Asymptotic and Finite-Time Semistability for Nonlinear Discrete-Time Systems With Application to Network Consensus

Wassim M. Haddad, *Fellow, IEEE*, Junsoo Lee, and Sanjay P. Bhat

***Abstract*—This article focuses on semistability and finite-time semistability analysis and synthesis of discrete-time dynamical systems having a continuum of equilibria. Semistability is the property whereby the solutions of a dynamical system converge to Lyapunov stable equilibrium points determined by the system initial conditions. In this article, we build on the theories of semistability and finite-time semistability for continuous-time dynamical systems to develop a rigorous framework for discrete semistability and discrete finite-time semistability. Specifically, Lyapunov and converse Lyapunov theorems for semistability and finite-time semistability are developed, and the regularity properties of the Lyapunov function establishing finite-time semistability are shown to be related to the settling time function capturing the finite settling time behavior of the dynamical system. These results are then used to develop a general framework for designing semistable and finite-time semistable consensus protocols for discrete dynamical networks for achieving multiagent coordination tasks asymptotically and in finite time. The proposed controller architectures involve the exchange of generalized energy state information between agents guaranteeing that the closed-loop dynamical network is semistable to an equipartitioned equilibrium representing a state of consensus consistent with basic thermodynamic principles.**

***Index Terms*—Consensus control, discrete systems, finite-time semistability, Lyapunov theorems, nonlinear networks, semistability, thermodynamic protocols.**

I. Introduction

FOR continuous-time dynamical systems, the authors in [1]–[4] developed a unified stability analysis framework for systems having a continuum of equilibria. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable nor finite-time stable. Hence, asymptotic and finite-time stability are not the appropriate notions of stability for systems having a continuum of equilibria. Two notions that are of particular relevance to such systems are *convergence* and *semistability*. Convergence is the property whereby every system solution converges (asymptotically or in finite time) to a limit point that may depend on the system initial conditions. Semistability (resp., finite-time semistability) is the additional requirement that all solutions converge asymptotically (resp., in finite time) to limit points that are Lyapunov stable. Semistability (resp., finite-time semistability) for an equilibrium, thus, implies Lyapunov stability, and is implied by asymptotic (resp., finite time) stability.

It is important to note that semistability and set stability of an equilibrium set are independent notions. Indeed, as shown in [1], it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense [3]. For continuous-time systems, this is further discussed in [1]–[3] and [5], with [5] characterizing limit sets within curves where trajectories converge to a continuum of not necessarily Lyapunov stable fixed points.

In this article, we build on the theories of semistability and finite-time semistability for continuous-time dynamical systems developed in [1]–[4] to develop a rigorous framework for semistability and finite-time semistability for discrete-time systems. First, in Section III, we develop the notion of semistability for discrete-time systems and give several alternative equivalent characterizations for semistability. Then, in Section IV, we develop the theory of discrete semistability by presenting Lyapunov theorems as well as converse Lyapunov theorems for discrete semistability, which hold with continuous Lyapunov functions whose Lyapunov difference decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the distance to the set of equilibria.

Next, in Sections V and VI, we develop the notion of finite-time semistability and establish finite-time semistability theory for discrete-time nonlinear dynamical systems. Specifically, using existence and uniqueness of solutions, we define a settling-time function for a finite-time semistable system and establish a lower semicontinuity property of this function. Then, we develop sufficient Lyapunov stability theorems for

Manuscript received 1 March 2021; revised 26 October 2021; accepted 27 December 2021. Date of publication 11 January 2022; date of current version 30 January 2023. This work was supported by the Air Force Office of Scientific Research under Grant FA9550-20-1-0038. Recommended by Associate Editor Giuseppe Notarstefano. *(Corresponding author: Wassim M. Haddad.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TAC.2022.3142189.

Digital Object Identifier 10.1109/TAC.2022.3142189

finite-time semistability and establish a relationship between finite-time convergence and finite-time semistability. In addition, we present the first converse Lyapunov theorems for discrete finite-time semistability, which are shown to hold for lower semicontinuous Lyapunov functions.

A sizable body of work has emerged in recent years that addresses the distributed consensus problem using the tools of algebraic graph theory (see, for example, [6]–[16] for continuous-time networks and [17]–[23] for discrete-time networks). In [4] and [24]–[26], the authors present an alternative perspective to the distributed consensus problem based on *dynamical thermodynamics* [27], [28]; a framework that unifies the foundational disciplines of thermodynamics and dynamical systems theory. Dynamical thermodynamics was developed in [27] and [28] to address the formulation of equilibrium and nonequilibrium thermodynamics in a dynamical systems setting. Dynamical thermodynamics has also been used to apply thermodynamic principles to the analysis and control design of dynamical systems using an energy- and entropy-based hybrid stabilization framework [29]–[31].

By generalizing the notions of temperature, energy, and entropy, dynamical thermodynamics is used in [4] and [24]–[26] to develop a design procedure for distributed consensus controllers that induce networked dynamical systems that emulate a thermodynamic behavior. In particular, for network systems with an undirected communication graph topology, system thermodynamic notions are used to show that every control law protocol of a symmetric Fourier type, with information (or communication) transfer playing the role of energy flow, achieves information consensus [4], [24]–[26].

Unlike most of the distributed nonlinear consensus protocols presented in the literature that merely guarantee system convergence, the thermodynamics-based control framework for network systems [4], [24]–[26] addresses both convergence and Lyapunov stability. From a practical viewpoint, it is not sufficient for a nonlinear control protocol to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable [1], [3], [24].

In Section VII, we use the results of Sections IV and VI to develop consensus protocols for multiagent systems with nonlinear discrete dynamics. Specifically, we use our discrete-time semistability and discrete-time finite-time semistability frameworks to design distributed asymptotic and finite-time consensus control protocols for nonlinear dynamical networks with bidirectional communication graph topologies. The proposed controller architectures are predicated on the recently developed notion of discrete dynamical thermodynamics [28] resulting in controller architectures involving the exchange of generalized energy state information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles. Finally, we note that even though some of the proofs of the results in this article are similar to their continuous-time counterparts given in [28], for completeness of exposition we provide self-contained proofs here.

We begin by establishing notation, definitions, and mathematical preliminaries in Section II.

II. Mathematical Preliminaries

In this section, we establish notation, definitions, and present some key results needed for developing the main results of this article. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of positive integers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denote transpose. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε and $\|\cdot\|$ and $\|\cdot\|_\infty$ for the Euclidean and infinity vector norms in $\mathbb{R}^n$, respectively.

Consider the discrete-time nonlinear dynamical system

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \tag{1}$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the system state vector, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathcal{D}$ is continuous on $\mathcal{D}$, and $\Delta f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = x\}$ is nonempty. We denote the solution to (1) with initial condition $x(0) = x_0$ by $s(\cdot, x_0)$ so that the *map* of the dynamical system given by $s : \overline{\mathbb{Z}}_+ \times \mathcal{D} \to \mathcal{D}$ is continuous on $\mathcal{D}$ and satisfies the *consistency* property $s(0, x_0) = x_0$ and the *semigroup* property $s(\kappa, s(k, x_0)) = s(k + \kappa, x_0)$ for all $x_0 \in \mathcal{D}$ and $k, \kappa \in \overline{\mathbb{Z}}_+$. We use the notation $s(k, x_0)$, $k \in \overline{\mathbb{Z}}_+$, and $x(k)$, $k \in \overline{\mathbb{Z}}_+$, interchangeably as the solution of the nonlinear discrete-time dynamical system (1) with initial condition $x(0) = x_0$. By a *solution* to (1) with initial condition $x(0) = x_0$, we mean a function $x : \overline{\mathbb{Z}}_+ \to \mathcal{D}$ that satisfies (1). Given $k \in \overline{\mathbb{Z}}_+$ and $x \in \mathcal{D}$, we denote the map $s(k, \cdot) : \mathcal{D} \to \mathcal{D}$ by s_k and the map $s(\cdot, x) : \overline{\mathbb{Z}}_+ \to \mathcal{D}$ by s^x.

If $f(\cdot)$ is continuous, then it follows that $f(s(k-1, \cdot))$ is also continuous since it is constructed as a composition of continuous functions. Hence, $s(k, \cdot)$ is continuous on $\mathcal{D}$. If $f(\cdot)$ is such that $f : \mathbb{R}^n \to \mathbb{R}^n$, then we can construct the *solution sequence* or *discrete trajectory* $x(k) = s(k, x_0)$ to (1) iteratively by setting $x(0) = x_0$ and using $f(\cdot)$ to define $x(k)$ recursively by $x(k+1) = f(x(k))$. This iterative process can be continued indefinitely, and hence, a solution to (1) exists for all $k \geq 0$.

Alternatively, if $f(\cdot)$ is such that $f : \mathcal{D} \to \mathbb{R}^n$, then the solution may cease to exist at some point if $f(\cdot)$ maps $x(k)$ into some point $x(k+1)$ outside the domain of $f(\cdot)$. In this case, the solution sequence $x(k) = s(k, x_0)$ will be defined on the maximal interval of existence $x(k)$, $k \in \mathcal{I}_{x_0}^+ \subset \overline{\mathbb{Z}}_+$. Note that the solution sequence $x(k)$, $k \in \mathcal{I}_{x_0}^+$, is uniquely defined for every initial condition $x_0 \in \mathcal{D}$ irrespective of whether or not $f(\cdot)$ is a continuous function. That is, any other solution sequence $y(k)$ starting from x_0 at $k = 0$ will take exactly the same values as $x(k)$ and can be continued to the same interval as $x(k)$. It is important to note that if $k \in \overline{\mathbb{Z}}_+$, then uniqueness of solutions backward in time need not necessarily hold. This is due to the fact that $s(k, x_0) = f^{-1}(s(k+1, x_0))$, $k \in \overline{\mathbb{Z}}_+$, and there is no guarantee that $f(\cdot)$ is invertible for all $k \in \overline{\mathbb{Z}}_+$. However, if $f : \mathcal{D} \to \mathcal{D}$ is a homeomorphism for all $k \in \overline{\mathbb{Z}}_+$, then the solution sequence is unique for all $k \in \mathbb{Z}$.

In light of the abovementioned, the following theorem is immediate.

Theorem 2.1 (see [3]): Consider the nonlinear dynamical system (1) and assume that $f : \mathcal{D} \to \mathcal{D}$. Then, for every $x_0 \in \mathcal{D}$, there exists $\mathcal{I}_{x_0}^+ \subseteq \overline{\mathbb{Z}}_+$ such that (1) has a unique solution $x : \mathcal{I}_{x_0}^+ \to \mathbb{R}^n$. Moreover, if $f(\cdot)$ is continuous, then the solution $s(k, \cdot)$ is continuous for each $k \in \mathcal{I}_{x_0}^+$. If, in addition, $f(\cdot)$ is a homeomorphism of $\mathcal{D}$ onto $\mathbb{R}^n$, then the solution $x : \mathcal{I}_{x_0}^+ \to \mathbb{R}^n$ is unique in all $\mathcal{I}_{x_0} \subseteq \mathbb{Z}$ and $s(k, \cdot)$ is continuous for all $k \in \mathcal{I}_{x_0}$. Finally, if $\mathcal{D} = \mathbb{R}^n$, then $\mathcal{I}_{x_0} = \mathbb{Z}$.

The following definition introduces the notion of class $\mathcal{W}_{\rm d}$ functions involving *nondecreasing* functions.

