_{\mathrm{s}}(\cdot)$, then the zero solution $x(t)\equiv 0$ of the undisturbed ($u(t)\equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

iii) If $\mathcal{G}$ is strongly zero-state observable and strongly nonexpansive with locally Lipschitz continuous, regular, and positive definite storage function $V_{\mathrm{s}}(\cdot)$, then the zero solution $x(t)\equiv 0$ of the undisturbed ($u(t)\equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

iv) If $\mathcal{G}$ is strongly exponentially nonexpansive with a locally Lipschitz continuous, regular, and positive definite storage function $V_{\mathrm{s}}(\cdot)$, then the zero solution $x(t)\equiv 0$ of the undisturbed ($u(t)\equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.

Proof. Statements *i*)–*iv*) are immediate and follow from (42)–(44) using Lyapunov and invariant set stability arguments given by Theorems 3.1 and 3.2, respectively. □

7. STABILITY OF FEEDBACK INTERCONNECTIONS OF DISSIPATIVE DISCONTINUOUS DYNAMICAL SYSTEMS

In this section, we consider feedback interconnections of dissipative discontinuous dynamical systems. Specifically, using the notions of dissipativity and exponential dissipativity for discontinuous dynamical systems, with appropriate storage functions and supply rates, we construct (not necessarily smooth) Lyapunov functions for interconnected discontinuous dynamical systems by appropriately combining the storage functions for each subsystem.

We begin by considering the nonlinear discontinuous dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{103}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{104}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and $J : \mathbb{R}^n \to \mathbb{R}^{l\times m}$, with the nonlinear feedback discontinuous system $\mathcal{G}_\mathrm{c}$ given by

$$\dot{x}_\mathrm{c}(t) = f_\mathrm{c}(x_\mathrm{c}(t)) + G_\mathrm{c}(u_\mathrm{c}(t), x_\mathrm{c}(t))u_\mathrm{c}(t), \quad x_\mathrm{c}(0) = x_{\mathrm{c}0}, \quad \text{a.e.} \quad t \geq 0, \tag{105}$$

$$y_\mathrm{c}(t) = h_\mathrm{c}(u_\mathrm{c}(t), x_\mathrm{c}(t)) + J_\mathrm{c}(u_\mathrm{c}(t), x_\mathrm{c}(t))u_\mathrm{c}(t), \tag{106}$$

where $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$, $u_\mathrm{c} \in \mathbb{R}^l$, $y_\mathrm{c} \in \mathbb{R}^m$, $f_\mathrm{c} : \mathbb{R}^{n_\mathrm{c}} \to \mathbb{R}^{n_\mathrm{c}}$, $G_\mathrm{c} : \mathbb{R}^l \times \mathbb{R}^{n_\mathrm{c}} \to \mathbb{R}^{n_\mathrm{c}\times l}$, $h_\mathrm{c} : \mathbb{R}^l \times \mathbb{R}^{n_\mathrm{c}} \to \mathbb{R}^m$, and $J_\mathrm{c} : \mathbb{R}^l \times \mathbb{R}^{n_\mathrm{c}} \to \mathbb{R}^{m\times l}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, $J(\cdot)$, $f_\mathrm{c}(\cdot)$, $G_\mathrm{c}(\cdot)$, $h_\mathrm{c}(\cdot,\cdot)$, and $J_\mathrm{c}(\cdot,\cdot)$ are Lebesgue measurable and locally essentially bounded, (105) and (106) has at least one equilibrium point, and the required properties for the existence of solutions of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are satisfied. Note that with the negative feedback interconnection given by Figure 3, $u_\mathrm{c} = y$ and $y_\mathrm{c} = -u$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is well posed, that is, $\det[I_m + J_\mathrm{c}(y, x_\mathrm{c})J(x)] \neq 0$ for all y, x, and x_c.

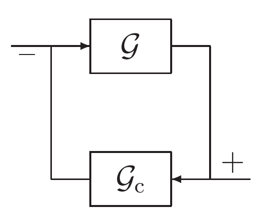

FIGURE 3. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$.

The following results give sufficient conditions for Lyapunov, asymptotic, and exponential stability of the feedback interconnection given by Figure 3. In this section, we assume that the forward path $\mathcal{G}$ and the feedback path $\mathcal{G}_{\mathrm{c}}$ in Figure 3 are strongly dissipative systems. This assumption holds when the closed-loop system (103)–(106) admits a unique solution and is only made for notational convenience. Finally, we also note that the obtained stability results also hold for the case where $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are weakly dissipative. In this case, however, the set-valued Lie derivative operator should be replaced with the upper right Dini directional derivative in the proofs of the stability theorems.

The following lemma is necessary for the next theorem.

Lemma 7.1 ([25]). *Let $x : [t_0, t] \to \mathbb{R}^q$ be a Filippov solution of the discontinuous dynamical system* (23) *and let $V : \mathbb{R}^q \to \mathbb{R}$ be locally Lipschitz continuous and regular. Then $\frac{\mathrm{d}}{\mathrm{d}\sigma}V(x(\sigma))$ exists for almost all $\sigma \in [t_0, t]$ and $\frac{\mathrm{d}}{\mathrm{d}\sigma}V(x(\sigma)) \in \mathcal{L}_f V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$.*

Theorem 7.1. *Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by* (103) *and* (104)*, and $\mathcal{G}_{\mathrm{c}}$ given by* (105) *and* (106) *with input-output pairs (u, y) and $(u_{\mathrm{c}}, y_{\mathrm{c}})$, respectively, and with $u_{\mathrm{c}} = y$ and $y_{\mathrm{c}} = -u$. Assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly zero-state observable, strongly completely reachable, and strongly dissipative with respect to the supply rates $s(u, y)$ and $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ and with locally Lipschitz continuous, regular, and radially unbounded storage functions $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$, respectively, such that $V_{\mathrm{s}}(0) = 0$ and $V_{\mathrm{sc}}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $s(u, y) + \sigma s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) \le 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_{\mathrm{c}} \in \mathbb{R}^l$, $y_{\mathrm{c}} \in \mathbb{R}^m$ such that $u_{\mathrm{c}} = y$ and $y_{\mathrm{c}} = -u$. Then the following statements hold:*

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_{\mathrm{c}}$ is strongly exponentially dissipative with respect to supply rate $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ and $\operatorname{rank}[G_{\mathrm{c}}(u_{\mathrm{c}}, 0)] = m$, $u_{\mathrm{c}} \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally strongly asymptotically stable.

iii) If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly exponentially dissipative with respect to supply rates $s(u, y)$ and $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$, respectively, and $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$ are such that there exist constants $\alpha, \alpha_{\mathrm{c}}, \beta$, and $\beta_{\mathrm{c}} > 0$ such that

$$\alpha\|x\|^2 \le V_{\mathrm{s}}(x) \le \beta\|x\|^2, \qquad x \in \mathbb{R}^n, \tag{107}$$

$$\alpha_{\mathrm{c}}\|x_{\mathrm{c}}\|^2 \le V_{\mathrm{sc}}(x_{\mathrm{c}}) \le \beta_{\mathrm{c}}\|x_{\mathrm{c}}\|^2, \quad x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}, \tag{108}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally strongly exponentially stable.

Proof. *i*) Note that the closed-loop dynamics of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ has a form given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\mathrm{c}}(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), x_{\mathrm{c}}(t)) \\ f_2(x(t), x_{\mathrm{c}}(t)) \end{bmatrix} \triangleq \tilde{f}(x(t), x_{\mathrm{c}}(t)), \quad \begin{bmatrix} x(t_0) \\ x_{\mathrm{c}}(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{\mathrm{c}0} \end{bmatrix}, \text{ a.e. } t \ge t_0. \tag{109}$$

Now, consider the Lyapunov function candidate $V(x,x_\mathrm{c}) = V_\mathrm{s}(x) + \sigma V_\mathrm{sc}(x_\mathrm{c})$. Since $\mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c}) \subseteq \mathcal{L}_{\tilde{f}}V_\mathrm{s}(x) + \sigma\mathcal{L}_{\tilde{f}}V_\mathrm{sc}(x_\mathrm{c})$ for almost all $(x,x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}$, it follows that

$$\begin{aligned}\max\mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c}) &\leq \max\{\mathcal{L}_{f_1}V_\mathrm{s}(x) + \sigma\mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c})\}\\ &\leq \max\mathcal{L}_{f_1}V_\mathrm{s}(x) + \sigma\max\mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c}).\end{aligned}$$

Next, since $s(u,y) + \sigma s_\mathrm{c}(u_\mathrm{c},y_\mathrm{c}) \leq 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_\mathrm{c} \in \mathbb{R}^l$, $y_\mathrm{c} \in \mathbb{R}^m$, $\frac{\mathrm{d}}{\mathrm{d}t}V_\mathrm{s}(x(t)) \in \mathcal{L}_{f_1}V_\mathrm{s}(x(t))$, a.e. $t \geq 0$, and $\frac{\mathrm{d}}{\mathrm{d}t}V_\mathrm{sc}(x_\mathrm{c}(t)) \in \mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c}(t))$, a.e. $t \geq 0$, there exist u', y', u'_c and y'_c such that

$$\max\mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c}) \leq \max\mathcal{L}_{f_1}V_\mathrm{s}(x) + \sigma\max\mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c}) \leq s(u',y') + \sigma s_\mathrm{c}(u'_\mathrm{c},y'_\mathrm{c}) \leq 0$$

for almost all $x \in \mathbb{R}^n$ and $x_\mathrm{c} \in \mathbb{R}^{n_\mathrm{c}}$. Now, it follows from Theorem 3.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_\mathrm{c}$ is strongly exponentially dissipative it follows that there exist u', y', u'_c and y'_c and a scalar $\varepsilon_\mathrm{c} > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x,x_\mathrm{c}) &\leq \max\mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c})\\ &\leq \max\mathcal{L}_{f_1}V_\mathrm{s}(x) + \sigma\max\mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c})\\ &\leq -\sigma\varepsilon_\mathrm{c}V_\mathrm{sc}(x_\mathrm{c}) + s(u',y') + \sigma s_\mathrm{c}(u'_\mathrm{c},y'_\mathrm{c})\\ &\leq -\sigma\varepsilon_\mathrm{c}V_\mathrm{sc}(x_\mathrm{c}), \quad \text{a.e.} \quad (x,x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}.\end{aligned}$$