Definition 2.1: A function $w : \mathbb{R} \to \mathbb{R}$ is of *class* $\mathcal{W}_{\rm d}$ if $w(z') \leq w(z'')$ for all $z', z'' \in \mathbb{R}$ such that $z' \leq z''$.

To develop the theory for finite-time semistability of discrete autonomous systems, we will require several key results on difference inequalities and the discrete-time comparison principle. Consider the scalar discrete-time nonlinear dynamical system given by

$$z(k+1) = w(z(k))), \quad z(k_0) = z_0, \quad k \in \mathcal{I}_{z_0} \tag{2}$$

where $z(k) \in \mathcal{Q} \subseteq \mathbb{R}$, $k \in \mathcal{I}_{z_0}$, is the system state vector, $\mathcal{I}_{z_0} \subseteq \mathbb{Z}$ is the maximal interval of existence of a solution $z(k)$ to (2), $\mathcal{Q}$ is an open set, $0 \in \mathcal{Q}$, and $w : \mathcal{Q} \to \mathbb{R}$ is a continuous function on $\mathcal{Q}$.

Theorem 2.2 (see [32]): Consider the discrete-time nonlinear dynamical system (2). Assume that the function $w : \mathcal{Q} \subseteq \mathbb{R} \to \mathbb{R}$ is continuous and $w(\cdot)$ is of class $\mathcal{W}_{\rm d}$. If there exists a continuous function $V : \mathcal{I}_{z_0} \to \mathcal{Q}$ such that

$$V(k+1) \leq w(V(k)), \quad k \in \mathcal{I}_{z_0} \tag{3}$$

then

$$V(k_0) \leq z_0, \quad z_0 \in \mathcal{Q} \tag{4}$$

implies

$$V(k) \leq z(k), \quad k \in \mathcal{I}_{z_0} \tag{5}$$

where $z(k)$, $k \in \mathcal{I}_{z_0}$, is the solution to (2).

The following result is a direct corollary of Theorem 2.2.

Corollary 2.1 (see [32]): Consider the discrete-time nonlinear dynamical system (1). Assume there exists a continuous function $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathcal{Q}$ such that

$$V(f(x)) \leq w(V(x)), \quad x \in \mathcal{D} \tag{6}$$

where $w : \mathcal{Q} \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous, $w(\cdot) \in \mathcal{W}_{\rm d}$, and

$$z(k+1) = w(z(k)), \quad z(k_0) = z_0, \quad k \in \mathcal{I}_{z_0}. \tag{7}$$

If $\{k_0, \ldots, k_0 + \tau\} \subseteq \mathcal{I}_{x_0} \cap \mathcal{I}_{z_0}$, then

$$V(x_0) \leq z_0, \quad z_0 \in \mathcal{Q} \tag{8}$$

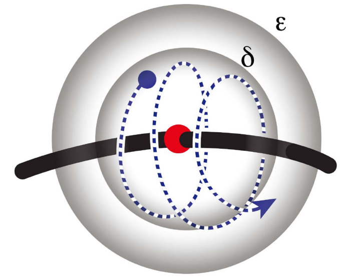
implies

$$V(x(k)) \leq z(k), \quad k \in \{k_0, \ldots, k_0 + \tau\}. \tag{9}$$

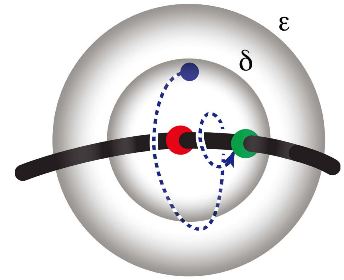
Note that if the solutions to (1) and (7) are globally defined for all $x_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}$, then Corollary 2.1 holds for all $k \geq k_0$. For the remainder of this article, we assume, without loss of generality that $k_0 = 0$.

III. Semistability of Discrete Autonomous Systems

In this section and in the following section, we develop a stability analysis framework for discrete-time systems having a continuum of equilibria and present necessary and sufficient conditions for *discrete-time semistability*. To develop semistability theory for discrete-time systems, we need some additional notation and definitions.



(a)



(b)

Fig. 1. (a) Lyapunov stable nonisolated equilibrium point. The perturbed trajectory need not converge to a new equilibrium. (b) Semistable nonisolated equilibrium point. Semistability guarantees convergence of the perturbed trajectory to a nearby Lyapunov stable equilibrium point, and is a stronger property than Lyapunov stability.

A set $\mathcal{M} \subset \mathcal{D} \subseteq \mathbb{R}^n$ is a *positively invariant set* with respect to the nonlinear dynamical system (1) if $s_k(\mathcal{M}) \subseteq \mathcal{M}$, for all $k \in \overline{\mathbb{Z}}_+$, where $s_k(\mathcal{M}) \triangleq \{s_k(x) : x \in \mathcal{M}\}$. A set $\mathcal{M} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ is an *invariant set* with respect to the dynamical system (1) if $s_k(\mathcal{M}) = \mathcal{M}$ for all $k \in \overline{\mathbb{Z}}_+$. A point $p \in \mathcal{D}$ is a *limit point* of the trajectory or solution sequence $s(\cdot, x)$ of (1) if there exists a monotonic sequence $\{k_n\}_{n=0}^\infty$ of positive integers, with $k_n \to \infty$ as $n \to \infty$, such that $s(k_n, x) \to p$ as $n \to \infty$. The set of all limit points of $s(k, x)$, $k \in \overline{\mathbb{Z}}_+$, is the *limit set* $\omega(x)$ of $s(\cdot, x)$ of (1). Finally, for $k \geq 0$, we define the *positive orbit* through the point $x_0 \in \mathcal{D}$ as the motion along the solution sequence $\mathcal{O}_{x_0}^+ \triangleq \{x \in \mathcal{D} : x = s(k, x_0), k \geq 0\}$.

The following definition introduces the notion of semistability for discrete-time systems (see Fig. 1).

Definition 3.1: An equilibrium point $x_{\rm e} \in \mathcal{D} \subseteq \mathbb{R}^n$ of (1) is *Lyapunov stable* if, for all $\varepsilon > 0$, there exists $\delta = \delta(x_{\rm e}) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then $x(k) \in \mathcal{B}_\varepsilon(x_{\rm e})$, $k \in \overline{\mathbb{Z}}_+$. An equilibrium point $x_{\rm e} \in \mathcal{D} \subseteq \mathbb{R}^n$ of (1) is *semistable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then $\lim_{k\to\infty} s(k, x) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of (1). An equilibrium point $x_{\rm e} \in \mathbb{R}^n$ is *globally semistable* if it is Lyapunov stable and, for every $x_0 \in \mathbb{R}^n$, $\lim_{k\to\infty} x(k) = y$, where $y \in \mathbb{R}^n$ is Lyapunov stable equilibrium point of (1). System (1) is *semistable* if every equilibrium point of (1) is semistable. Finally, (1) is *globally semistable* if every equilibrium point of (1) is globally semistable.

The following proposition gives a sufficient condition for a trajectory or solution sequence of (1) to converge to a limit. For this result, $\mathcal{D}_{\rm c} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ denotes a positively invariant set with respect to (1) and $s_k(\mathcal{D}_{\rm c})$ denotes the image of $\mathcal{D}_{\rm c} \subset \mathcal{D}$ under the map $s_k : \mathcal{D}_{\rm c} \to \mathcal{D}$; that is, $s_k(\mathcal{D}_{\rm c}) \triangleq \{y : y = s_k(x_0)$ for some $x(0) = x_0 \in \mathcal{D}_{\rm c}\}$.

Proposition 3.1: Consider the nonlinear discrete-time dynamical system (1) and let $x \in \mathcal{D}_{\rm c}$. If the limit set $\omega(x)$ of (1) contains a Lyapunov stable equilibrium point y, then $y = \lim_{k\to\infty} s(k,x)$, that is, $\omega(x) = \{y\}$.

Proof: Suppose $y \in \omega(x)$ is Lyapunov stable and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ be an open neighborhood of y. Since y is Lyapunov stable, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_{\rm c}$ of y such that $s_k(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for every $k \in \mathbb{Z}_+$. Now, since $y \in \omega(x)$, it follows that there exists $\kappa \in \overline{\mathbb{Z}}_+$ such that $s(\kappa, x) \in \mathcal{N}_\delta$. Hence, $s(k + \kappa, x) = s_k(s(\kappa, x)) \in s_k(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for every $k > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_{\rm c}$ is arbitrary, it follows that $y = \lim_{k\to\infty} s(k,x)$. Thus, $\lim_{n\to\infty} s(k_n, x) = y$ for every increasing sequence $\{k_n\}_{n=1}^\infty$, and hence, $\omega(x) = \{y\}$.

Next, we present alternative equivalent characterizations of semistability of (1). For this result, the following definition is required.

Definition 3.2: The *domain of semistability* is the set of points $x_0 \in \mathcal{D}$ such that if $x(k)$ is a solution to (1) with $x(0) = x_0$, $k \in \overline{\mathbb{Z}}_+$, then $x(k)$ converges to a Lyapunov stable equilibrium point in $\mathcal{D}$.

Note that if (1) is semistable, then its domain of semistability contains the set of equilibria in its interior. For the next result recall the definitions of class $\mathcal{K}$ and class $\mathcal{L}$ functions (see[28, p. 162]), and recall $\Delta f^{-1}(0) = \{x \in \mathcal{D} : f(x) = x\}$.

Proposition 3.2: Consider the nonlinear discrete-time dynamical system $\mathcal{G}$ given by (1). Then, the following statements are equivalent.

i) $\mathcal{G}$ is semistable.

ii) For each $x_{\rm e} \in \Delta f^{-1}(0)$, there exist class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, respectively, and $\delta = \delta(x_{\rm e}) > 0$, such that if $\|x_0 - x_{\rm e}\| < \delta$, then $\|x(k) - x_{\rm e}\| \le \alpha(\|x_0 - x_{\rm e}\|)$, $k \in \mathbb{Z}_+$, and $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \beta(k)$, $k \in \mathbb{Z}_+$.

iii) For each $x_{\rm e} \in \Delta f^{-1}(0)$, there exist class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, a class $\mathcal{L}$ function $\beta(\cdot)$, and $\delta = \delta(x_{\rm e}) > 0$, such that if $\|x_0 - x_{\rm e}\| < \delta$, then $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \alpha_1(\|x(k) - x_{\rm e}\|)\beta(k) \le \alpha_2(\|x_0 - x_{\rm e}\|)\beta(k)$, $k \in \overline{\mathbb{Z}}_+$.

Proof: See Appendix A. ■

IV. Lyapunov and Converse Lyapunov Theorems for Semistability

In this section, we present Lyapunov and converse Lyapunov theorems for discrete-time semistability. For the results in this section, define $\Delta V(x) \triangleq V(f(x)) - V(x)$ for a given continuous function $V : \mathcal{D} \to \mathbb{R}$ and $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x\in\mathcal{M}} \|p - x\|$.

Theorem 4.1: Consider the nonlinear discrete-time dynamical system (1). Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be an open neighborhood of $\Delta f^{-1}(0)$ and assume that there exists a continuous function $V : \mathcal{Q} \to \mathbb{R}$ such that

$$\Delta V(x) < 0, \quad x \in \mathcal{Q} \setminus \Delta f^{-1}(0). \tag{10}$$

If every equilibrium point of (1) is Lyapunov stable, then (1) is semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (1) is globally semistable.

Proof: Since every equilibrium point of (1) is Lyapunov stable by assumption, for every $z \in \Delta f^{-1}(0)$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s(\overline{\mathbb{Z}}_+ \times \mathcal{V}_z)$ is bounded and contained in $\mathcal{Q}$. The set $\mathcal{V} \triangleq \bigcup_{z\in\Delta f^{-1}(0)} \mathcal{V}_z$ is an open neighborhood of $\Delta f^{-1}(0)$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}$ so that there exists $z \in \Delta f^{-1}(0)$ such that $x \in \mathcal{V}_z$ and $s(k,x) \in \mathcal{V}_z$, $k \in \overline{\mathbb{Z}}_+$. Since $\mathcal{V}_z$ is bounded it follows that the positive limit set of x is nonempty and invariant. Furthermore, it follows from (10) that $\Delta V(x) \le 0$, $k \in \overline{\mathbb{Z}}_+$, and hence, it follows from [28, Th. 13.3] that $s(k,x) \to \mathcal{M}$ as $k \to \infty$, where $\mathcal{M}$ is the largest invariant set contained in the set $\mathcal{R} = \{y \in \mathcal{V}_z : \Delta V(x) = 0\}$. Note that $\mathcal{R} = \Delta f^{-1}(0)$ is invariant, and hence, $\mathcal{M} = \mathcal{R}$, which implies that $\lim_{k\to\infty} \mathrm{dist}(s(k,x), \Delta f^{-1}(0)) = 0$. Finally, since every point in $\Delta f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 3.1 that $\lim_{k\to\infty} s(k,x) = x^*$, where $x^* \in \Delta f^{-1}(0)$ is Lyapunov stable. Hence, by definition, (1) is semistable.

Finally, if $\mathcal{Q} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then global semistability follows using standard arguments. ■

Next, we present a slightly more general theorem for semistability wherein we do not assume that all points in $\Delta V^{-1}(0) \triangleq \{x \in \mathcal{Q} : V(f(x)) = V(x)\}$ are Lyapunov stable but rather we assume that all points in the largest invariant subset of $\Delta V^{-1}(0)$ are Lyapunov stable.

Theorem 4.2: Consider the nonlinear discrete-time dynamical system (1) and let $\mathcal{Q}$ be an open neighborhood of $\Delta f^{-1}(0)$. Suppose the positive orbit $\mathcal{O}_x$ of (1) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{Q} \to \mathbb{R}$ such that

$$\Delta V(x) \le 0, \quad x \in \mathcal{Q}. \tag{11}$$

If every point in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : \Delta V(x) = 0\}$ is Lyapunov stable, then (1) is semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (1) is globally semistable.

Proof: Since every solution of (1) is bounded, it follows from the hypotheses on $V(\cdot)$ that, for every $x \in \mathcal{Q}$, the positive limit set $\omega(x)$ of (1) is nonempty and contained in the largest invariant subset $\mathcal{M}$ of $\{x \in \mathcal{Q} : \Delta V(x) = 0\}$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium, it follows from Proposition 3.1 that $\omega(x)$ contains a single point for every $x \in \mathcal{Q}$ and $\lim_{k\to\infty} s(k,x)$ exists for every $x \in \mathcal{Q}$. Now, since $\lim_{k\to\infty} s(k,x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{Q}$, it follows from the definition of semistability that every equilibrium point in $\mathcal{M}$ is semistable.

Finally, if $\mathcal{Q} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, the global semistability follows using standard arguments. ■

Finally, we provide a converse Lyapunov theorem for semistability.

Theorem 4.3: Consider the nonlinear discrete-time dynamical system (1). Suppose (1) is discrete-time semistable with the domain of semistability $\mathcal{D}_0$. Then, there exist a continuous nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}$ function $\alpha(\cdot)$ such that:

i) $V(x) = 0$, $x \in \Delta f^{-1}(0)$;

ii) $V(x) \ge \alpha(\mathrm{dist}(x, \Delta f^{-1}(0)))$, $x \in \mathcal{D}_0$; and

iii) $\Delta V(x) < 0$, $x \in \mathcal{D}_0 \backslash \Delta f^{-1}(0)$.

Proof: See Appendix B. ■

V. Finite-Time Semistability of Discrete Autonomous Systems

In this section and in the following section, we extend the results of Sections III and IV to address *finite-time semistability* of discrete-time nonlinear dynamical systems. The notion of finite-time semistability involves finite-time convergence along with Lyapunov stability as detailed in the following definition.

Definition 5.1: Consider the nonlinear dynamical system (1). An equilibrium point $x_{\rm e} \in \Delta f^{-1}(0)$ of (1) is *finite-time semistable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of $x_{\rm e}$ and a function $K : \mathcal{N}\backslash\Delta f^{-1}(0) \to \mathbb{Z}_+$, called the *settling-time function*, such that the following statements hold.

i) *Finite-time convergence.* For every $x \in \mathcal{N}\backslash\Delta f^{-1}(0)$, $s^x(k) \in \mathcal{N}\backslash\Delta f^{-1}(0)$ is defined on $k \in \{0,\ldots,K(x)-1\}$ and $s^x(k)$, $k \geq K(x)$, is contained in $\mathcal{N} \cap \Delta f^{-1}(0)$.

ii) *Semistability.* $x_{\rm e} \in \Delta f^{-1}(0)$ is semistable.

An equilibrium point $x_{\rm e} \in \Delta f^{-1}(0)$ of (1) is *globally finite-time semistable* if it is finite-time semistable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$. System (1) is said to be *finite-time semistable* if every equilibrium point in $\Delta f^{-1}(0)$ is finite-time semistable. Finally, (1) is said to be *globally finite-time semistable* if every equilibrium point in $\Delta f^{-1}(0)$ is globally finite-time semistable.

Note that the definition of finite-time convergence in Definition 5.1 is simpler than the corresponding definition in the case of continuous-time systems [33]. In the case of continuous-time systems, the usual sufficient conditions for existence and uniqueness of solutions necessarily fail to hold at a finite-time stable equlibrium. Since discrete-time systems possess existence and uniqueness of solutions without any additional assumptions on $f(\cdot)$, the definition of finite-time convergence can be stated in a manner simpler than in the case of continuous-time systems.

It is easy to see from Definition 5.1 that

$$K(x) = \min\{k \in \mathbb{Z}_+ : f(s(k,x)) = s(k,x)\}, \quad x \in \mathcal{N}. \tag{12}$$

In particular, $K(x_{\rm e}) = 0$ for any equilibrium point $x_{\rm e} \in \Delta f^{-1}(0)$ of (1).

The following definition is needed for the following result.

Definition 5.2: Let $\mathcal{D} \subseteq \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}$, and $x \in \mathcal{D}$. The function g is *lower semicontinuous at* $x \in \mathcal{D}$ if for every sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$, $g(x) \leq \liminf_{n\to\infty} g(x_n)$. The function g is *lower semicontinuous on* $\mathcal{D}$ if g is lower semicontinuous at every point $x \in \mathcal{D}$.

The next proposition shows that if the settling-time function of a finite-time semistable system is lower semicontinuous at each $x_{\rm e} \in \mathcal{N} \cap \Delta f^{-1}(0)$, then it is lower semicontinuous on $\mathcal{N}$.

Proposition 5.1: Consider the nonlinear dynamical system (1). Assume that every equilibrium point $x_{\rm e} \in \Delta f^{-1}(0)$ of (1) is finite-time semistable, let $\mathcal{N} \subseteq \mathcal{D}$ be as in Definition 5.1, and let $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ be the settling-time function. Then, $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$.

Proof: Let $y \in \mathcal{N}$, consider the sequence $\{y_n\}_{n=0}^{\infty}$ in $\mathcal{N}$ converging to y, and let $\tau^- = \liminf_{n\to\infty} K(y_n)$. Let $\{y_m^-\}_{m=0}^{\infty}$ be a subsequence of $\{y_n\}_{n=0}^{\infty}$ such that $K(y_m^-) \to \tau^-$ as $m \to \infty$. Since K only takes integer values, it follows that there exists $M > 0$ such that $K(y_m^-) = \tau^-$ for all $m > M$. Since $s(k,\cdot)$ is continuous for each k, and since $K(y_m^-) = \tau^{-1}$ for $m > M$, it follows that $s(K(y_m^-), y_m^-) \to s(\tau^-, y)$ as $m \to \infty$. Now, it follows from (12) that $s(K(y_m^-), y_m^-) \in \Delta f^{-1}(0)$ for each m. Since the set $\Delta f^{-1}(0)$ is closed, we conclude that $s(\tau^-, y) \in \Delta f^{-1}(0)$. Equation (12) now implies that

$$K(y) \leq \tau^- = \liminf_{n\to\infty} K(y_n) \tag{13}$$

which implies that $K(\cdot)$ is lower semicontinuous at y. Since $y \in \mathcal{N}$ was chosen arbitrarily, the assertion follows. ■

Remark 5.1: In the case of continuous-time systems, it is known that the settling-time function $K(\cdot)$ of a finite-time stable equilibrium is continuous in the domain of convergence if and only if it is continuous at the equilibrium (see [33, Prop. 2.4] and [4, Lemma 4.1]). In the case of discrete-time systems, the integer-valued function $K(\cdot)$ is continuous at a point only if it is locally constant. Thus, if $K(\cdot)$ is continuous at an equilibrium point $x_{\rm e}$, then $x_{\rm e}$ necessarily has to lie in the interior of $\Delta f^{-1}(0)$. On the other hand, the set of equilibrium points is closed. Hence, $K(\cdot)$ can be continuous at all equilibrium points only in the uninteresting case where the set of equilibria is either empty or the whole state space.

VI. Lyapunov and Converse Lyapunov Theorems for Finite-Time Semistability

In this section, we present necessary and sufficient conditions for finite-time semistability. For these results, we assume $f : \mathcal{D} \to \mathcal{D}$ and, for every $x_0 \in \mathcal{D}$, (1) is forward complete. The first result establishes a relationship between finite-time convergence and finite-time semistability.

Theorem 6.1: Consider the nonlinear discrete-time dynamical system (1). Assume that there exist a continuous nonnegative function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $\Delta V^{-1}(0) = \Delta f^{-1}(0)$, an open neighborhood $\mathcal{Q} \subseteq \mathcal{D}$ such that $\mathcal{Q} \cap \Delta f^{-1}(0)$ is nonempty, and

$$V(f(x)) \leq w(V(x)), \quad x \in \mathcal{Q} \setminus \Delta f^{-1}(0) \tag{14}$$

where $w : \mathbb{R} \to \mathbb{R}$ is continuous, $w(0) = 0$, and

$$z(k+1) = w(z(k)), \quad z(0) = z_0, \quad k \geq 0. \tag{15}$$

If (15) is finite-time convergent to the origin for $\overline{\mathbb{R}}_+$ and every point in $\mathcal{Q} \cap \Delta f^{-1}(0)$ is a Lyapunov stable equilibrium point of (1), then every equilibrium point $x_{\rm e} \in \mathcal{Q} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable. Moreover, the settling-time function of (1) is lower semicontinuous on an open neighborhood of $\mathcal{Q} \cap \Delta f^{-1}(0)$. If, in addition, $\mathcal{Q} = \mathcal{D}$, then (1) is finite-time semistable. Finally, if $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (14) holds on $\mathbb{R}^n$, then every equilibrium point $x_{\rm e} \in \Delta f^{-1}(0)$ of (1) is globally finite-time semistable.