Now, let $\mathcal{R} \triangleq \{(x,x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}} : \frac{\mathrm{d}}{\mathrm{d}t}V(x,x_\mathrm{c}) = 0 \in \mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c})\}$ and, since $V_\mathrm{sc}(x_\mathrm{c})$ is positive definite, note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x,x_\mathrm{c}) = 0$ if and only if $x_\mathrm{c} = 0$. Now, since $\operatorname{rank}[G_\mathrm{c}(u_\mathrm{c},0)] = m$, $u_\mathrm{c} \in \mathbb{R}^l$, it follows that on every invariant set $\mathcal{M}$ contained in $\mathcal{R}$, $u_\mathrm{c}(t) = y(t) \equiv 0$, and hence, by (106), $u(t) \equiv 0$ so that $\dot{x}(t) = f(x(t))$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{M} = \{(0,0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 3.2 that $\operatorname{dist}(\psi(t),\mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (109). Now, global strong asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ follows from the fact that $V_\mathrm{s}(\cdot)$ and $V_\mathrm{sc}(\cdot)$ are, by assumption, radially unbounded.

iii) Finally, if $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly exponentially dissipative it follows that there exist u', y', u'_c and y'_c, and scalars $\varepsilon > 0$ and $\varepsilon_\mathrm{c} > 0$ such that

$$\begin{aligned}\max\mathcal{L}_{\tilde{f}}V(x,x_\mathrm{c}) &\leq \max\mathcal{L}_{f_1}V_\mathrm{s}(x) + \sigma\max\mathcal{L}_{f_2}V_\mathrm{sc}(x_\mathrm{c})\\ &\leq -\varepsilon V_\mathrm{s}(x) - \sigma\varepsilon_\mathrm{c}V_\mathrm{sc}(x_\mathrm{c}) + s(u',y') + \sigma s_\mathrm{c}(u'_\mathrm{c},y'_\mathrm{c})\\ &\leq -\min\{\varepsilon,\varepsilon_\mathrm{c}\}V(x,x_\mathrm{c}), \quad (x,x_\mathrm{c}) \in \mathbb{R}^n \times \mathbb{R}^{n_\mathrm{c}}.\end{aligned}$$

Hence, it follows from Theorem 3.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is globally strongly exponentially stable. $\square$

The next result presents Lyapunov, asymptotic, and exponential stability of dissipative discontinuous feedback systems with supply rate maps consisting of quadratic supply rates.

Theorem 7.2. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, $Q_\mathrm{c} \in \mathbb{S}^m$, $S_\mathrm{c} \in \mathbb{R}^{m\times l}$, and $S_\mathrm{c} \in \mathbb{S}^l$. Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by* (103) *and* (104) *and $\mathcal{G}_\mathrm{c}$ given by* (105) *and* (106)*, and assume $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly zero-state observable. Furthermore, assume $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^\mathrm{T}Qy + 2y^\mathrm{T}Su + u^\mathrm{T}Ru$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_\mathrm{s}(\cdot)$, and $\mathcal{G}_\mathrm{c}$ is strongly dissipative with respect to the supply rate $s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) = y_\mathrm{c}^\mathrm{T}Q_\mathrm{c}y_\mathrm{c} + 2y_\mathrm{c}^\mathrm{T}S_\mathrm{c}u_\mathrm{c} + u_\mathrm{c}^\mathrm{T}R_\mathrm{c}u_\mathrm{c}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_\mathrm{sc}(\cdot)$. Finally, assume there exists $\sigma > 0$ such that*

$$\hat{Q} \triangleq \begin{bmatrix} Q + \sigma R_\mathrm{c} & -S + \sigma S_\mathrm{c}^\mathrm{T} \\ -S^\mathrm{T} + \sigma S_\mathrm{c} & R + \sigma Q_\mathrm{c} \end{bmatrix} \le 0. \tag{110}$$

Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is strongly Lyapunov stable.

ii) If $\mathcal{G}_\mathrm{c}$ is strongly exponentially dissipative with respect to supply rate $s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c})$ and $\operatorname{rank}[G_\mathrm{c}(u_\mathrm{c}, 0)] = m$, $u_\mathrm{c} \in \mathbb{R}^l$*, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is globally strongly asymptotically stable.*

iii) If $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly exponentially dissipative with respect to supply rates $s(u,y)$ and $s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c})$ and there exist constants $\alpha, \beta, \alpha_\mathrm{c}$, and $\beta_\mathrm{c} > 0$ such that (107) *and* (108) *hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is globally strongly exponentially stable.*

iv) If $\hat{Q} < 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is globally strongly asymptotically stable.

Proof. Statements *i*)–*iii*) are a direct consequence of Theorem 7.1 by noting that

$$s(u,y) + \sigma s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) = \begin{bmatrix} y \\ y_\mathrm{c} \end{bmatrix}^\mathrm{T} \hat{Q} \begin{bmatrix} y \\ y_\mathrm{c} \end{bmatrix},$$

and hence, $s(u,y) + \sigma s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) \le 0$.

To show *iv*) consider the Lyapunov function candidate $V(x, x_\mathrm{c}) = V_\mathrm{s}(x) + \sigma V_\mathrm{sc}(x_\mathrm{c})$. Now, since $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are strongly dissipative it follows that there exist u', y', u'_c and y'_c with $u'_\mathrm{c} = y'$ and $y'_\mathrm{c} = -u'$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x, x_\mathrm{c}) &\le \max \mathcal{L}_{\tilde{f}} V(x, x_\mathrm{c}) \\ &\le \max \mathcal{L}_{f_1} V_\mathrm{s}(x) + \sigma \max \mathcal{L}_{f_2} V_\mathrm{sc}(x_\mathrm{c}) \\ &\le s(u,y) + \sigma s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) \end{aligned}$$

$$
\begin{aligned}
&= y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru + \sigma(y_{\mathrm{c}}^{\mathrm{T}}Q_{\mathrm{c}}y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}}S_{\mathrm{c}}u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}}R_{\mathrm{c}}u_{\mathrm{c}}) \\
&= \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix}^{\mathrm{T}} \hat{Q} \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix} \\
&\leq 0, \quad \text{a.e.} \quad (x, x_{\mathrm{c}}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}},
\end{aligned}
$$

which implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly Lyapunov stable. Next, let $\mathcal{R} \triangleq \{(x, x_{\mathrm{c}}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\mathrm{c}}} : \frac{\mathrm{d}}{\mathrm{d}t}V(x, x_{\mathrm{c}}) = 0 \in \mathcal{L}_{\tilde{f}}V(x, x_{\mathrm{c}})\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x, x_{\mathrm{c}}) = 0$ if and only if $(y, y_{\mathrm{c}}) = (0, 0)$. Now, since $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly zero-state observable it follows that $\mathcal{M} = \{(0, 0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 3.2 that $\mathrm{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (109). Finally, global strong asymptotic stability follows from the fact that $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$ are, by assumption, radially unbounded, and hence, $V(x, x_{\mathrm{c}}) \to \infty$ as $||(x, x_{\mathrm{c}})|| \to \infty$. □

The following corollary to Theorem 7.2 is necessary for the results in Section 11.

Corollary 7.1. *Consider the closed-loop system consisting of the discontinuous nonlinear dynamical systems $\mathcal{G}$ given by* (103) *and* (104)*, and $\mathcal{G}_{\mathrm{c}}$ given by* (105) *and* (106)*. Let $\alpha, \beta, \alpha_{\mathrm{c}}, \beta_{\mathrm{c}}, \delta \in \mathbb{R}$ be such that $\beta > 0$, $0 < \alpha + \beta$, $0 < 2\delta < \beta - \alpha$, $\alpha_{\mathrm{c}} = \alpha + \delta$, and $\beta_{\mathrm{c}} = \beta - \delta$, let $M \in \mathbb{R}^{m \times m}$ be positive definite, and assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly zero-state observable. If $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = u^{\mathrm{T}}My + \frac{\alpha\beta}{\alpha+\beta}y^{\mathrm{T}}My + \frac{1}{\alpha+\beta}u^{\mathrm{T}}Mu$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\mathrm{s}}(\cdot)$, and $\mathcal{G}_{\mathrm{c}}$ is strongly dissipative with respect to the supply rate $s_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = u_{\mathrm{c}}^{\mathrm{T}}My_{\mathrm{c}} - \frac{1}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}}y_{\mathrm{c}}^{\mathrm{T}}My_{\mathrm{c}} - \frac{\alpha_{\mathrm{c}}\beta_{\mathrm{c}}}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}}u_{\mathrm{c}}^{\mathrm{T}}Mu_{\mathrm{c}}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{\mathrm{sc}}(\cdot)$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally strongly asymptotically stable.*

Proof. The proof is a direct consequence of Theorem 7.2 with $Q = \frac{\alpha\beta}{\alpha+\beta}M$, $S = \frac{1}{2}M$, $R = \frac{1}{\alpha+\beta}M$, $Q_{\mathrm{c}} = -\frac{1}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}}M$, $S_{\mathrm{c}} = \frac{1}{2}M$, and $R_{\mathrm{c}} = -\frac{\alpha_{\mathrm{c}}\beta_{\mathrm{c}}}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}}M$. Specifically, let $\sigma > 0$ be such that

$$
\sigma\left(\frac{\delta^2}{(\alpha+\beta)^2} - \frac{1}{4}\right) + \frac{1}{4} > 0.
$$

In this case, $\hat{Q}$ given by (110) satisfies

$$
\hat{Q} = \begin{bmatrix} (\frac{\alpha\beta}{\alpha+\beta} - \frac{\sigma\alpha_{\mathrm{c}}\beta_{\mathrm{c}}}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}})M & \frac{\sigma-1}{2}M \\ \frac{\sigma-1}{2}M & (\frac{1}{\alpha+\beta} - \frac{\sigma}{\alpha_{\mathrm{c}}+\beta_{\mathrm{c}}})M \end{bmatrix} < 0,
$$

so that all the conditions of Theorem 7.2 are satisfied. □

The following corollary is a direct consequence of Theorem 7.2. Note that if a nonlinear discontinuous dynamical system $\mathcal{G}$ is strongly dissipative with respect to a supply rate $s(u, y) = u^{\mathrm{T}}y - \varepsilon u^{\mathrm{T}}u - \hat{\varepsilon}y^{\mathrm{T}}y$, where $\varepsilon, \hat{\varepsilon} \geq 0$, then with $\kappa(y) = ky$, where

$k \in \mathbb{R}$ is such that $k(1-\varepsilon k) < \hat{\varepsilon}$, $s(u,y) = [k(1-\varepsilon k) - \hat{\varepsilon}]y^{\mathrm{T}}y < 0$, $y \neq 0$. Hence, if $\mathcal{G}$ is strongly zero-state observable it follows from Theorem 3.2 of [46] that all storage functions of $\mathcal{G}$ are positive definite.