Proof: Consider $x_{\rm e} \in \mathcal{Q} \cap \Delta f^{-1}(0)$. Since $x(k) \equiv x_{\rm e}$ is Lyapunov stable, it follows that there exists an open positively invariant set $\mathcal{V} \subseteq \mathcal{Q}$ such that $x_{\rm e} \in \mathcal{V}$. Next, it follows from (14) that

$$V(s(k+1,x)) \leq w(V(s(k,x))), \quad x \in \mathcal{V}, \quad k \in \overline{\mathbb{Z}}_+. \tag{16}$$

Now, it follows from Corollary 2.1 that

$$V(s(k,x)) \leq \psi(k, V(x_0)), \quad x \in \mathcal{V}, \quad k \in \overline{\mathbb{Z}}_+ \tag{17}$$

where $\psi : \overline{\mathbb{Z}}_+ \times \mathbb{R} \to \mathbb{R}$ is the global semiflow of (15). Since (15) is finite-time convergent to the origin for $\overline{\mathbb{R}}_+$, it follows from (17) and the nonnegativity of $V(\cdot)$ that

$$V(s(k,x)) = 0, \quad k \geq \hat{K}(V(x)), \quad x \in \mathcal{V} \tag{18}$$

where $\hat{K}(\cdot)$ denotes the settling-time function of (15).

Next, since $s(0,x) = x$, $s(k,\cdot)$ is continuous, and $\Delta V(s(k,x)) = 0$ is equivalent to $f(s(k,x)) = s(k,x)$ on $\mathcal{V}$, it follows that $\min\{k \in \overline{\mathbb{Z}}_+ : f(s(k,x)) = s(k,x)\} > 0$,

$x \in \mathcal{V} \setminus \Delta f^{-1}(0)$. Furthermore, it follows from (18) that $\min\{k \in \overline{\mathbb{Z}}_+ : f(s(k,x)) = s(k,x)\} < \infty, \quad x \in \mathcal{V}$. Now, define $K : \mathcal{V} \setminus \Delta f^{-1}(0) \to \mathbb{Z}_+$ by using (12). Then, it follows that every point in $\mathcal{V} \cap \Delta f^{-1}(0)$ is finite-time semistable and, by Proposition 5.1, K is a lower semicontinuous settling-time function on $\mathcal{V}$. Furthermore, it follows from (18) that $K(x) \leq \hat{K}(V(x))$, $x \in \mathcal{V}$.

Moreover, if $\mathcal{Q} = \mathcal{D}$, then $\mathcal{Q}$ is positively invariant by the fact that (1) with $f : \mathcal{D} \to \mathcal{D}$ is forward complete with unique solutions, and hence, the preceeding arguments hold with $\mathcal{V} = \mathcal{Q} = \mathcal{D}$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then global finite-time semistability follows using identical arguments. ■

The following definition and lemma are needed for the following results of this article.

Definition 6.1: A continuous function $w : \mathbb{R} \to \mathbb{R}$ is a *generalized deadzone function* if i) $|w(z)| < |z|$, $z \in \mathbb{R}$, and ii) there exists $\varepsilon > 0$ such that $w(z) = 0$ for all $z \in \mathcal{B}_\varepsilon(0)$.

Lemma 6.1: Consider the scalar nonlinear discrete-time dynamical system (2) with $\mathcal{Q} = \mathbb{R}$, $k_0 = 0$, and $\mathcal{I}_{z_0} = \mathbb{Z}$. Then, the zero solution $z(k) \equiv 0$ to (2) is a globally finite-time stable equilibrium point of (2) if and only if $w : \mathbb{R} \to \mathbb{R}$ is a generalized deadzone function.

Proof: To show sufficiency, suppose $w : \mathbb{R} \to \mathbb{R}$ is a generalized deadzone function. Then, the zero solution $z(k) \equiv 0$ to (2) is an equilibrium point of (2). Let $|z(0)| > 0$ and consider the solution sequence $\{z(k)\}_{k=0}^{\infty}$ generated by (2). Suppose, *ad absurdum* that $|z(k)| \geq \varepsilon$, $k \in \overline{\mathbb{Z}}$. Since $|w(z)| < |z|$, $z \in \mathbb{R}$, it follows that $|z(k+1)| < |z(k)|$. Thus, since $|z(k)|$, $k \in \overline{\mathbb{Z}}_+$, is a decreasing sequence that is bounded from below, there exists $z^* > 0$ such that $|z(k)| \to z^* \geq \varepsilon > 0$ as $k \to \infty$. Now, since w is continuous, it follows that

$$w(z^*) = w\left(\min_{k\to\infty} |z(k)|\right) = \lim_{k\to\infty} w(|z(k)|) = z^*$$

which is a contradiction. Hence, there exists k such that $|z(k)| < \varepsilon$, and hence, $z(k+1) = 0$. Thus, the zero solution $z(k) \equiv 0$ to (2) is globally finite-time convergent. Lyapunov stability now follows trivially since $w(z) = 0$, $z \in \mathcal{B}_\varepsilon(0)$.

Conversely, to show necessity suppose that the zero solution $z(k) \equiv 0$ to (2) is a globally finite-time stable equilibrium point of (2). Let $z(0) \in \mathbb{R}$, consider the solution sequence $\{z(k)\}_{k=0}^{\infty}$ generated by (2), and let $\kappa = \min\{k : z(k) = 0\} - 1$. It follows from finite-time stability that $\kappa < \infty$. By the definition of κ, it follows that $z(\kappa+1) = 0$, while $z(\kappa) \neq 0$. Now, since $|w(z)| < |z|$, $z \in \mathbb{R}$, it also follows that $w(z) = 0$ for all $z \in \mathcal{B}_{|z(\kappa)|}(0)$. Hence, there exists $\varepsilon = |z(\kappa)| > 0$ such that $w(z) = 0$, $z \in \mathcal{B}_\varepsilon(0)$. ■

Next, using Lemma 6.1, we present two concrete forms for $w(\cdot)$ in Theorem 6.1 for establishing finite-time semistability. For the statement of the following results, we write $\lceil \alpha \rceil$ for the ceiling function denoting the smallest integer greater than or equal to α.

Corollary 6.1: Consider the nonlinear discrete-time dynamical system (1). Assume that there exist a continuous nonnegative function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $\Delta V^{-1}(0) = \Delta f^{-1}(0)$, real numbers $\alpha \in (0,1)$ and $c > 0$, an open neighborhood $\mathcal{Q} \subseteq \mathcal{D}$ such that $\mathcal{Q} \cap \Delta f^{-1}(0)$ is nonempty, and

$$\Delta V(x) \leq -c \min\left\{\frac{V(x)}{c}, V(x)^\alpha\right\}, \quad x \in \mathcal{Q} \setminus \Delta f^{-1}(0). \tag{19}$$

If every equilibrium point $x_\mathrm{e} \in \mathcal{Q} \cap \Delta f^{-1}(0)$ is a Lyapunov stable equilibrium point of (1), then every equilibrium point $x_\mathrm{e} \in \mathcal{Q} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable. Moreover, there exist an open neighborhood $\mathcal{N}$ of $\mathcal{Q} \cap \Delta f^{-1}(0)$ and a settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ such that either

$$K(x_0) \leq \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathcal{N}$$
$$V(x_0) > c^{\frac{1}{1-\alpha}} \tag{20}$$

or

$$K(x_0) = 1, \quad x_0 \in \mathcal{N} \setminus \Delta f^{-1}(0), \quad V(x_0) \leq c^{\frac{1}{1-\alpha}} \tag{21}$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{Q} = \mathcal{D}$, then (1) is finite-time semistable. Finally, if $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (19) holds on $\mathbb{R}^n$, then every equilibrium point $x_\mathrm{e} \in \Delta f^{-1}(0)$ of (1) is globally finite-time semistable.

Proof: Consider the scalar discrete-time nonlinear dynamical system given by

$$z(k+1) = z(k) - c\,\mathrm{sign}(z(k))\min\left\{\frac{|z(k)|}{c}, |z(k)|^\alpha\right\}$$
$$z(0) = z_0, \quad k \geq 0 \tag{22}$$

where $z(k) \in \mathbb{R}, k \in \overline{\mathbb{Z}}_+, \mathrm{sign}(z) \triangleq z/|z|, z \neq 0, \mathrm{sign}(0) \triangleq 0$, $\alpha \in (0,1)$, and $c > 0$. Note that the right-hand side of (22) is a generalized deadzone function, and hence, by Lemma 6.1, the zero solution $z(k) \equiv 0$ to (22) is globally finite-time stable. Furthermore, note that if $|z(k)| \leq c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then $z(k+1) = 0$, and if $|z(k)| > c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then

$$|z(k)| = \left|z(k-1)\left(1 - c|z(k-1)|^{\alpha-1}\right)\right| < |z(k-1)|$$
$$k \in \overline{\mathbb{Z}}_+. \tag{23}$$

Since $\alpha \in (0,1)$, $|z(k)|^{\alpha-1} > |z(k-1)|^{\alpha-1}$, $k \in \overline{\mathbb{Z}}_+$, and

$$1 - c\,|z(k)|^{\alpha-1} < 1 - c\,|z(k-1)|^{\alpha-1}, \quad k \in \overline{\mathbb{Z}}_+. \tag{24}$$

Next, it follows from (23), (24), and $|z(k)| > c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$ that

$$\begin{aligned} |z(k)| &= |z(k-1)|\left(1 - c\,|z(k-1)|^{\alpha-1}\right) \\ &\vdots \\ &= |z_0|\left(1 - c\,|z_0|^{\alpha-1}\right)\cdots\left(1 - c\,|z(k-1)|^{\alpha-1}\right) \\ &< |z_0|\left(1 - c\,|z_0|^{\alpha-1}\right)^k, \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{25}$$

Now, if $|z_0(1-c|z_0|^{\alpha-1})^k| \le c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, then $|z(k)| < c^{\frac{1}{1-\alpha}}$, $k \in \overline{\mathbb{Z}}_+$, which implies $z(k+1)=0$ for

$$k \ge \log_{[1-c|z_0|^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{|z_0|}, \quad |z_0| > c^{\frac{1}{1-\alpha}}$$

and hence, i) of Definition 5.1 is satisfied with $\mathcal{N}=\mathcal{D}=\mathbb{R}$ and with the settling-time function $\hat{K}(z_0)$ given by either

$$\hat{K}(z_0) \le \left\lceil \log_{[1-c|z_0|^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{|z_0|} \right\rceil + 1, \quad |z_0| > c^{\frac{1}{1-\alpha}} \tag{26}$$

or

$$\hat{K}(z_0) = 1, \quad |z_0| \le c^{\frac{1}{1-\alpha}}, \quad z_0 \ne 0. \tag{27}$$

Next, with $z=V(x)$ and $w(z)=w(V(x))=V(x)-c\min\{\frac{V(x)}{c}, V(x)^\alpha\}$, it follows from Corollary 2.1 and (26) that

$$\hat{K}(V(x_0)) \le \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1$$
$$x_0 \in \mathcal{B}_\delta(x_e), \quad V(x_0) > c^{\frac{1}{1-\alpha}}. \tag{28}$$

Hence, it follows from (18) that $K(x) \le \hat{K}(V(x))$, and hence

$$x(k) \in \mathcal{Q} \cap \Delta f^{-1}(0), \quad k \ge \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1$$
$$x_0 \in \mathcal{B}_\delta(x_e), \quad V(x_0) > c^{\frac{1}{1-\alpha}} \tag{29}$$

which implies finite-time convergence of the trajectories of (1) for all $x_0 \in \mathcal{B}_\delta(x_e)$ such that $V(x_0) > c^{\frac{1}{1-\alpha}}$. Alternatively, if $V(x_0) \le c^{\frac{1}{1-\alpha}}$, then it follows from (27) that the equilibrium point $x(k) \equiv x_e$ is finite-time convergent with the settling-time function $K(x_0)=1$.

Now, since every point in $\mathcal{Q} \cap \Delta f^{-1}(0)$ is a Lyapunov stable equilibrium point of (1) it follows from Theorem 6.1 that every equilibrium point $x_e \in \mathcal{Q} \cap \Delta f^{-1}(0)$ of (1) with $\mathcal{N} \triangleq \mathcal{B}_\delta(x_e)$ is finite-time semistable. The remainder of the proof now follows as in the proof of Theorem 6.1. ■

Example 6.1: Consider the discrete-time collective dynamics of two agents on $\mathbb{R}^2$ described by

$$x_i(k+1) = x_i(k) + u_i(k), \; x_i(0) = x_{i0}, \; k \in \overline{\mathbb{Z}}_+, i=1,2 \tag{30}$$

where for $k \in \overline{\mathbb{Z}}_+$, $x_1(k), x_2(k) \in \mathbb{R}$, and $u_1(k)$ and $u_2(k)$ are given by

$$u_1(k) = -c\,\mathrm{sign}\,(x_1(k)-x_2(k)) \cdot \min\left\{\frac{|x_1(k)-x_2(k)|}{2c}, \left[\frac{|x_1(k)-x_2(k)|}{2}\right]^\alpha\right\} \tag{31}$$

$$u_2(k) = -c\,\mathrm{sign}\,(x_2(k)-x_1(k)) \cdot \min\left\{\frac{|x_2(k)-x_1(k)|}{2c}, \left[\frac{|x_2(k)-x_1(k)|}{2}\right]^\alpha\right\} \tag{32}$$

where $\alpha \in (0,1)$ and $c \in \mathbb{R}_+$.

First, note that $\mathcal{M} = \{x \in \mathbb{R}^2 : x = \beta \mathbf{e}_2, \beta \in \mathbb{R}\}$, where $\mathbf{e}_2 \triangleq [1\ 1]^{\mathrm{T}}$, is the set of equilibria for (30) with (31) and (32), $u_1(k) = -u_2(k)$, and $x_1(k)+x_2(k) = 2\beta$, $k \in \overline{\mathbb{Z}}_+$. Now, consider the Lyapunov function candidate $V(x) = (x-\beta\mathbf{e}_2)^{\mathrm{T}}(x-\beta\mathbf{e}_2)$. Note that $V(x)>0$, $x \in \mathbb{R}^2$, $x \ne \beta\mathbf{e}_2$, and $V(x)=0$ if and only if $x_1=x_2=\beta$. Furthermore, note that $V(\cdot)$ is radially unbounded and since $x_1(k)+x_2(k)=2\beta$, $k \in \overline{\mathbb{Z}}_+$, it follows that $|x_1-x_2|^2 = 2V(x)$.

Next, note that

$$\begin{aligned}
\Delta V(x) &= (x_1+u_1-\beta)^2 + (x_2+u_2-\beta)^2 \\
&\quad - (x_1-\beta)^2 + (x_2-\beta)^2 \\
&= u_1(2x_1+u_1-2\beta) + u_2(2x_2+u_2-2\beta) \\
&= 2u_1[x_1-x_2+u_1] \\
&= -2c\min\left\{\frac{|x_1-x_2|}{2c}, \left[\frac{|x_1-x_2|}{2}\right]^\alpha\right\} \\
&\quad \cdot \left[|x_1-x_2| - c\min\left\{\frac{|x_1-x_2|}{2c}, \left[\frac{|x_1-x_2|}{2}\right]^\alpha\right\}\right] \\
&\le -2c^2\left[\min\left\{\frac{|x_1-x_2|}{2c}, \left[\frac{|x_1-x_2|}{2}\right]^\alpha\right\}\right]^2 \\
&= -2c^2\min\left\{\frac{|x_1-x_2|^2}{4c^2}, \left[\frac{|x_1-x_2|}{2}\right]^{2\alpha}\right\} \\
&= -2c^2\min\left\{\frac{V(x)}{2c^2}, \left[\frac{V(x)}{2}\right]^\alpha\right\} \\
&\le 0, \quad x \in \mathbb{R}^2
\end{aligned} \tag{33}$$

and hence, it follows from Corollary 6.1 that (30), with control inputs (31) and (32), is finite-time semistable with the settling-time function given by either

$$K(x_0) \le \left\lceil \log_{[1-2^{(1-\alpha)}c^2V(x_0)^{\alpha-1}]} \frac{2c^{\frac{2}{1-\alpha}}}{V(x_0)} \right\rceil + 1$$
$$V(x_0) > 2c^{\frac{1}{1-\alpha}} \tag{34}$$

or

$$K(x_0) = 1, \quad V(x_0) \le 2c^{\frac{1}{1-\alpha}}. \tag{35}$$

The system trajectory and control profile of (30), with control inputs (31) and (32), for the initial condition $x_0 = [0\quad 20]^{\mathrm{T}}$, $c=2$, and $\alpha = 0.5$ are shown in Fig. 2 . The guaranteed settling-time function is given by $K(x_0)=5$, whereas the achieved finite-time convergence step is 4. △

Corollary 6.2: Consider the nonlinear discrete-time dynamical system (1). Assume that there exist a continuous nonnegative function $V : \mathcal{D} \to \overline{\mathbb{R}}_+$ such that $\Delta V^{-1}(0) = \Delta f^{-1}(0)$, a real number $c>0$, an open neighborhood $\mathcal{Q} \subseteq \mathcal{D}$ such that $\mathcal{Q} \cap \Delta f^{-1}(0)$ is nonempty, and

$$\Delta V(x) \le -\min\{V(x), c\}, \quad x \in \mathcal{Q} \setminus \Delta f^{-1}(0). \tag{36}$$

If every equilibrium point $x_e \in \mathcal{Q} \cap \Delta f^{-1}(0)$ is a Lyapunov stable equilibrium point of (1), then every equilibrium point $x_e \in \mathcal{Q} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable. Moreover,

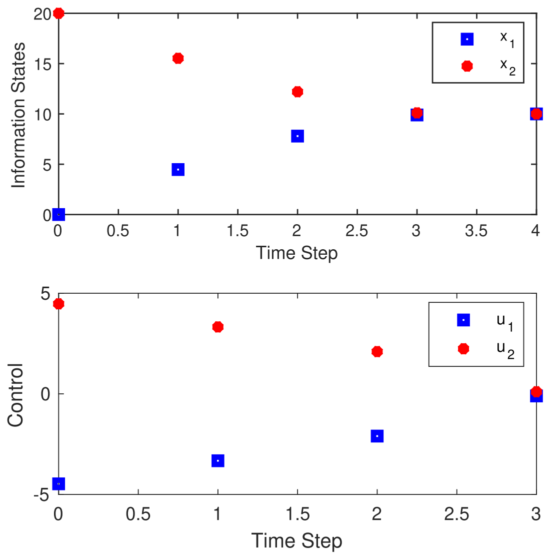


Fig. 2. Information states and control inputs versus time of (30) with control inputs (31) and (32). The two agents achieve finite-time consensus in 4 s.

there exist an open neighborhood $\mathcal{N}$ of $\mathcal{Q}\cap\Delta f^{-1}(0)$ and a settling-time function $K:\mathcal{N}\to\overline{\mathbb{Z}}_+$ such that

$$K(x_0)\leq\left\lceil\frac{V(x_0)}{c}\right\rceil,\quad x_0\in\mathcal{N} \tag{37}$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{Q}=\mathcal{D}$, then (1) is finite-time semistable. Finally, if $\mathcal{D}=\mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (36) holds on $\mathbb{R}^n$, then every equilibrium point $x_{\mathrm{e}}\in\Delta f^{-1}(0)$ of (1) is globally finite-time semistable.

Proof: Consider the scalar discrete-time nonlinear dynamical system given by

$$z(k+1)=z(k)-\mathrm{sign}(z(k))\mathrm{min}\{|z(k)|,c\}$$
$$z(0)=z_0,\quad k\geq 0 \tag{38}$$

where $z(k)\in\mathbb{R}$, $k\in\overline{\mathbb{Z}}_+$, and $c>0$. Note that the right-hand side of (38) is a generalized deadzone function, and hence, by Lemma 6.1 the zero solution $z(k)\equiv 0$ to (38) is globally finite-time stable. Furthermore, note that if $|z(k)|\leq c$, $k\in\overline{\mathbb{Z}}_+$, then $z(k+1)=z(k)-\mathrm{sign}(z(k))|z(k)|=0$, and hence, i) of Definition 5.1 is satisfied with $\mathcal{N}=\mathcal{D}=\mathbb{R}$ and with the settling-time function $\hat{K}(z_0)$ given by

$$\hat{K}(z_0)=\left\lceil\frac{|z_0|}{c}\right\rceil. \tag{39}$$

Next, using identical arguments as in the proof of Corollary 6.1 with $z=V(x)$ and $w(z)=w(V(x))=V(x)-\min\{V(x),c\}$, it follows that the equilibrium point $x_{\mathrm{e}}\in\mathcal{Q}\cap\Delta f^{-1}(0)$ of (1) is finite-time semistable with settling-time function

$$K(x_0)\leq\left\lceil\frac{V(x_0)}{c}\right\rceil,\quad x_0\in\mathcal{N}. \tag{40}$$

The remainder of the proof now follows as in the proof of Theorem 6.1. ■

Example 6.2: Consider the two agent system given by (30) with control inputs $u_1(k)$ and $u_2(k)$ given by

$$u_1(k)=-\mathrm{sign}\,(x_1(k)-x_2(k))\,\min\left\{\frac{|x_1(k)-x_2(k)|}{2},c\right\} \tag{41}$$

$$u_2(k)=-\mathrm{sign}\,(x_2(k)-x_1(k))\,\min\left\{\frac{|x_2(k)-x_1(k)|}{2},c\right\} \tag{42}$$

where $c\in\mathbb{R}_+$. First, note that $\mathcal{M}=\{x\in\mathbb{R}^2:x=\beta\mathbf{e}_2,\beta\in\mathbb{R}\}$ is the set of equilibria for (30) with (31) and (32), $u_1(k)=-u_2(k)$, and $x_1(k)+x_2(k)=2\beta$, $k\in\overline{\mathbb{Z}}_+$. Now, consider the Lyapunov function candidate $V(x)=(x-\beta\mathbf{e}_2)^{\mathrm{T}}(x-\beta\mathbf{e}_2)$. Note that $V(x)>0$, $x\in\mathbb{R}^2$, $x\neq\beta\mathbf{e}_2$, and $V(x)=0$ if and only if $x_1=x_2$. Furthermore, note that $V(\cdot)$ is radially unbounded and since $x_1(k)+x_2(k)=2\beta, k\in\overline{\mathbb{Z}}_+$, it follows that $|x_1-x_2|^2=2V(x)$.