Corollary 7.2. *Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by* (103) *and* (104) *and $\mathcal{G}_{\mathrm{c}}$ given by* (105) *and* (106)*, and assume $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly zero-state observable. Then the following statements hold:*

i) If $\mathcal{G}$ is strongly passive, $\mathcal{G}_{\mathrm{c}}$ is strongly exponentially passive, and $\operatorname{rank}[G_{\mathrm{c}}(u_{\mathrm{c}},0)] = m$, $u_{\mathrm{c}} \in \mathbb{R}^l$, *then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly asymptotically stable.*

ii) If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly exponentially passive with storage functions $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$, respectively, such that (107) *and* (108) *hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly exponentially stable.*

iii) If $\mathcal{G}$ is strongly nonexpansive with gain $\gamma > 0$, $\mathcal{G}_{\mathrm{c}}$ is strongly exponentially nonexpansive with gain $\gamma_{\mathrm{c}} > 0$, $\operatorname{rank}[G_{\mathrm{c}}(u_{\mathrm{c}},0)] = m$, $u_{\mathrm{c}} \in \mathbb{R}^l$, *and $\gamma\gamma_{\mathrm{c}} \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly asymptotically stable.*

iv) If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are strongly exponentially nonexpansive with storage functions $V_{\mathrm{s}}(\cdot)$ and $V_{\mathrm{sc}}(\cdot)$, respectively, such that (107) *and* (108) *hold, and with gains $\gamma > 0$ and $\gamma_{\mathrm{c}} > 0$, respectively, such that $\gamma\gamma_{\mathrm{c}} \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is strongly exponentially stable.*

Proof. The proof is a direct consequence of Theorem 7.2. Specifically, i) and ii) follow from Theorem 7.2 with $Q = Q_{\mathrm{c}} = 0$, $S = S_{\mathrm{c}} = I_m$, and $R = R_{\mathrm{c}} = 0$, whereas iii) and iv) follow from Theorem 7.2 with $Q = -I_l$, $S = 0$, $R = \gamma^2 I_m$, $Q_{\mathrm{c}} = -I_{l_{\mathrm{c}}}$, $S_{\mathrm{c}} = 0$, and $R_{\mathrm{c}} = \gamma_{\mathrm{c}}^2 I_{m_{\mathrm{c}}}$. □

Example 7.1. Consider the nonlinear mechanical system $\mathcal{G}$ with a discontinuous spring force given by

$$\ddot{x}(t) + \operatorname{sign}(x(t)) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad \text{a.e.} \quad t \geq 0, \tag{111}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{112}$$

or, equivalently,

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{113}$$

$$\dot{x}_2(t) = -\operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \tag{114}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{115}$$

and the continuous nonlinear second-order dynamic controller $\mathcal{G}_{\mathrm{c}}$ given by

$$\dot{x}_{\mathrm{c}1}(t) = -\frac{1}{2}x_{\mathrm{c}1}(t) - x_{\mathrm{c}2}(t), \quad x_{\mathrm{c}1}(0) = x_{\mathrm{c}10}, \quad t \geq 0, \tag{116}$$

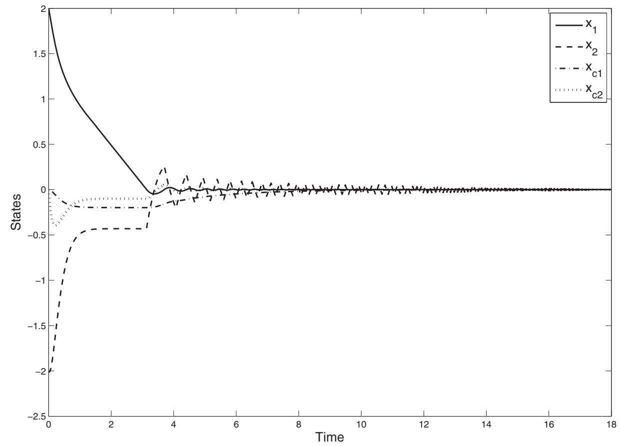

FIGURE 4. State trajectories of the closed-loop system versus time for the full-order controller.

$$\dot{x}_{c2}(t) = -10x_{c1}^3(t) - 10x_{c2}(t) + 5u_c(t), \quad x_{c2}(0) = x_{c20}, \tag{117}$$

$$y_c(t) = 10x_{c2}(t). \tag{118}$$

Furthermore, consider the feedback interconnection of (113)–(118) given by $u = -y_c$ and $u_c = y$. Next, let $V_s(x) = |x_1| + \frac{1}{2}x_2^2$ and note that, for almost all $x \in \mathbb{R}^2$,

$$\partial V_s(x_1, x_2) = \begin{cases} \{\mathrm{sign}(x_1)\} \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, \\ [-1, 1] \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0. \end{cases}$$

Hence, $\mathcal{L}_f V_s(x_1, x_2) = \{0\}$ and $\mathcal{L}_G V_s(x_1, x_2) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_s(x) = \max \mathcal{L}_f V_s(x) = 0$ for almost all $x \in \mathbb{R}^2$. Now, with $\varepsilon = 0$, $\ell(x) = 0$, and $\mathcal{W}(x) = 0$, (73)–(76) are satisfied. Hence, it follows from Corollary 6.1 that $\mathcal{G}$ is weakly passive.

Next, note that with $V_{sc}(x_c) = 10x_{c1}^4 + 2x_{c2}^2$, $\varepsilon \in (0, 2]$, $\ell(x_c) = \pm\sqrt{10x_{c1}^4(2-\varepsilon) + 2x_{c2}^2(20-\varepsilon)}$, and $\mathcal{W}(x_c) \equiv 0$, it follows from Corollary 6.1 that $\mathcal{G}_c$ is exponentially passive. Furthermore, $\mathrm{rank}[G_c(u_c, 0)] = 1$, $u_c \in \mathbb{R}$. Now, it follows from *ii*) of Theorem 7.2 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable. Figure 4 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $x_c(0) = 0$.

Alternatively, we consider the reduced-order dynamic controller $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = -10x_c(t) + 20u_c(t), \quad x_c(0) = x_{c0}, \quad t \geq 0, \tag{119}$$

$$y_c(t) = 12x_c(t). \tag{120}$$

Note that with $V_{sc}(x_c) = \frac{3}{5}x_c^2$, $\varepsilon = 20$, $\ell(x_c) \equiv 0$, and $\mathcal{W}(x_c) \equiv 0$, it follows from Corollary 6.1 that $\mathcal{G}_c$ is exponentially passive. Moreover, $\mathrm{rank}[G_c(u_c, 0)] = 1$, $u_c \in \mathbb{R}$. Hence, it follows from *ii*) of Theorem 7.2 that the negative feedback interconnection

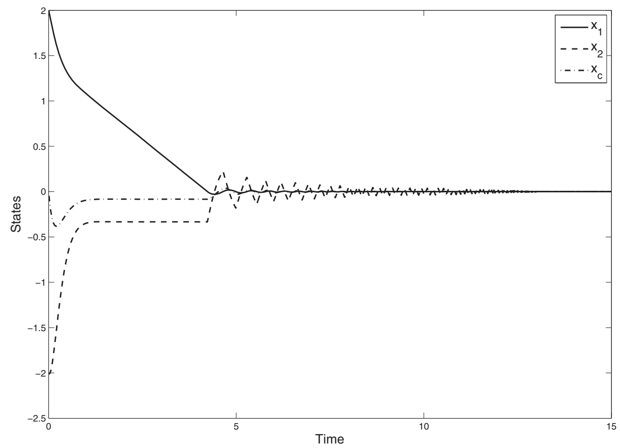

FIGURE 5. State trajectories of the closed-loop system versus time for the reduced-order controller.

of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable. Figure 5 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $x_c(0) = 0$. $\triangle$

8. FINITE-TIME STABILITY OF FEEDBACK INTERCONNECTIONS

In this section, we develop finite-time stability conditions for feedback interconnections of dissipative discontinuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ given by (103) and (104), and (105) and (106), respectively. Here, for simplicity of exposition, we assume that $J(x) \equiv 0$ and $J_c(u_c, x_c) \equiv 0$, and $\mathcal{G}_c$ is strictly strongly dissipative with respect to the supply rate $s_c(u_c, y_c)$, and if $x_c \equiv 0$, then $u_c \equiv 0$. The following definition is needed for the main result of this section.

Definition 8.1. Consider the closed-loop nonlinear dynamical system $\tilde{\mathcal{G}}$ consisting of the nonlinear dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ with closed-loop system state $\tilde{x} = [x^{\mathrm{T}}, x_c^{\mathrm{T}}]^{\mathrm{T}}$, where $x \in \mathbb{R}^n$ and $x_c \in \mathbb{R}^{n_c}$. The zero solution $\tilde{x}(\cdot) = 0$ of $\tilde{\mathcal{G}}$ is *partially finite-time stable with respect to* x_c if the zero solution $\tilde{x}(\cdot) = 0$ of $\tilde{\mathcal{G}}$ is asymptotically stable and there exists $T \in [0, \infty)$ such that $x_c(t) = 0$ for all $t \geq T$.

It follows from Definition 8.1 that if the zero solution $\tilde{x}(\cdot) = 0$ of $\tilde{\mathcal{G}}$ is partially finite-time stable, then the zero solution $\tilde{x}(\cdot) = 0$ of $\tilde{\mathcal{G}}$ is asymptotically stable. However, the converse is not necessarily true. The following result gives partial finite-time stability and finite-time stability results for feedback interconnected discontinuous dynamical systems.

Theorem 8.1. *Consider the closed-loop system consisting of the nonlinear dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ with input-output pairs (u, y) and (u_c, y_c), respectively, and with*

$u = -y_\mathrm{c}$ and $u_\mathrm{c} = y$. Assume that $\mathcal{G}$ is zero-state observable, completely reachable, and weakly dissipative with respect to the supply rate $s(u, y)$ and with a locally Lipschitz continuous and regular storage function $V_\mathrm{s}(\cdot)$. Furthermore, assume that $\mathcal{G}_\mathrm{c}$ is strictly strongly dissipative with respect to the set-valued supply rate $s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c})$ and with a locally Lipschitz continuous regular storage function $V_\mathrm{sc}(\cdot)$, and with $x_\mathrm{c} \equiv 0$ implying $u_\mathrm{c} \equiv 0$. If there exists a scalar $\kappa > 0$ such that $s(u, y) + \kappa s_\mathrm{c}(u_\mathrm{c}, y_\mathrm{c}) \leq 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_\mathrm{c} \in \mathbb{R}^l$, $y_\mathrm{c} \in \mathbb{R}^m$ such that $u_\mathrm{c} = y$ and $y_\mathrm{c} = -u$, then the zero solution of the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is partially finite-time stable with respect to x_c. If, alternatively, $\mathcal{G}$ is strictly strongly dissipative with respect to the supply rate $s(u, y)$, then the zero solution of the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is finite-time stable.