Next, note that

$$\begin{aligned}\Delta V(x)&=(x_1+u_1-\alpha)^2+(x_2+u_2-\alpha)^2\\&\quad-(x_1-\alpha)^2+(x_2-\alpha)^2\\&=2u_1[x_1-x_2+u_1]\\&=-2\mathrm{min}\left\{\frac{|x_1-x_2|}{2},c\right\}\\&\quad\cdot\left[|x_1-x_2|-\min\left\{\frac{|x_1-x_2|}{2},c\right\}\right]\\&\leq-2\left[\min\left\{\frac{|x_1-x_2|}{2},c\right\}\right]^2\\&=-2\min\left\{\frac{|x_1-x_2|^2}{4},c^2\right\}\\&=-2\min\left\{\frac{V(x)}{2},c^2\right\}\\&\leq 0,\quad x\in\mathbb{R}^2\end{aligned} \tag{43}$$

and hence, it follows from Corollary 6.2 that (30), with control inputs (41) and (42), is finite-time semistable with the settling-time function

$$K(x_0)\leq\left\lceil\frac{V(x_0)}{2c^2}\right\rceil. \tag{44}$$

The system trajectory and control profile of (30) with control inputs (41) and (42) for the initial condition $x_0=[0\quad 20]^{\mathrm{T}}$ and $c=3$ are shown in Fig. 3 . The guaranteed settling-time function is given by $K(x_0)=12$, whereas the achieved finite-time convergence step is 4. △

Finally, we present partial converse theorems to Theorem 6.1 and Corollaries 6.1 and 6.2.

Theorem 6.2: Consider the nonlinear discrete-time dynamical system (1) and let $\mathcal{N}$ be as in Definition 5.1. If every equilibrium point $x_{\mathrm{e}}\in\mathcal{N}\cap\Delta f^{-1}(0)$ of (1) is finite-time semistable, then there exist a nonnegative lower semicontinuous function $V:\mathcal{N}\to\overline{\mathbb{R}}_+$ and a continuous function $w:\mathbb{R}\to\mathbb{R}$ such that $V(f(x))\leq w(V(x))$, $x\in\mathcal{N}$, where $w(\cdot)$ is of class $\mathcal{W}_{\mathrm{d}}$, and the zero solution $z(k)\equiv 0$ to (15) is finite-time convergent.

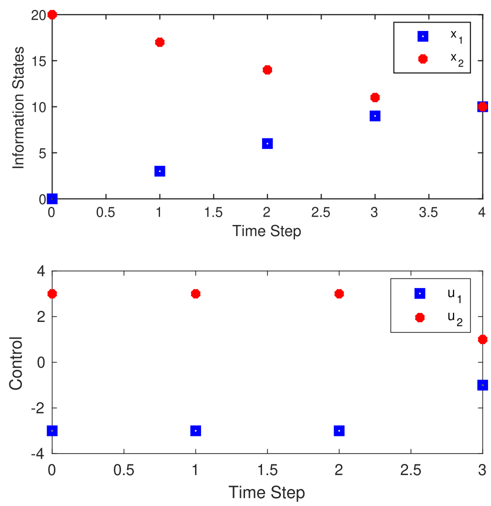


Fig. 3. Information states and control inputs versus time of (30) with control inputs (41) and (42). The two agents achieve finite-time consensus in 4 s.

Proof: First, note that it follows from Proposition 5.1 that the settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ is lower semicontinuous on $\mathcal{N}$. Next, define $V : \mathcal{N} \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{k\geq 0} \frac{1+bk}{1+ak} \left[K(s(k,x))\right]^{\beta} \tag{45}$$

where $\beta > 2$, $\beta \in \mathbb{Z}_+$, and $b > a > 0$. Note that $V(\cdot)$ is lower semicontinuous and nonnegative, and, by $s(K(x)+k,x) \in \mathcal{N} \cap \Delta f^{-1}(0)$ for all $x \in \mathcal{N}$ and $k \in \mathbb{Z}_+$, $\Delta V(x) = 0$, $x \in \mathcal{N} \cap \Delta f^{-1}(0)$. Now, note that it follows from the definition of $K(\cdot)$ that the supremum in the definition of $V(s(1,x))$ is reached at some time $\hat{k}$ such that $0 \leq \hat{k} \leq K(x)$. If $\hat{k} < K(x)$, then

$$\begin{aligned} V(s(1,x)) &= \frac{1+b\hat{k}}{1+a\hat{k}} \left[K(s(\hat{k}+1,x))\right]^{\beta} \\ &= \left[1 - \frac{b-a}{(1+b\hat{k}+b)(1+a\hat{k})}\right] \\ &\quad \frac{1+b\hat{k}+b}{1+a\hat{k}+a} \left[K(s(\hat{k}+1,x))\right]^{\beta} \\ &\leq \left[1 - \frac{a(b-a)}{b\left[1+aK(x)\right]^2}\right] V(x), \quad K(x) > \hat{k}. \end{aligned} \tag{46}$$

Alternatively, if $\hat{k} = K(x)$, then $V(s(1,x)) = 0$, which implies $x \in \mathcal{N} \cap \Delta f^{-1}(0)$.

Next, if $x \notin \mathcal{N} \cap \Delta f^{-1}(0)$, then

$$V(x) = \sup_{k\geq 0} \frac{1+bk}{1+ak} \left[K(s(k,x))\right]^{\beta} \geq \left[K(x)\right]^{\beta} \geq 1 \tag{47}$$

and hence, $[1+aK(x)]^{\beta} \leq (1+a)^{\beta} V(x)$. Now, (46) yields

$$\begin{aligned} &V(f(x)) - V(x) \\ &\leq -\frac{a(b-a)}{b} V(x) \left(1+K(x)\right)^{-2} \\ &\leq -\frac{a(b-a)}{b} V(x) \left[(1+a)^{\beta} V(x)\right]^{-\frac{2}{\beta}} \\ &= -\frac{a(b-a)}{b(1+a)^2} V(x)^{\frac{\beta-2}{\beta}} \\ &\leq -\frac{a(b-a)}{b(1+a)^2} \min\left\{\frac{b(1+a)^2}{a(b-a)} V(x), V(x)^{\frac{\beta-2}{\beta}}\right\}. \end{aligned} \tag{48}$$

Now, letting $\alpha = \frac{\beta-2}{\beta} \in (0,1)$ and $c = \frac{a(b-a)}{b(1+a)^2} > 0$, (48) becomes

$$V(f(x)) \leq V(x) - c\min\left\{\frac{V(x)}{c}, V(x)^{\alpha}\right\}. \tag{49}$$

Finally, using Lemma 6.1, it follows that the zero solution $z(k) \equiv 0$ to (15), with the class $\mathcal{W}_{\rm d}$ function $w(z) = z - c\,\mathrm{sign}(z)\min\{\frac{|z|}{c}, |z|^{\alpha}\}$, is finite-time convergent. ■

Theorem 6.3: Consider the nonlinear discrete-time dynamical system (1), let $\alpha \in (0,1)$, and let $\mathcal{N}$ be as in Definition 5.1. If every equilibrium point $x_{\rm e} \in \mathcal{N} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable, then there exist a nonnegative lower semicontinuous function $V : \mathcal{N} \to \overline{\mathbb{R}}_+$ and real numbers $\alpha \in (0,1)$ and $c > 0$ such that

$$V(f(x)) \leq V(x) - c\min\left\{\frac{V(x)}{c}, V(x)^{\alpha}\right\}, \quad x \in \mathcal{N}. \tag{50}$$

Proof: The proof is identical to that of Theorem 6.2 with class $\mathcal{W}_{\rm d}$ function $w : \mathbb{R} \to \mathbb{R}$ given by $w(z) = z - c\,\mathrm{sign}(z)\min\{\frac{|z|}{c}, |z|^{\alpha}\}$. ■

Theorem 6.4: Consider the nonlinear discrete-time dynamical system (1) and let $\mathcal{N}$ be as in Definition 5.1. If every equilibrium point $x_{\rm e} \in \mathcal{N} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable, then there exist a nonnegative lower semicontinuous function $V : \mathcal{N} \to \mathbb{R}$ and a real number $c > 0$ such that

$$V(f(x)) \leq V(x) - \min\left\{V(x), c\right\}, \quad x \in \mathcal{N}. \tag{51}$$

Proof: First, note that it follows from Proposition 5.1 that the settling-time function $K : \mathcal{N} \to \overline{\mathbb{Z}}_+$ is lower semicontinuous on $\mathcal{N}$. Next, define $V : \mathcal{N} \to \overline{\mathbb{R}}_+$ by $V(x) \triangleq cK(x)$, where $c > 0$. Note that $V(\cdot)$ is lower semicontinuous and nonnegative, and, by $s(K(x)+k,x) \in \mathcal{N} \cap \Delta f^{-1}(0)$ for all $x \in \mathcal{N}$ and $k \in \mathbb{Z}_+$, $\Delta V(x) = 0$, $x \in \mathcal{N} \cap \Delta f^{-1}(0)$. Now, since every equilibrium point $x_{\rm e} \in \mathcal{N} \cap \Delta f^{-1}(0)$ of (1) is finite-time semistable and $K(s(1,x)) = K(x) - 1$, it follows that

$$V(f(x)) = cK(s(1,x)) = c(K(x)-1) = V(x) - c \tag{52}$$

for $x \notin \mathcal{N} \cap \Delta f^{-1}(0)$, and hence

$$V(f(x)) - V(x) = -c \leq -\min\{V(x), c\}. \tag{53}$$

Finally, using Lemma 6.1, it follows that the zero solution $z(k) \equiv 0$ to (15) with the class $\mathcal{W}_{\rm d}$ function $w(z) = z - \mathrm{sign}(z)\min\{z, c\}$, is finite-time convergent. ■

VII. Thermodynamic-Based Architecture for Asymptotic Network Consensus

In this section, we develop a thermodynamically motivated information consensus framework for discrete-time nonlinear network systems to achieve semistability and state equipartition. The consensus problem we address in this section appears frequently in coordination of multiagent network systems and involves finding a dynamic algorithm that enables a group of agents in a network to agree upon certain quantities of interest with undirected and directed information flow. Here, we use graph-theoretic notions to represent a dynamical network and present solutions to the consensus problem for networks with *undirected communication graph topologies* (or information flows).

Specifically, let $\mathfrak{G}(\mathcal{C}) = (\mathcal{V}, \mathcal{E})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of *edges* $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and a *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ such that $\mathcal{C}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, while $\mathcal{C}_{(i,j)} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent j can receive information from agent i, but not necessarily vice versa.

A *graph* or *undirected graph* $\mathfrak{G}(\mathcal{C})$ associated with the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. A graph is *all-to-all connected* if every node of $\mathfrak{G}(\mathcal{C})$ is connected to every other node of $\mathfrak{G}(\mathcal{C})$. Weighted graphs can also be considered here; however, since this extension does not alter any of the conceptual results in this article, we do not consider this extension for simplicity of exposition. Finally, we denote the value of the node $i \in \{1, \ldots, q\}$ at discrete-time instant k by $x_i(k) \in \mathbb{R}$. The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is, $\lim_{k\to\infty} x_i(k) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, q$, where α depends on the system initial conditions.

Consider the q discrete-time dynamical agents $\mathcal{G}_i$ with dynamics given by

$$x_i(k+1) = x_i(k) + u_i(k), \quad x_i(0) = x_{i0}, \quad k \in \overline{\mathbb{Z}}_+ \tag{54}$$

where for each agent $i \in \{1, \ldots, q\}$, $x_i(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, denotes the information state of agent $\mathcal{G}_i$ and $u_i(k) \in \mathbb{R}$, $k \in \overline{\mathbb{Z}}_+$, denotes the information control input of agent $\mathcal{G}_i$. The nonlinear consensus protocol is given by

$$u_i(k) = \sum_{j=1, j\neq i}^{q} \phi_{ij}\left(x_i(k), x_j(k)\right), \quad i = 1, \ldots, q \tag{55}$$

where $\phi_{ij}(\cdot,\cdot)$, $i, j = 1, \ldots, q$, are continuous functions characterizing the information exchange between agents $\mathcal{G}_j$ and $\mathcal{G}_i$.

In this case, the closed-loop system (54) and (55) is given by

$$x_i(k+1) = x_i(k) + \sum_{j=1, j\neq i}^{q} \phi_{ij}\left(x_i(k), x_j(k)\right), \quad x_i(0) = 0$$
$$k \in \overline{\mathbb{Z}}_+, \quad i = 1, \ldots, q \tag{56}$$

or, equivalently, in vector form

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \tag{57}$$

where $x(k) \triangleq [x_1(k), \ldots, x_q(k)]^{\mathrm{T}}$ and $f = [f_1, \ldots, f_q]^{\mathrm{T}} : \mathcal{D} \subseteq \mathbb{R}^q \to \mathbb{R}^q$ is such that

$$f_i(x(k)) = x_i(k) + \sum_{j=1, j\neq i}^{q} \phi_{ij}\left(x_i(k), x_j(k)\right), \quad i = 1, \ldots, q. \tag{58}$$

Note that $\mathcal{G}$ given by (57) describes an interconnected network where information states are updated using a distributed controller involving neighbor-to-neighbor interaction between agents.

Remark 7.1: Although our results can be directly extended to the case where (54) and (55) describe the dynamics of an aggregate multiagent system with an aggregate state vector $x(k) = [x_1^{\mathrm{T}}(k), \ldots, x_q^{\mathrm{T}}(k)]^{\mathrm{T}} \in \mathbb{R}^{Nq}$, where $x_i(k) \in \mathbb{R}^N$ and $u_i(k) \in \mathbb{R}^N$, $i = 1, \ldots, q$, by using Kronecker calculus, for simplicity of exposition, we focus on individual agent states evolving in $\mathbb{R}$ (i.e., $N = 1$).

The following definition and assumptions are needed for the main result of this section.

Definition 7.1 (see [34]): A directed graph $\mathfrak{G}(\mathcal{C})$ is *strongly connected* if for any ordered pair of vertices (i, j), $i \neq j$, there exists a *path* (i.e., sequence of arcs) leading from i to j.

Recall that $\mathcal{C} \in \mathbb{R}^{q\times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{C}$ is cogredient to a lower block triangular matrix, if and only if $\mathfrak{G}(\mathcal{C})$ is strongly connected (see [34, Th. 2.7]). Furthermore, note that for an undirected graph $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$, and hence, every undirected graph is balanced.

To ensure a thermodynamically consistent information flow model, we make the following assumptions on the information flow functions $\phi_{ij}(\cdot,\cdot)$, $i = 1, \ldots, q$.

Assumption 7.1: The connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent dynamical system $\mathcal{G}$ given by (57) is defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \phi_{ij}(x_i, x_j) \equiv 0 \\ 1, & \text{otherwise} \end{cases} \quad i \neq j, \quad i, j = 1, \ldots, q$$

and $\mathcal{C}_{(i,i)} \triangleq -\sum_{m=1, m\neq i}^{q} \mathcal{C}_{(i,m)}$, $i = 1, \ldots, q$, with $\operatorname{rank} \mathcal{C} = q - 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i \neq j$, $\phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$.

Assumption 7.2: For $i, j = 1, \ldots, q$, $(x_i - x_j)\phi_{ij}(x_i, x_j) \leq 0$, $x_i, x_j \in \mathbb{R}$.

Assumption 7.3: For $i, j = 1, \ldots, q$, $\frac{\Delta x_i - \Delta x_j}{x_i - x_j} \geq -1$, $x_i \neq x_j$, where $\Delta x_m(k) \triangleq x_m(k+1) - x_m(k)$.

The condition that $\phi(x_i, x_j) = 0$ if and only if $x_i = x_j$, $i \neq j$, implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *connected*, and hence, can share information; alternatively, $\phi_{ij}(x_i, x_j) \equiv 0$ implies that agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are *disconnected*, and hence, cannot share information.

Assumption 7.1 implies that if the energies or information in the connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$ are equal, then energy or information exchange between these agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and $\operatorname{rank} \mathcal{C} = q - 1$, then it follows that the connectivity matrix $\mathcal{C}$ is irreducible, which implies that for any pair of agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, of $\mathcal{G}$ there exists a sequence of information connectors (information arcs) of $\mathcal{G}$ that connect agents $\mathcal{G}_i$ and $\mathcal{G}_j$.

Assumption 7.2 implies that energy or information flows from more energetic or information rich agents to less energetic or information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (i.e., energy in transition) must flow in the direction of lower temperatures. Finally, Assumption 7.3 implies that the energy or information difference between any consecutive time instants is monotonic for any pair of connected agents $\mathcal{G}_i$ and $\mathcal{G}_j$, $i \neq j$, that is, $[x_i(k+1) - x_j(k+1)][x_i(k) - x_j(k)] \geq 0$ for all $x_i(k) \neq x_j(k)$, $k \geq 0$, $i, j = 1, \ldots, q$. It is important to note here that both finite-time consensus controllers in Examples 6.1 and 6.2 satisfy Assumptions 7.1–7.3, and hence, satisfy basic thermodynamic principles. For further details on Assumptions 7.1–7.3, see [28].

For the next result, $\mathbf{e}_n$ or $\mathbf{e} \in \mathbb{R}^n$ denote the ones vector of order n, that is, $\mathbf{e}_n \triangleq [1, \ldots, 1]^{\mathrm{T}}$.

Theorem 7.1: Consider the discrete-time multiagent dynamical system (56) or, equivalently, (57). Assume that Assumptions 7.1–7.3 hold, and $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$. Then, for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}$ is a globally semistable equilibrium state of (57). Furthermore, $x(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ as $k \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a globally semistable equilibrium state.

Proof: See Appendix C. ■

Note that in the special case of an all-to-all communication graph topology Assumption 7.3 can always be satisfied. Specifically, consider the consensus protocol given by

$$\begin{aligned} u_i(k) &= \sum_{j=1, j\neq i}^{q} \phi_{ij}\left(x_i(k), x_j(k)\right) \\ &= \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\left[\sigma(x_j(k)) - \sigma(x_i(k))\right] \\ i &= 1, \ldots, q \end{aligned} \tag{59}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is such that $\sigma(z) = \beta z$, with $\beta > 0$, and assume $\mathcal{C}_{(i,j)} = 1$ for all $i, j = 1, \ldots, q$, $i \neq j$. In this case, if $\beta \leq 1/q$, then Assumption 7.3 holds. To see this, first note that if $x_i(k) > x_j(k)$, $i \neq j$, $i, j = 1, \ldots, q$, $k \geq 0$, then

$$\begin{aligned} &\Delta x_i(k) - \Delta x_j(k) \\ &\quad = u_i(x(k)) - u_j(x(k)) \\ &\quad = \sum_{h=1}^{q}\left[\sigma(x_h(k)) - \sigma(x_i(k)) - \sigma(x_h(k)) + \sigma(x_j(k))\right] \\ &\quad = -q\beta\left(x_i(k) - x_j(k)\right) \\ &\quad \geq -\left(x_i(k) - x_j(k)\right), \quad i \neq j, \quad i, j = 1, \ldots, q \end{aligned} \tag{60}$$

and hence, Assumption 7.3 holds. Alternatively, if $x_i(k) - x_j(k) < 0$, $i \neq j$, $i = 1, \ldots, q$, $k \geq 0$, then analogously it can be shown that Assumption 7.3 holds.

Next, we provide explicit connections of the proposed thermodynamic-based consensus control architecture developed in this section to the recently developed notion of discrete thermodynamics [28]. To develop these connections the following definition of entropy is needed.

Definition 7.2: For the distributed discrete-time consensus protocol $\mathcal{G}$ given by (57), a function $\mathcal{S} : \mathbb{R}^q \to \mathbb{R}$ satisfying

$$\mathcal{S}(x(k_2)) \geq \mathcal{S}(x(k_1)), \quad k_2 \geq k_1 \geq 0 \tag{61}$$

is called an *entropy* of $\mathcal{G}$.

The following theorem gives an explicit expression for the entropy function of the closed-loop, discrete-time multiagent dynamical system $\mathcal{G}$ given by (57).

Theorem 7.2: Consider the closed-loop, discrete-time multiagent dynamical system $\mathcal{G}$ given by (57) and assume that Assumptions 7.2 and 7.3 hold. Then, the function $\mathcal{S} : \mathbb{R}^q \to \mathbb{R}$ given by

$$\mathcal{S}(x) = \mathbf{e}^{\mathrm{T}}\mathbf{log}_e(c\,\mathbf{e} + x) - q\log_e c, \quad x \in \mathbb{R}^q \tag{62}$$

where $\mathbf{log}_e(c\,\mathbf{e} + x)$ denotes the vector natural logarithm given by $[\log_e(c + x), \ldots, \log_e(c + x)]^{\mathrm{T}}$ and $c > \|x\|_\infty$, is an entropy function of $\mathcal{G}$.

Proof: Since $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$, $i \neq j$, $i, j = 1, \ldots, q$, and $c > \|x\|_\infty$, it follows that

$$\begin{aligned} &\Delta\mathcal{S}(x(k)) \\ &\quad = \sum_{i=1}^{q}\log_e\left[1 + \frac{\Delta x_i(k)}{c + x_i(k)}\right] \\ &\quad \geq \sum_{i=1}^{q}\left[\frac{\Delta x_i(k)}{c + x_i(k)}\right]\left[1 + \frac{\Delta x_i(k)}{c + x_i(k)}\right]^{-1} \\ &\quad = \sum_{i=1}^{q}\frac{\Delta x_i(k)}{c + x_i(k) + \Delta x_i(k)} \\ &\quad = \sum_{i=1}^{q}\frac{\Delta x_i(k)}{c + x_i(k+1)} \\ &\quad = \sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q}\frac{\phi_{ij}(x_i(k), x_j(k))}{c + x_i(k+1)} \\ &\quad = \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}\left[\frac{\phi_{ij}(x_i(k), x_j(k))}{c + x_i(k+1)} - \frac{\phi_{ij}(x_i(k), x_j(k))}{c + x_j(k+1)}\right] \\ &\quad = \sum_{i=1}^{q-1}\sum_{j=1, j\neq i}^{q}\frac{\phi_{ij}(x_i(k), x_j(k))[x_j(k+1) - x_i(k+1)]}{[c + x_i(k+1)]\,[c + x_j(k+1)]} \\ &\quad \geq 0, \quad k \in \overline{\mathbb{Z}}_+ \end{aligned} \tag{63}$$

where in (63), we use the fact that $\log_e(1 + x) \geq \frac{x}{x+1}$, $x > -1$. Now, summing (63) over $\{k_1, \ldots, k_2 - 1\}$ yields (61). ■

Note that it follows from (63) that the entropy function given by (62) satisfies (61) as an equality for an equilibrium (equipartitioned) process and as a strict inequality for a nonequilibrium (nonequipartitioned) process. The entropy expression given by (62) is identical in form to the Boltzmann entropy for statistical thermodynamics and the Shannon entropy characterizing the amount of information [28]. In addition, note that $\mathcal{S}(x)$ given by (62) achieves a maximum when all the information states x_i, $i = 1, \ldots, q$, are equal [28]. Inequality (61) is a generalization of Clausius' inequality for equilibrium and nonequilibrium thermodynamics as well as reversible and irreversible thermodynamics as applied to adiabatically isolated discrete-time thermodynamic systems. For details, see [28].