Proof. Consider the Lyapunov function candidate $\tilde{V}(\tilde{x}) \triangleq V_\mathrm{s}(x) + \kappa V_\mathrm{sc}(x_\mathrm{c})$, where $\tilde{x} \triangleq [x^\mathrm{T}, x_\mathrm{c}^\mathrm{T}]^\mathrm{T}$. Now, it follows that

$$\begin{aligned}
\tilde{V}(\tilde{x}(t)) - \tilde{V}(\tilde{x}(0)) &\leq \int_0^t [s(u(\sigma), y(\sigma)) + \kappa s_\mathrm{c}(u_\mathrm{c}(\sigma), y_\mathrm{c}(\sigma)) - \varepsilon_\mathrm{c}]\mathrm{d}\sigma \\
&\leq \int_0^t (-\varepsilon_\mathrm{c})\mathrm{d}\sigma \\
&\leq 0, \quad t \geq 0, \qquad (121)
\end{aligned}$$

which implies that the closed-loop system is Lyapunov stable.

Next, we show that $x_\mathrm{c}(t)$, $t \geq 0$, converges to zero in finite time. To see this, suppose, *ad absurdum*, that this is not the case. Then it follows from (121) that

$$\tilde{V}(\tilde{x}(t)) \leq \tilde{V}(\tilde{x}(0)) - \varepsilon_\mathrm{c} t, \quad \tilde{x}(s) \neq 0, \quad 0 \leq s \leq t. \qquad (122)$$

Letting $t \to \infty$ in (122) yields $\tilde{V}(\tilde{x}(t)) \to -\infty$, which contradicts that $\tilde{V}(\tilde{x}) \geq 0$, $\tilde{x} \in \mathbb{R}^{n+n_\mathrm{c}}$. Hence, there exists $T \geq 0$ such that $x_\mathrm{c}(t) = 0$ for all $t \geq T$.

Next, let $\mathcal{R} \triangleq \{\tilde{x} \in \mathbb{R}^{n+n_\mathrm{c}} : \max \mathcal{L}_{\tilde{f}} \tilde{V}(\tilde{x}) = 0\}$ and let $\mathcal{M}$ be the largest weakly invariant set contained in $\mathcal{M}$. Since $\max \mathcal{L}_{\tilde{f}} \tilde{V}(\tilde{x}) < 0$ for $x_\mathrm{c} \neq 0$, it follows that $\mathcal{R} \subseteq \{\tilde{x} \in \mathbb{R}^{n+n_\mathrm{c}} : x_\mathrm{c} = 0\}$. On $\mathcal{M}$, $x_\mathrm{c}(t) \equiv 0$ implies that $u_\mathrm{c}(t) = 0 = y(t)$ and $0 = h_\mathrm{c}(x_\mathrm{c}(t)) = y_\mathrm{c}(t) = -u(t)$. By complete reachability and zero-state observability, it follows that $x(t) = 0$ on $\mathcal{M}$. Hence, $\mathcal{M} = \{0\}$. Now, it follows from Theorem 3.2 that $\tilde{x}(t) \to \mathcal{M}$ as $t \to \infty$, and hence, $\tilde{x}(t) \to 0$ as $t \to \infty$, which implies that the closed-loop system is asymptotically stable. Thus, the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ is partially finite-time stable with respect to x_c. The proof of the second assertion is similar and, hence, is omitted. □

The following corollary is a direct consequence of Theorem 8.1. For this result we assume that all storage functions of $\mathcal{G}$ and $\mathcal{G}_\mathrm{c}$ are positive definite.

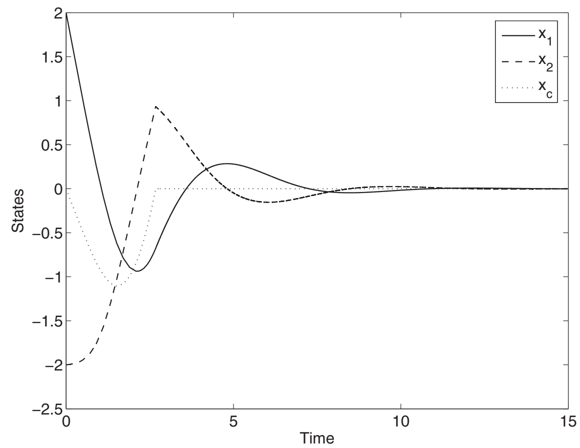

FIGURE 6. State trajectories versus time

Corollary 8.1. *Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ with input-output pairs (u, y) and $(u_{\rm c}, y_{\rm c})$, respectively, and with $u = -y_{\rm c}$ and $u_{\rm c} = y$. Furthermore, assume that $\mathcal{G}$ is zero-state observable and completely reachable. If $\mathcal{G}$ is weakly passive and $\mathcal{G}_{\rm c}$ is strictly strongly passive with $x_{\rm c} \equiv 0$ implying $u_{\rm c} \equiv 0$, then the zero solution of the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is partially finite-time stable with respect to $x_{\rm c}$. If, alternatively, $\mathcal{G}$ is strictly strongly passive, then the zero solution of the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is finite-time stable.*

Proof. The proof is a direct consequence of Theorem 8.1. □

Example 8.1. Consider the nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{123}$$

$$\dot{x}_2(t) = -\tanh x_1(t) + u(t), \quad x_2(0) = x_{20}, \tag{124}$$

$$y(t) = x_2(t), \tag{125}$$

and the discontinuous dynamic controller $\mathcal{G}_{\rm c}$ given by

$$\dot{x}_{\rm c}(t) = -\mathrm{sign}(x_{\rm c}(t)) + u_{\rm c}(t), \quad x_{\rm c}(0) = x_{\rm c0}, \quad \text{a.e.} \quad t \geq 0, \tag{126}$$

$$y_{\rm c}(t) = \mathrm{sign}(x_{\rm c}(t)). \tag{127}$$

Consider the feedback interconnection of (123)–(127) given by $u = -y_{\rm c}$ and $u_{\rm c} = y$. It is easy to verify that $\mathcal{G}$ is passive and $\mathcal{G}_{\rm c}$ is strictly strongly passive with storage functions $V_{\rm s}(x) = \int_0^{x_1} \tanh(\sigma)\mathrm{d}\sigma + \frac{1}{2}x_2^2$ and $V_{\rm sc}(x_{\rm c}) = |x_{\rm c}|$, respectively. Now, it follows from *i*) of Corollary 8.1 that the closed-loop system given by $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is partially finite-time stable with respect to $x_{\rm c}$. Figure 6 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\rm T}$ and $x_{\rm c}(0) = 0$. △

9. STABILITY MARGINS FOR DISCONTINUOUS FEEDBACK REGULATORS

To develop relative stability margins for discontinuous nonlinear regulators consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{128}$$

$$y(t) = -\phi(x(t)), \tag{129}$$

where $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded, and $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is a discontinuous feedback controller such that $\mathcal{G}$ is weakly (resp., strongly) asymptotically stable with $u = -y$. Furthermore, assume that the system $\mathcal{G}$ is weakly (resp., strongly) zero-state observable. Next, we define the relative stability margins for $\mathcal{G}$ given by (128) and (129). Specifically, let $u_\mathrm{c} \triangleq -y$, $y_\mathrm{c} \triangleq u$, and consider the negative feedback interconnection $u = \Delta(-y)$ of $\mathcal{G}$ and $\Delta(\cdot)$ given in Figure 7, where $\Delta(\cdot)$ is either a linear operator $\Delta(u_\mathrm{c}) = \Delta u_\mathrm{c}$, a nonlinear static operator $\Delta(u_\mathrm{c}) = \sigma(u_\mathrm{c})$, or a dynamic nonlinear operator $\Delta(\cdot)$ with input u_c and output y_c. Furthermore, we assume that in the nominal case $\Delta(\cdot) = I(\cdot)$ so that the nominal closed-loop system is weakly (resp., strongly) asymptotically stable.

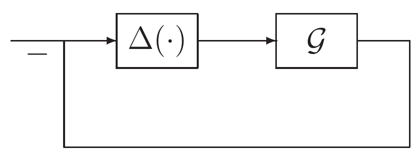

FIGURE 7. Multiplicative input uncertainty of $\mathcal{G}$ and input operator $\Delta(\cdot)$.

Definition 9.1. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (128) and (129) is said to have a *weak* (resp., *strong*) *gain margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_\mathrm{c}) = \Delta u_\mathrm{c}$ is globally weakly (resp., strongly) asymptotically stable for all $\Delta = \mathrm{diag}[k_1, \ldots, k_m]$, where $k_i \in (\alpha, \beta)$, $i = 1, \ldots, m$.

Definition 9.2. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (128) and (129) is said to have a *weak* (resp., *strong*) *sector margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(u_\mathrm{c}) = \sigma(u_\mathrm{c})$ is globally weakly (resp., strongly) asymptotically stable for all nonlinearities $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ such that $\sigma(0) = 0$, $\sigma(u_\mathrm{c}) = [\sigma_1(u_{\mathrm{c}1}), \ldots, \sigma_m(u_{\mathrm{c}m})]^\mathrm{T}$, and $\alpha u_{\mathrm{c}i}^2 < \sigma_i(u_{\mathrm{c}i})u_{\mathrm{c}i} < \beta u_{\mathrm{c}i}^2$, for all $u_{\mathrm{c}i} \neq 0$, $i = 1, \ldots, m$.

Definition 9.3. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \leq 1 \leq \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (128) and (129) is said to have a *weak* (resp., *strong*) *disk margin* (α, β) if the negative feedback interconnection of

$\mathcal{G}$ and $\Delta(\cdot)$ is globally weakly (resp., strongly) asymptotically stable for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is weakly (resp., strongly) zero-state observable and weakly (resp., strongly) dissipative with respect to the supply rate $s(u_{\rm c}, y_{\rm c}) = u_{\rm c}^{\rm T} y_{\rm c} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{\rm c}^{\rm T} y_{\rm c} - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{\rm c}^{\rm T} u_{\rm c}$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Definition 9.4. Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha \le 1 \le \beta < \infty$. Then the discontinuous nonlinear dynamical system $\mathcal{G}$ given by (128) and (129) is said to have a *weak* (resp., *strong*) *structured disk margin* (α, β) if the negative feedback interconnection of $\mathcal{G}$ and $\Delta(\cdot)$ is globally weakly (resp., strongly) asymptotically stable for all dynamic operators $\Delta(\cdot)$ such that $\Delta(\cdot)$ is weakly (resp., strongly) zero-state observable, $\Delta(u_{\rm c}) = \operatorname{diag}[\delta_1(u_{\rm c1}), \ldots, \delta_m(u_{{\rm c}m})]$, and $\delta_i(\cdot)$, $i = 1, \ldots, m$, is weakly (resp., strongly) dissipative with respect to the supply rate $s(u_{{\rm c}i}, y_{{\rm c}i}) = u_{{\rm c}i} y_{{\rm c}i} - \frac{1}{\hat{\alpha}+\hat{\beta}} y_{{\rm c}i}^2 - \frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}+\hat{\beta}} u_{{\rm c}i}^2$, where $\hat{\alpha} = \alpha + \delta$, $\hat{\beta} = \beta - \delta$, and $\delta \in \mathbb{R}$ such that $0 < 2\delta < \beta - \alpha$.

Remark 9.1. Note that if $\mathcal{G}$ has a weak (resp., strong) disk margin (α, β), then $\mathcal{G}$ has weak (resp., strong) gain and sector margins (α, β).

10. NONLINEAR-NONQUADRATIC OPTIMAL REGULATORS FOR DISCONTINUOUS DYNAMICAL SYSTEMS

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear-nonquadratic cost functional. To address the optimal control problem let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open set and let $U \subseteq \mathbb{R}^m$, where $0 \in \mathcal{D}$ and $0 \in U$. Next, consider the controlled nonlinear discontinuous dynamical system (22), where $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$ for almost all $t \ge 0$ and the constraint set U is given. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop dynamical system shown in the Figure 8 is given by (23).

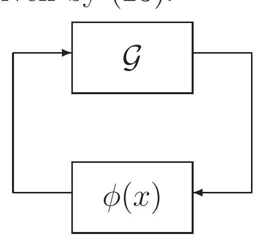

FIGURE 8. Nonlinear closed-loop feedback system.

Next, we present a main theorem for characterizing feedback controllers that guarantee stability of the controlled discontinuous dynamical system $\mathcal{G}$ and minimize a nonlinear-nonquadratic performance functional. For the statement of this result let $L : \mathcal{D} \times U \to \mathbb{R}$ be Lipschitz continuous and define the set of regulation controllers by

$$\mathcal{S}(x_0) \quad \triangleq \quad \{u(\cdot) \in U : u(\cdot) \text{ is measurable and locally essentially bounded,}$$

and $x(\cdot)$ driven by (1) satisfies $x(t) \to 0$ as $t \to \infty\}$.

Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_0)$, that is, inputs corresponding to null convergent solutions, can be interpreted as incorporating a system detectability condition through the cost.

Theorem 10.1. *Consider the controlled discontinuous nonlinear dynamical system* (22) *with performance functional*[5]

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t, \tag{130}$$

where (130) *is defined with respect to absolutely continuous state arcs* $x(\cdot)$ *and measurable control functions* $u : [0, \infty) \to U$. *Assume that there exists a locally Lipschitz continuous and regular function* $V : \mathcal{D} \to \mathbb{R}$ *and a control law* $\phi : \mathcal{D} \to U$ *such that*

$$V(0) = 0, \tag{131}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{132}$$

$$\phi(0) = 0, \tag{133}$$

$$\max \mathcal{L}_{F(\cdot,\phi(\cdot))}V(x) < 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \quad x \neq 0, \tag{134}$$

$$\mathcal{H}(x, \phi(x)) = 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \tag{135}$$

$$\mathcal{H}(x, u) \geq 0, \quad \text{a.e.} \quad x \in \mathcal{D}, \quad u \in U, \tag{136}$$

where

$$\mathcal{H}(x, u) \triangleq L(x, u) + \min \mathcal{L}_{F(\cdot,u)}V(x). \tag{137}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, *the zero Filippov solution* $x(t) \equiv 0$ *of the closed-loop system* (23) *is locally strongly asymptotically stable and there exists a neighborhood of the origin* $\mathcal{D}_0 \subseteq \mathcal{D}$ *such that*

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathcal{D}_0. \tag{138}$$

In addition, if $x_0 \in \mathcal{D}_0$, *then the feedback control* $u(\cdot) = \phi(x(\cdot))$ *minimizes* $J(x_0, u(\cdot))$ *in the sense that*

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{139}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, *and*

$$V(x) \to \infty \text{ as } ||x|| \to \infty, \tag{140}$$

then the zero Filippov solution $x(t) \equiv 0$ *of the closed-loop system* (23) *is globally strongly asymptotically stable.*

[5]Since solutions to (22) are not necessarily unique, $J(x_0, u(\cdot))$ given by (130) depends on the particular state trajectory $x(\cdot)$ along which we integrate. Alternatively, if we assume that $f(\cdot, u)$ is essentially one-sided Lipschitz on $\mathcal{B}_\delta(x)$ for some $\delta > 0$, then there exists a unique Filippov solution to (22) with initial condition $x(t_0) = x_0$ and $u(t) \in U$ [33].

Proof. Local and global strong asymptotic stability follow from (131)–(134) by applying Theorem 3.1 to the closed-loop system (23). Next, with $u(t) \equiv \bar{u}(t)$, where $\bar{u}(\cdot)$ is measurable and locally essentially bounded, let $\bar{\psi}(t)$, $t \geq 0$, be any Filippov solution of (22). Then, it follows that $\mathcal{L}_{F(\cdot,\bar{u}(\cdot))}V(\bar{\psi}(t)) \subseteq \mathcal{L}_{F(\cdot,u)}V(\bar{\psi}(t))$ for almost every $t \geq 0$. Moreover, it follows from Lemma 6.1 that $\frac{\mathrm{d}}{\mathrm{d}t}V(\bar{\psi}(t)) \in \mathcal{L}_{F(\cdot,\bar{u}(\cdot))}V(\bar{\psi}(t))$ for almost every $t \geq 0$. Now, since $\bar{u}(t)$ and $\bar{\psi}(t)$ are arbitrary, it follows that

$$\begin{aligned}\min \mathcal{L}_{F(\cdot,u)}V(x(\sigma)) &\leq \frac{\mathrm{d}}{\mathrm{d}\sigma}V(x(\sigma)) \\ &\leq \max \mathcal{L}_{F(\cdot,u)}V(x(\sigma)), \quad \text{a.e.} \quad \sigma \in [0,t], \;\; u \in U. \end{aligned} \tag{141}$$

Next, let $x_0 \in \mathcal{D}_0$, let $u(\cdot) \in \mathcal{S}(x_0)$, and let $x(t)$ for almost all $t \geq 0$ be the Filippov solution of (1). Then, it follows from (141) that

$$\begin{aligned} L(x(t),u(t)) &\geq -\dot{V}(x(t)) + L(x(t),u(t)) + \min \mathcal{L}_{F(\cdot,u)}V(x(t)) \\ &= -\dot{V}(x(t)) + \mathcal{H}(x(t),u(t)), \quad \text{a.e.} \quad t \geq 0. \end{aligned} \tag{142}$$

Furthermore, note that

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}V(x(\sigma))\mathrm{d}\sigma, \tag{143}$$

where the integral in (143) is the Lebesgue integral. Now, using (136), (142), (143), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, it follows that

$$\begin{aligned} J(x_0,u(\cdot)) &\geq \int_0^\infty [-\dot{V}(x(t)) + \mathcal{H}(x(t),u(t))]\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x(t)) + V(x_0) + \int_0^\infty \mathcal{H}(x(t),u(t))\mathrm{d}t \\ &= V(x_0) + \int_0^\infty \mathcal{H}(x(t),u(t))\mathrm{d}t \\ &\geq V(x_0) \\ &= J(x_0,\phi(x(\cdot)), \end{aligned}$$

which yields (139). □

Note that (135) is the steady-state Hamilton-Jacobi-Bellman equation for the discontinuous dynamical system (22) with the cost $J(x_0,u(\cdot))$. Since we are not imposing that solutions to (135) be smooth, the Hamilton-Jacobi-Bellman equation (135) should be interpreted in the viscosity sense (i.e., a viscosity supersolution) [47, 48] or, equivalently, as in the proximal analysis formalism of [49]. Specifically, since $\underline{\partial}V(x) \subseteq \partial V(x)$, where

$$\underline{\partial}V(x) \triangleq \left\{ p \in \mathbb{R}^n : \liminf_{\|h\|\to 0} \frac{V(x+h) - V(x) - p^{\mathrm{T}}h}{\|h\|} \geq 0 \right\},$$

denotes the subdifferential of $V(\cdot)$ at x [24,49], it follows from (136) that $V(x)$ is a viscosity supersolution of (135). However, in general, $V(x)$ is not a viscosity subsolution of (135), which shows that the equivalence between optimal regulation, solvability of the Hamilton-Jacobi-Bellman equation, and feedback stabilizability breaks down for nonsmooth value functions $V(\cdot)$. It is important to note that Theorem 10.1 provides constructive *sufficient* conditions for optimality of a feedback controller. Furthermore, this controller is stabilizing and its optimality is independent of the system initial condition x_0. Finally, *necessary* conditions for optimality of nonsmooth regulation and existence of viscosity solutions of the resulting Hamilton-Jacobi-Bellman equation are discussed in [50, 51].

Next, we specialize Theorem 10.1 to discontinuous affine dynamical systems. Specifically, we construct discontinuous nonlinear feedback controllers using an optimal control framework that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the total generalized derivative of the Lyapunov function is negative along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing viscosity supersolutions to the Hamilton-Jacobi-Bellman equation. Thus, these results provide a family of globally stabilizing controllers parameterized by the cost functional that is minimized.

The controllers obtained in this section are predicated on an *inverse optimal control problem* [16,52]. In particular, to avoid the complexity in solving the steady-state Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law.

Consider the discontinuous nonlinear affine dynamical system given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{144}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. We assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u, \tag{145}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}, L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{P}^m$ with $\mathbb{P}^m$ denoting the set of $m \times m$ positive definite matrices, so that (130) becomes

$$J(x_0, u(\cdot)) \;=\; \int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t. \tag{146}$$

Theorem 10.2. *Consider the discontinuous nonlinear controlled affine dynamical system* (144) *with performance functional* (146). *Assume that there exists a locally Lipschitz continuous and regular function* $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$V(0) \;=\; 0, \tag{147}$$

$$L_2(0) \;=\; 0, \tag{148}$$

$$V(x) \quad > \quad 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{149}$$

$$\max \mathcal{L}_{[f(x)-\frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x)-\frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)]} V(x) < 0, \qquad \text{a.e.} \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{150}$$

and

$$V(x) \to \infty \text{ as } ||x|| \to \infty. \tag{151}$$

Then the zero Filippov solution $x(t) \equiv 0$ of the closed-loop discontinuous dynamical system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{152}$$

is globally strongly asymptotically stable with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \tag{153}$$

and the performance functional (146)*, with*

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - \min \mathcal{L}_f V(x), \tag{154}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \tag{155}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{156}$$

Proof. The result is a direct consequence of Theorem 10.1 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u$, and $f(x,u) = f(x) + G(x)u$. Specifically, with (145) the Hamiltonian has the form

$$\mathcal{H}(x,u) \quad = \quad L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u + \min \mathcal{L}_f V(x) + \mathcal{L}_G V(x)u.$$

Now, the feedback control law (153) is obtained by setting $\frac{\partial \mathcal{H}}{\partial u} = 0$. With (153), it follows that (147), (149), (150), and (151) imply (131), (132), (134), and (140), respectively. Next, since $V(\cdot)$ is locally Lipschitz continuous and regular, and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $\mathcal{L}_G V(0) = 0$, and hence, since by assumption $L_2(0) = 0$, it follows that $\phi(0) = 0$, which implies (133). Next, with $L_1(x)$ given by (154) and $\phi(x)$ given by (153), (135) holds. Finally, since $\mathcal{H}(x,u) = \mathcal{H}(x,u) - \mathcal{H}(x,\phi(x)) = [u - \phi(x)]^{\mathrm{T}}R_2(x)[u - \phi(x)]$ and $R_2(x)$ is positive definite for almost all $x \in \mathbb{R}^n$, condition (136) holds. The result now follows as a direct consequence of Theorem 10.1. $\square$

Example 10.1. To illustrate the utility of Theorem 10.2 we consider a controlled nonsmooth harmonic oscillator with nonsmooth friction given by [25]

$$\dot{x}_1(t) = -\operatorname{sign}(x_2(t)) - \frac{1}{2}\operatorname{sign}(x_1(t)), \quad x_1(0) = x_{10}, \quad \text{a.e.} \quad t \geq 0, \tag{157}$$

$$\dot{x}_2(t) = \operatorname{sign}(x_1(t)) + u(t), \quad x_2(0) = x_{20}, \tag{158}$$

where $\operatorname{sign}(\sigma) \triangleq \frac{\sigma}{|\sigma|}$, $\sigma \neq 0$, and $\operatorname{sign}(0) \triangleq 0$. To construct an inverse optimal globally stabilizing control law for (157) and (88) let $V(x) = |x_1| + |x_2|$ and note that

$$\partial V(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \overline{\operatorname{co}}\{(1,1), (-1,1), (-1,-1), (1,-1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V(x_1, x_2) = \begin{cases} \{-\frac{1}{2}\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

$$\mathcal{L}_G V(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

which implies that $\max \mathcal{L}_f V(x) = 0$, $\min \mathcal{L}_f V(x) = -\frac{1}{2}$, and $\mathcal{L}_G V(x) = \{\operatorname{sign}(x_2)\}$ for almost all $x \in \mathbb{R}^2$.

Next, it follows that

$$\mathcal{L}_{\tilde{f}} V(x_1, x_2) = \begin{cases} \{-1 - \frac{1}{2}L_2(x_1, x_2)\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{-1 - \frac{1}{2}L_2(x_1, x_2)\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

where $\tilde{f} \triangleq f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)$ with $R_2(x) \equiv 1$. Let $L(x, u) = L_1(x) + L_2(x)u + u^2$. Now, $L_2(x) = x_2$ satisfies (150) so that the inverse optimal control law (153) is given by

$$\phi(x) = -\frac{1}{2}[\operatorname{sign}(x_2) + x_2], \quad \text{a.e.} \quad x \in \mathbb{R}^2. \tag{159}$$

In this case, the performance functional (146), with

$$L_1(x) = \frac{1}{4}[\operatorname{sign}(x_2) + x_2]^2 + \frac{1}{2}, \quad \text{a.e.} \quad x \in \mathbb{R}^2, \tag{160}$$

is minimized in the sense of (155). Furthermore, using the feedback control law (159) it follows that

$$\mathcal{L}_{\tilde{f}}V(x_1,x_2) = \begin{cases} \{-1-\frac{1}{2}|x_2|\}, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2\neq 0, \\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2 : x_1\neq 0, x_2 = 0, \\ \{-1-\frac{1}{2}|x_2|\}, & (x_1,x_2)\in\mathbb{R}^2 : x_2\neq 0, x_1 = 0, \\ \{0\}, & (x_1,x_2) = (0,0). \end{cases}$$

Note that $\max \mathcal{L}_{\tilde{f}}V(x) \leq 0$. Now, let $\mathcal{R} \triangleq \{x\in\mathbb{R}^2 : \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0 \in \mathcal{L}_{\tilde{f}}V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$ if and only if $x = 0$. Hence, since $\mathcal{M} = \{(0,0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$, it follows from Theorem 3.2 that $\mathrm{dist}(\psi(t),\mathcal{M}) \to 0$ as $t\to\infty$ for all Filippov solutions $\psi(\cdot)$ of (157) and (88). Now, since $V(x)$ is radially unbounded, the feedback control law (159) is globally strongly stabilizing. △

11. GAIN, SECTOR, AND DISK MARGINS OF NONLINEAR-NONQUADRATIC OPTIMAL REGULATORS FOR DISCONTINUOUS DYNAMICAL SYSTEMS

In this section, we derive guaranteed gain, sector, and disk margins for nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion for discontinuous dynamical systems. Specifically, sufficient conditions that guarantee gain, sector, and disk margins are given in terms of the state, control, and cross-weighting nonlinear-nonquadratic weighting functions. In particular, we consider the discontinuous nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad \text{a.e.} \quad t\geq 0, \tag{161}$$

$$y(t) = -\phi(x(t)), \tag{162}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$, $G:\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $\mathcal{D}=\mathbb{R}^n$, $U=\mathbb{R}^m$, and $\phi:\mathbb{R}^n\to\mathbb{R}^m$, with a nonquadratic performance criterion

$$J(x_0,u(\cdot)) = \int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t, \tag{163}$$

where $L_1:\mathbb{R}^n\to\mathbb{R}$, $L_2:\mathbb{R}^n\to\mathbb{R}^{1\times m}$, and $R_2:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ are given such that $R_2(x)>0$, $x\in\mathbb{R}^n$, and $L_2(0)=0$. Once again, we assume that $f(\cdot)$ and $G(\cdot)$ are Lebesgue measurable and locally essentially bounded. In this case, the optimal nonlinear feedback controller $u=\phi(x)$ that minimizes the nonlinear-nonquadratic performance criterion (163) is given by the following result.

Theorem 11.1. *Consider the discontinuous nonlinear dynamical system given by* (161) *and* (162) *with performance functional* (163)*. Assume that there exists a locally Lipschitz continuous and regular function* $V:\mathbb{R}^n\to\mathbb{R}$ *such that*

$$V(0) = 0, \tag{164}$$

$$\begin{aligned} V(x) &> 0, \quad x \in \mathbb{R}^n, \ x \neq 0, && (165) \\ L_2(0) &= 0, && (166) \end{aligned}$$

$$\max \mathcal{L}_{[f(x)-\frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x)-\frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)]} V(x) < 0, \quad \text{a.e. } x \in \mathbb{R}^n, \ x \neq 0, \tag{167}$$

$$L_1(x)+\min \mathcal{L}_f V(x)-\tfrac{1}{4}[\mathcal{L}_G V(x)+L_2(x)]R_2^{-1}(x)[\mathcal{L}_G V(x)+L_2(x)]^{\mathrm{T}} = 0, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{168}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{169}$$

Then the zero Filippov solution $x(t) \equiv 0$ *of the closed-loop discontinuous dynamical system*

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0, \tag{170}$$

is globally strongly asymptotically stable with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \tag{171}$$

and the performance functional (163) *is minimized in the sense that*

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}^n. \tag{172}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{173}$$

Proof. The proof is a direct consequence of Theorem 10.1. □

The following key lemma is needed.

Lemma 11.1. *Consider the discontinuous nonlinear dynamical system* $\mathcal{G}$ *given by* (161) *and* (162) *where* $\phi(x)$ *is a strongly stabilizing feedback control law given by* (171)*. Suppose* $V(x)$, $x \in \mathbb{R}^n$, *satisfies*

$$0 = \min \mathcal{L}_f V(x) + L_1(x) - \tfrac{1}{4}[\mathcal{L}_G V(x) + L_2(x)]R_2^{-1}(x)[\mathcal{L}_G V(x) + L_2(x)]^{\mathrm{T}}, \tag{174}$$

$$[\max \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x)] \leq L_1(x) - \tfrac{1}{4(1-\theta^2)} L_2(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x), \quad \text{a.e.} \quad x \in \mathbb{R}^n, \tag{175}$$

with $\theta \in \mathbb{R}$ *such that* $0 < \theta < 1$*. Then, for almost all* $u(t) \in U$ *and* $t_1, t_2 \geq 0$, $t_1 < t_2$, *the solution* $x(t)$, $t \geq 0$, *to (161) satisfies*

$$V(x(t_2)) \leq \int_{t_1}^{t_2} \big\{[u(t) + y(t)]^{\mathrm{T}} R_2(x(t))[u(t) + y(t)] - \theta^2 u^{\mathrm{T}}(t)R_2(x(t))u(t)\big\}\mathrm{d}t + V(x(t_1)). \tag{176}$$

Proof. Note that it follows from (171), (174), and (175) that for almost all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\begin{aligned}
\theta^2 u^{\mathrm{T}} R_2(x) u \;&\le\; \theta^2 u^{\mathrm{T}} R_2(x) u + \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right] \\
&\qquad \times R_2(x) \left[\frac{1}{2\sqrt{1-\theta^2}} L_2(x) R_2^{-1}(x) + \sqrt{1-\theta^2} u^{\mathrm{T}}\right]^{\mathrm{T}} \\
&=\; u^{\mathrm{T}} R_2(x) u + \frac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) + L_2(x) u \\
&\le\; u^{\mathrm{T}} R_2(x) u + L_2(x) u + L_1(x) - [\max \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x)] \\
&=\; u^{\mathrm{T}} R_2(x) u + [L_2(x) + \mathcal{L}_G V(x)] u + \min \mathcal{L}_f V(x) - \min \mathcal{L}_f V(x) \\
&\qquad + \phi^{\mathrm{T}}(x) R_2(x) \phi(x) - \max \mathcal{L}_f V(x) - \mathcal{L}_G V(x) u \\
&=\; [u+y]^{\mathrm{T}} R_2(x) [u+y] - \max \mathcal{L}_f V(x) - \mathcal{L}_G V(x) u. \qquad (177)
\end{aligned}$$

Next, using the sum rule for the generalized gradient of locally Lipschitz continuous functions [40] it follows that $\mathcal{L}_{f+Gu} V(x) \subseteq \mathcal{L}_f V(x) + \mathcal{L}_{Gu} V(x)$ for almost all $x \in \mathbb{R}^n$. Now, it follows from Lemma 6.1 that $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \in \mathcal{L}_{f+Gu} V(x(t)) \subseteq \mathcal{L}_f V(x(t)) + \mathcal{L}_{Gu} V(x(t))$ for almost all $t \ge 0$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &\le \max \mathcal{L}_{f+Gu} V(x(t)) \\
&\le \max\, [\mathcal{L}_f V(x(t)) + \mathcal{L}_G V(x(t)) u(t)] \\
&= \max \mathcal{L}_f V(x(t)) + \mathcal{L}_G V(x(t)) u(t), \quad \text{a.e. } t \ge 0,\ u(t) \in U. \qquad (178)
\end{aligned}$$

It follows from (177) and (178) that, for all $u(t) \in U$ and almost all $t \ge 0$,

$$\theta^2 u^{\mathrm{T}}(t) R_2(x(t)) u(t) \;\le\; [u(t)+y(t)]^{\mathrm{T}} R_2(x(t)) [u(t)+y(t)] - \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)).$$

Now, integrating over $[t_1, t_2]$ and using (143) yields (176). □

Note that with $R_2(x) \equiv I_m$ condition (176) is the counterpart, for discontinuous dynamical systems, of the return difference condition for continuous-time and discrete-time systems [31, 32, 53]. Next, using the extended nonlinear Kalman-Yakubovich-Popov conditions for discontinuous dynamical systems given by Theorem 6.1, we show that for a given nonlinear dynamical system $\mathcal{G}$ given by (161) and (162), there exists an equivalence between optimality and dissipativity. For the following result we assume that for the given discontinuous nonlinear system (161), if there exists a feedback control law $\phi(x)$ that minimizes the performance functional (163) with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \ge 0$, $x \in \mathbb{R}^n$, then there exists a locally Lipschitz continuous, regular, and positive-definite function $V(x)$, $x \in \mathbb{R}^n$, such that (174) and (175) are satisfied. Necessary and sufficient conditions such that the aforementioned statement holds, modulo (175) holding, are given in Theorem 3.7.6 of [24].

Theorem 11.2. *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (161) *and* (162). *The feedback control law $u = \phi(x)$ is optimal with respect to a performance functional* (163) *with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, if and only if the nonlinear system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a locally Lipschitz continuous, regular, positive-definite, and radially unbounded storage function $V(x)$, $x \in \mathbb{R}^n$.*

Proof. If the control law $\phi(x)$ is optimal with respect to a performance functional (163) with $R_2(x) \equiv I_m$, $L_2(x) \equiv 0$, and $L_1(x) \geq 0$, $x \in \mathbb{R}^n$, then, by assumption, there exists a locally Lipschitz continuous, regular, and positive-definite function $V(x)$ such that (174) and (175) are satisfied. Hence, it follows from Lemma 11.1 that the solution $x(t)$, $t \geq 0$, to (161) satisfies

$$V(x(t_2)) \leq \int_{t_1}^{t_2} \left\{[u(t)+y(t)]^{\mathrm{T}}[u(t)+y(t)] - u^{\mathrm{T}}(t)u(t)\right\}\mathrm{d}t + V(x(t_1)), \quad 0 \leq t_1 \leq t_2,$$

which implies that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$.

Conversely, if $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = y^{\mathrm{T}}y + 2u^{\mathrm{T}}y$ and has a locally Lipschitz continuous, regular, and positive-definite storage function, then, with $h(x) = -\phi(x)$, $J(x) \equiv 0$, $Q = I_m$, $R = 0$, and $S = I_m$, it follows from Theorem 6.1 that there exists a function $\ell : \mathbb{R}^n \to \mathbb{R}^p$ such that $\phi(x) = -\frac{1}{2}\mathcal{L}_G V^{\mathrm{T}}(x)$ and, for almost all $x \in \mathbb{R}^n$,

$$0 = \min \mathcal{L}_f V(x) - \tfrac{1}{4}\mathcal{L}_G V(x)\mathcal{L}_G V^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\ell(x).$$

Now, the result follows from Theorem 11.1 with $L_1(x) = \ell^{\mathrm{T}}(x)\ell(x)$. □

Example 11.1. Consider the controlled discontinuous dynamical system $\mathcal{G}$ representing a mass sliding on a horizontal surface subject to a Coulomb frictional force given in Example 6.1. Let $V(x) = x^2$ and note that $\mathcal{L}_f V(x) = \{-|x|\}$ and $\mathcal{L}_G V(x) = \{2x\}$ for almost all $x \in \mathbb{R}$. Next, it follows that

$$\mathcal{L}_{\tilde{f}} V(x) = -|x| - L_2(x)x - 2x^2,$$

where $\tilde{f} \triangleq f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)\mathcal{L}_G V^{\mathrm{T}}(x)$ with $R_2(x) \equiv 1$. Let $L(x,u) = L_1(x) + L_2(x)u + u^2$. Now, $L_2(x) = 2x$ satisfies $\max \mathcal{L}_{\tilde{f}} V(x) < 0$ for almost all $x \in \mathbb{R}$, $x \neq 0$, so that the inverse optimal control law is given by

$$\phi(x) = -\frac{1}{2}[2x + 2x] = -2x, \quad \text{a.e.} \quad x \in \mathbb{R}. \tag{179}$$

In this case, the performance functional $J(x_0, u(\cdot)) = \int_0^\infty L(x,u)\mathrm{d}t$, with

$$L_1(x) = 4x^2 + |x|, \quad \text{a.e.} \quad x \in \mathbb{R}, \tag{180}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad x_0 \in \mathbb{R}.$$

Furthermore, using the feedback control law (179) it follows that

$$\mathcal{L}_{\tilde{f}}V(x) = -|x| - 4x^2, \quad \text{a.e} \quad x \in \mathbb{R}.$$

Note that $\max \mathcal{L}_{\tilde{f}}V(x) \leq 0$. Now, let $\mathcal{R} \triangleq \{x \in \mathbb{R} : \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0 \in \mathcal{L}_{\tilde{f}}V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$ if and only if $x = 0$. Hence, since $\mathcal{M} = \{0\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$, it follows from Theorem 3.2 that $\mathrm{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (48). Now, since $V(x)$ is radially unbounded, the feedback control law (179) is globally strongly stabilizing.

Next, note that with $L_2(x) \equiv 0$ it follows from the above analysis that the optimal control law $\phi(x) = -x$ minimizes the cost functional

$$J(x_0, u(\cdot)) = \int_0^\infty [x^2(t) + |x(t)| + u^2(t)]\mathrm{d}t. \tag{181}$$

Now, it follows from Theorem 6.2 that the discontinuous nonlinear dynamical system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$, where $y = -\phi(x) = x$. To show this, consider the storage function $V_\mathrm{s}(x) = V(x) = x^2$. Next, with $J(x) \equiv 0$, $Q = 1$, $R = 0$, $S = 1$, and $\varepsilon = 0$, the extended Kalman-Yakubovich-Popov conditions given in Theorem 6.1 become

$$\begin{aligned} 0 &= \min \mathcal{L}_f V_\mathrm{s}(x) - h^2(x) + \ell^\mathrm{T}(x)\ell(x), && (182)\\ 0 &= \frac{1}{2}\mathcal{L}_G V_\mathrm{s}(x) - h(x) + \ell^\mathrm{T}(x)\mathcal{W}(x), && (183)\\ 0 &= -\mathcal{W}^\mathrm{T}(x)\mathcal{W}(x), && (184)\\ \ell^\mathrm{T}(x)\ell(x) &\geq [\max \mathcal{L}_f V_\mathrm{s}(x) - \min \mathcal{L}_f V_\mathrm{s}(x)]\,. && (185) \end{aligned}$$

Now, with $h(x) = -\phi(x) = x$, $\mathcal{W}(x) = 0$, and $L_1(x) = \ell^\mathrm{T}(x)\ell(x)$, conditions (182)–(184) are satisfied. Furthermore, (185) is equivalent to (175) which is satisfied since $\phi(x) = -x$ is optimal. Hence, it follows from Theorem 6.1 that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$. △

Example 11.2. Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given in Example 10.1. Note that with $R_2(x) \equiv 1$ and $L_2(x) \equiv 0$ it follows from the analysis given in Example 10.1 that the optimal control law $\phi(x) = -\frac{1}{2}\,\mathrm{sign}(x_2)$ minimizes the cost functional

$$J(x_0, u(\cdot)) = \int_0^\infty [\frac{1}{2} + \frac{1}{4}\,\mathrm{sign}^2(x_2(t)) + u^2(t)]\mathrm{d}t.$$

Now, it follows from Theorem 11.2 that the discontinuous nonlinear dynamical system $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u, y) = y^2 + 2uy$, where $y = -\phi(x) = \frac{1}{2}\,\mathrm{sign}(x_2)$. To show this, consider the storage function $V_\mathrm{s}(x) = V(x) =$

$|x_1|+|x_2|$. Next, with $J(x)\equiv 0$, $Q=1$, $R=0$, and $S=1$, the extended Kalman-Yakubovich-Popov conditions given in Theorem 6.1 become

$$\begin{aligned} 0 &= \min \mathcal{L}_f V_{\rm s}(x) - h^{\rm T}(x)h(x) + \ell^{\rm T}(x)\ell(x), &(186)\\ 0 &= \frac{1}{2}\mathcal{L}_G V_{\rm s}(x) - h^{\rm T}(x) + \ell^{\rm T}(x)\mathcal{W}(x), &(187)\\ 0 &= -\mathcal{W}^{\rm T}(x)\mathcal{W}(x), &(188)\\ \ell^{\rm T}(x)\ell(x) &\geq \left[\max \mathcal{L}_f V_{\rm s}(x) - \min \mathcal{L}_f V_{\rm s}(x)\right], \quad \text{a.e.} \quad x\in\mathbb{R}^2. &(189)\end{aligned}$$

Next, it was shown in Example 10.1 that $\max \mathcal{L}_f V_{\rm s}(x)=0$, $\min \mathcal{L}_f V_{\rm s}(x) = -\frac{1}{2}$, and $\mathcal{L}_G V_{\rm s}(x) = \{\operatorname{sign}(x_2)\}$. Now, with $h(x) = -\phi(x) = \frac{1}{2}\operatorname{sign}(x_2)$, $\mathcal{W}(x)=0$, and $L_1(x)=\ell^{\rm T}(x)\ell(x)$, conditions (186)–(188) are satisfied. Furthermore, (189) is equivalent to (175) which is satisfied since $\phi(x) = -\frac{1}{2}\operatorname{sign}(x_2)$ is optimal. Hence, it follows from Theorem 6.1 that $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y)=y^2+2uy$. △

Next, we present disk margins for the nonlinear-nonquadratic optimal regulator given by Theorem 11.1. First, we consider the case in which $R_2(x)$, $x\in\mathbb{R}^n$, is a constant diagonal matrix.

Theorem 11.3. *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (161) *and* (162) *where $\phi(x)$ is a strongly stabilizing feedback control law given by* (171) *and where $V(x)$, $x\in\mathbb{R}^n$, satisfies* (174) *and* (175) *with $\theta\in\mathbb{R}$ such that $0<\theta<1$. If $R_2(x)\equiv \operatorname{diag}[r_1,\ldots,r_m]$, where $r_i>0$, $i=1,\ldots,m$, then the discontinuous nonlinear system $\mathcal{G}$ has a strong structured disk margin $(\frac{1}{1+\theta},\frac{1}{1-\theta})$. If, in addition, $R_2(x)\equiv I_m$, then the discontinuous nonlinear system $\mathcal{G}$ has a strong disk margin $(\frac{1}{1+\theta},\frac{1}{1-\theta})$.*

Proof. Note that for all $u(t)\in U$ and almost all $t_1,t_2\geq 0$, $t_1<t_2$, it follows from Lemma 11.1 that the solution $x(t)$, $t\geq 0$, to (161) satisfies

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} \left\{[u(t)+y(t)]^{\rm T}R_2[u(t)+y(t)] - \theta^2 u^{\rm T}(t)R_2 u(t)\right\} {\rm d}t.$$

Hence, with the storage function $V_{\rm s}(x)=\frac{1}{2}V(x)$, $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = u^{\rm T}R_2 y + \frac{1-\theta^2}{2}u^{\rm T}R_2 u + \frac{1}{2}y^{\rm T}R_2 y$. Now, the result is a direct consequence of Corollary 7.1 and Definitions 9.4 and 9.3 with $\alpha=\frac{1}{1+\theta}$ and $\beta=\frac{1}{1-\theta}$. □

Next, we consider the case in which $R_2(x)$, $x\in\mathbb{R}^n$, is not a diagonal constant matrix. For the following result define

$$\bar{\gamma} \triangleq \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \sigma_{\max}(R_2(x)), \quad \underline{\gamma} \triangleq \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} \sigma_{\min}(R_2(x)), \tag{190}$$

where $R_2(x)$ is such that $\bar{\gamma}<\infty$ and $\underline{\gamma}>0$.

Theorem 11.4. *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (161) *and* (162) *where $\phi(x)$ is a strongly stabilizing feedback control law given by* (171) *and suppose $V(x)$, $x \in \mathbb{R}^n$, satisfies* (174) *and* (175) *with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. Then the discontinuous nonlinear system $\mathcal{G}$ has a strong disk margin $(\frac{1}{1+\eta\theta}, \frac{1}{1-\eta\theta})$, where $\eta \triangleq \sqrt{\underline{\gamma}/\bar{\gamma}}$.*

Proof. Note that for almost all $u(t) \in U$ and $t_1, t_2 \geq 0$, $t_1 < t_2$, it follows from Lemma 11.1 that the solution $x(t)$, $t \geq 0$, to (161) satisfies

$$\begin{aligned} V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} &\big\{[u(t)+y(t)]^{\mathrm{T}} R_2(x(t))[u(t)+y(t)] \\ &- \theta^2 u^{\mathrm{T}}(t) R_2(x(t)) u(t)\big\} \mathrm{d}t, \end{aligned}$$

which implies that

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} \big\{\bar{\gamma}[u(t)+y(t)]^{\mathrm{T}}[u(t)+y(t)] - \underline{\gamma}\theta^2 u^{\mathrm{T}}(t)u(t)\big\} \mathrm{d}t.$$

Hence, with the storage function $V_{\mathrm{s}}(x) = \frac{1}{2\underline{\gamma}}V(x)$, $\mathcal{G}$ is strongly dissipative with respect to the supply rate $s(u,y) = u^{\mathrm{T}}y + \frac{1-\eta^2\theta^2}{2}u^{\mathrm{T}}u + \frac{1}{2}y^{\mathrm{T}}y$. Now, the result is a direct consequence of Corollary 7.1 and Definition 9.3 with $\alpha = \frac{1}{1+\eta\theta}$ and $\beta = \frac{1}{1-\eta\theta}$. $\square$

Next, using Theorem 3.2 we provide an alternative result that guarantees sector and gain margins for the case in which $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal.

Theorem 11.5. *Consider the discontinuous nonlinear dynamical system $\mathcal{G}$ given by* (161) *and* (162) *where $\phi(x)$ is a strongly stabilizing feedback control law given by* (171) *and suppose $V(x)$, $x \in \mathbb{R}^n$, satisfies* (174) *and* (175) *with $\theta \in \mathbb{R}$ such that $0 < \theta < 1$. Furthermore, let $R_2(x) = \mathrm{diag}\,[r_1(x), \ldots, r_m(x)]$, where $r_i : \mathbb{R}^n \to \mathbb{R}$, $r_i(x) > 0$, $i = 1, \ldots, m$. If $\mathcal{G}$ is strongly zero-state observable, then the discontinuous nonlinear system $\mathcal{G}$ has a strong sector (and, hence, gain) margin $(\frac{1}{1+\theta}, \frac{1}{1-\theta})$.*

Proof. Let $\Delta(-y) = \sigma(-y)$, where $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is a static nonlinearity such that $\sigma(0) = 0$, $\sigma(v) = [\sigma_1(v_1), \ldots, \sigma_m(v_m)]^{\mathrm{T}}$, and $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, for all $v_i \neq 0$, $i = 1, \ldots, m$, where $\alpha = \frac{1}{1+\theta}$ and $\beta = \frac{1}{1-\theta}$; or, equivalently, $(\sigma_i(v_i) - \alpha v_i)(\sigma_i(v_i) - \beta v_i) < 0$, for all $v_i \neq 0$, $i = 1, \ldots, m$. In this case, the closed-loop discontinuous system (161) and (162) with $u = \sigma(-y)$ is given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\sigma(\phi(x(t))), \quad x(0) = x_0, \quad \text{a.e.} \quad t \geq 0. \tag{191}$$

Next, consider the locally Lipschitz continuous and regular Lyapunov function candidate $V(x)$, $x \in \mathbb{R}^n$. Now, it follows from (174), (175), and (178) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(x) &\leq \max \mathcal{L}_{f+G\sigma}V(x) \\ &\leq \max\,[\mathcal{L}_f V(x) + \mathcal{L}_{G\sigma}V(x)] \end{aligned}$$

$$
\begin{aligned}
&= \max \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) \\
&\leq \min \mathcal{L}_f V(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) + L_1(x) - \tfrac{1}{4(1-\theta^2)} L_2(x) R_2^{-1}(x) L_2^{\mathrm{T}}(x) \\
&\quad + (1-\theta^2)\left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)} R_2^{-1}(x) L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x) \\
&\quad \times \left[\sigma(\phi(x)) + \tfrac{1}{2(1-\theta^2)} R_2^{-1}(x) L_2^{\mathrm{T}}(x)\right] \\
&= \min \mathcal{L}_f V(x) + L_1(x) + \mathcal{L}_G V(x)\sigma(\phi(x)) + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x)) R_2(x)\sigma(\phi(x)) \\
&\quad + L_2(x)\sigma(\phi(x)) \\
&= \phi^{\mathrm{T}}(x) R_2(x)\phi(x) - 2\phi^{\mathrm{T}}(x) R_2(x)\sigma(\phi(x)) + (1-\theta^2)\sigma^{\mathrm{T}}(\phi(x)) R_2(x)\sigma(\phi(x)) \\
&= \sum_{i=1}^{m} r_i(x)(\tfrac{1}{\beta}\sigma_i(-y_i) + y_i)(\tfrac{1}{\alpha}\sigma_i(-y_i) + y_i) \\
&= \tfrac{1}{\alpha\beta} \sum_{i=1}^{m} r_i(x)\left(\sigma_i(-y_i) + \alpha y_i\right)\left(\sigma_i(-y_i) + \beta y_i\right) \\
&\leq 0, \quad \text{a.e.} \quad x \in \mathbb{R}^n,
\end{aligned}
$$

which implies that the closed-loop discontinuous system (191) is strongly Lyapunov stable.

Next, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^n : \frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0 \in \mathcal{L}_{f+G\sigma}V(x)\}$ and note that $\frac{\mathrm{d}}{\mathrm{d}t}V(x) = 0$ if and only if $y = 0$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{M} \triangleq \{x \in \mathbb{R}^n : \ x = 0\}$ is the largest weakly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 3.2 that $x(t) \to \mathcal{M} = \{0\}$ as $t \to \infty$. Thus, the closed-loop discontinuous system (191) is globally strongly asymptotically stable for all $\sigma(\cdot)$ such that $\alpha v_i^2 < \sigma_i(v_i)v_i < \beta v_i^2$, $v_i \neq 0$, $i = 1, \ldots, m$, which implies that the discontinuous nonlinear system $\mathcal{G}$ given by (161) and (162) has strong sector (and, hence, gain) margin (α, β). $\square$

Note that in the case where $R_2(x)$, $x \in \mathbb{R}^n$, is diagonal, Theorem 11.5 guarantees larger strong gain and sector margins to the strong gain and sector margin guarantees provided by Theorem 11.4. However, Theorem 11.5 does not provide strong disk margin guarantees.

12. **CONCLUSION**

In this paper, we extended the notions of stability theory and dissipativity theory for continuous dynamical systems with continuously differentiable flows to discontinuous dynamical systems whose solutions are characterized by Filippov set-valued maps. Furthermore, extended Kalman-Yakubovich-Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions were developed. In addition, using the concepts of dissipativity for discontinuous dynamical systems with appropriate storage functions and supply rates, general stability criteria for feedback interconnections of discontinuous dynamical systems were given. The notion of optimality for time-invariant discontinuous control systems on the infinite interval utilizing a steady-state Hamilton-Jacobi-Bellman approach for characterizing optimal discontinuous nonlinear feedback controllers was also considered. Moreover, sufficient conditions for gain, sector, and disk margin guarantees for discontinuous nonlinear systems controlled by nonlinear optimal and inverse optimal regulators that minimize a nonlinear-nonquadratic performance criterion were derived. Using these results, connections between dissipativity and optimality of discontinuous nonlinear systems were established. These results provide a generalization of the meaningful inverse optimal nonlinear regulator stability margins as well as the classical linear-quadratic optimal regulator gain and phase margins to discontinuous nonlinear regulators.

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