Example 7.1: Consider a network of six dynamical agents $\mathcal{G}$ with a weakly connected, undirected communication graph topology shown in Fig. 4 with dynamics given by (56). The

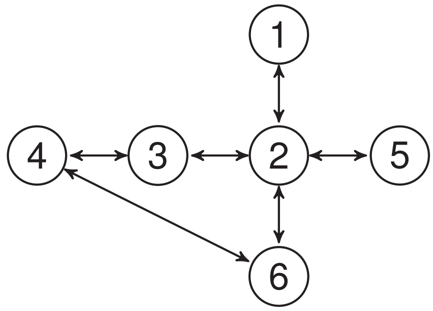


Fig. 4. Communication graph topology for the six mobile agents.

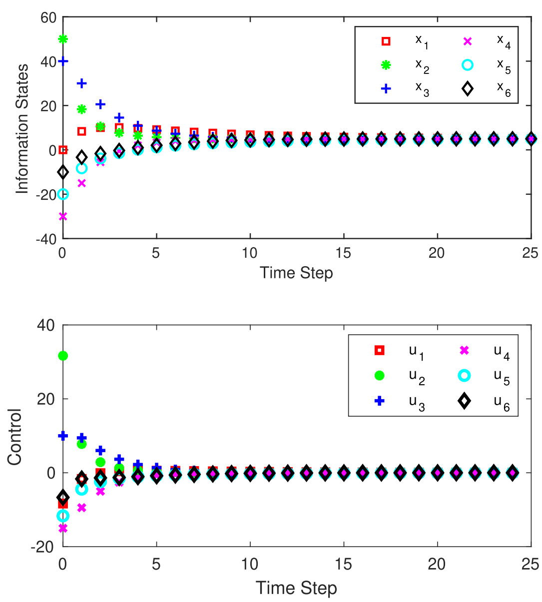


Fig. 5. Information states and control inputs versus time for the linear consensus protocol.

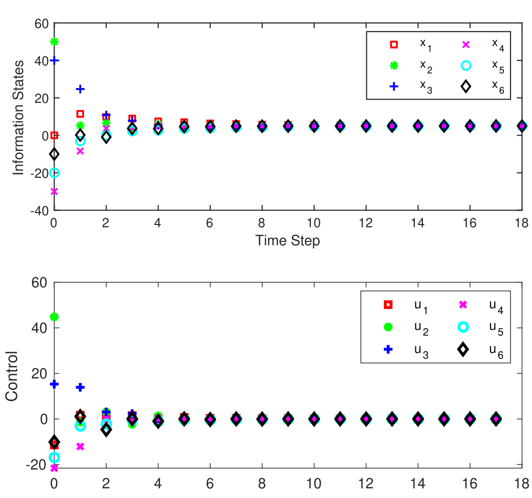


Fig. 6. Information states and control inputs versus time for the nonlinear consensus protocol. Note that the nonlinear consensus control protocol achieves significantly improved convergence as compared to the linear protocol.

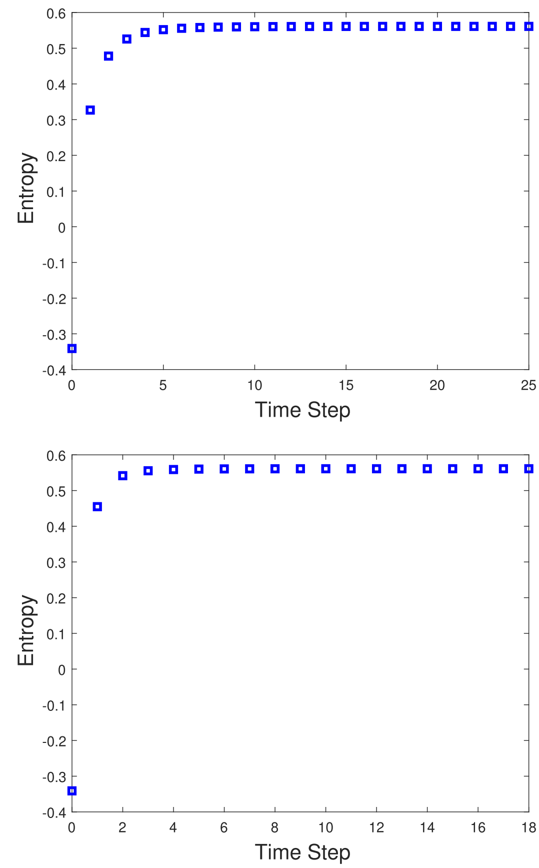


Fig. 7. Total agent entropies versus time (with $c = \|x\|_\infty + 1$) for both control protocols; linear on the top and nonlinear on the bottom.

corresponding connectivity matrix is given by

$$\mathcal{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 \end{bmatrix}.$$

Note that rank $\mathcal{C} = 5$.

Fig. 5 shows the information states and control inputs for the six agents versus time with the linear consensus protocol $\phi_{ij}(x_i, x_j) = -\frac{1}{6}(x_i - x_j)$, $i, j = 1, \ldots, 6$, and initial condition $x_0 = [0, 50, 40, -30, -20, -10]^{\mathrm{T}}$. Fig. 6 shows the information states and control inputs for the six agents versus time with the nonlinear consensus protocol $\phi_{ij}(x_i, x_j) = -\frac{1}{2}(\mathrm{sign}(x_i)|x_i|^{0.8} - \mathrm{sign}(x_j)|x_j|^{0.8})$, $i, j = 1, \ldots, 6$, and initial condition $x_0 = [0, 50, 40, -30, -20, -10]^{\mathrm{T}}$. Note that for both information flow functions $\phi_{ij}(x_i, x_j)$, $i, j = 1, \ldots, 6$, $i \neq j$, considered, Assumptions 7.1–7.3 hold and by Theorem 7.1 the information states converge to $x_{\mathrm{e}} = \frac{1}{6}\mathbf{e}_6\mathbf{e}_6^{\mathrm{T}} x_0 = 5\mathbf{e}_6$. Finally, Fig. 7 shows the total agent entropies versus time for both control protocols with $c = \|x\|_\infty + 1$. △

VIII. Conclusion

This article extends the notions semistability and finite-time semistability to discrete-time nonlinear dynamical systems having a continuum of equilibria. In particular, Lyapunov and converse Lyapunov theorems for semistability and finite-time semistability are established. These results are then used to develop a thermodynamic-based framework for addressing consensus problems for multiagent dynamical systems with discrete-time information transmission between agents in the network. Specifically, nonlinear network protocols are designed that guarantee asymptotic and finite-time convergence to Lyapunov stable equilibria over a discrete network of dynamic agents. Our analysis relies on several tools from algebraic graph theory, semistability, finite-time semistability, and dynamical thermodynamics [28]. Future research will explore extending the proposed framework to include directed communication graph topologies as well as developing hybrid information consensus algorithms for achieving finite-time coordination tasks with intermittent communication between agents.

Appendix A
Proof of Proposition 3.2

The proof of this result is similar to its continuous-time counterpart given in [28]. For completeness of exposition, however, we provide a self-contained proof here.

To show that i) implies ii), suppose (1) is semistable and let $x_{\rm e} \in \Delta f^{-1}(0)$. Since $x_{\rm e}$ is Lyapunov stable, it follows that there exists $\delta = \delta(x_{\rm e}) > 0$ and a class $\mathcal{K}$ function $\alpha(\cdot)$ such that if $\|x_0 - x_{\rm e}\| \le \delta$, then $\|x(k) - x_{\rm e}\| \le \alpha(\|x_0 - x_{\rm e}\|)$, $k \in \mathbb{Z}_+$. Without loss of generality, we may assume that δ is such that $\overline{\mathcal{B}_\delta(x_{\rm e})}$, where $\overline{(\ )}$ denotes closure, is contained in the domain of semistability of (1). Hence, for every $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, $\lim_{k\to\infty} x(k) = x^* \in \Delta f^{-1}(0)$ and, consequently, $\lim_{k\to\infty} \mathrm{dist}(x(k), \Delta f^{-1}(0)) = 0$.

For each $\varepsilon > 0$ and $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, define $K_{x_0}(\varepsilon)$ to be the infimum of K with the property that $\mathrm{dist}(x(k), \Delta f^{-1}(0)) < \varepsilon$ for all $k \ge K$, that is, $K_{x_0}(\varepsilon) \triangleq \inf\{K : \mathrm{dist}(x(k), \Delta f^{-1}(0)) < \varepsilon, k \ge K\}$. For each $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, the function $K_{x_0}(\varepsilon)$ is nonnegative and nonincreasing in ε, and $K_{x_0}(\varepsilon) = 0$ for sufficiently large ε.

Next, let $K(\varepsilon) \triangleq \sup\{K_{x_0}(\varepsilon) : x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}\}$. We claim that K is well defined. To show this, consider $\varepsilon > 0$ and $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$. Since $\mathrm{dist}(s(k, x_0), \Delta f^{-1}(0)) < \varepsilon$ for every $k > K_{x_0}(x_{\rm e})$, it follows from the continuity of s that, for every $\varepsilon > 0$, there exists an open neighborhood $\mathcal{U}$ of x_0 such that $\mathrm{dist}(s(k, z), \Delta f^{-1}(0)) < \varepsilon$ for every $z \in \mathcal{U}$, $k > K_{x_0}(\varepsilon)$. Hence, $\limsup_{z\to x_0} K_z(\varepsilon) \le K_{x_0}(\varepsilon)$ implying that the function $x_0 \longmapsto K_{x_0}(\varepsilon)$ is upper semicontinuous at the arbitrarily chosen point x_0, and hence, on $\overline{\mathcal{B}_\delta(x_{\rm e})}$. Since an upper semicontinuous function defined on a compact set achieves its supremum, it follows that $K(\varepsilon)$ is well defined. The function $K(\cdot)$ is the pointwise supremum of a collection of nonnegative and nonincreasing functions, and hence, is nonnegative and nonincreasing. Moreover, $K(\varepsilon) = 0$ for every $\varepsilon > \max\{\alpha(\|x_0 - x_{\rm e}\|) : x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}\}$.

Let $\psi(\varepsilon) \triangleq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} K(\sigma)\mathrm{d}\sigma + \frac{1}{\varepsilon} \ge K(\varepsilon) + \frac{1}{\varepsilon}$. Note that $K(\cdot)$ is measurable since it is upper semicontinuous, and hence, integrable. The function $\psi(\varepsilon)$ is positive, continuous, strictly decreasing, and $\psi(\varepsilon) \to 0$ as $\varepsilon \to \infty$. Choose $\beta(\cdot) = \psi^{-1}(\cdot)$. Then, $\beta(\cdot)$ is positive, continuous, strictly decreasing, and $\beta(\sigma) \to 0$ as $\sigma \to \infty$. Furthermore, $K(\beta(\sigma)) < \psi(\beta(\sigma)) = \sigma$. Hence, $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \beta(k)$, $k \in \mathbb{Z}_+$.

Next, to show that ii) implies iii), suppose ii) holds and let $x_{\rm e} \in \Delta f^{-1}(0)$. Then, $x_{\rm e}$ is Lyapunov stable. Choosing x_0 sufficiently close to $x_{\rm e}$, it follows from the inequality $\|x(k) - x_{\rm e}\| \le \alpha(\|x_0 - x_{\rm e}\|)$, $k \ge 0$ that trajectories of (1) starting sufficiently close to $x_{\rm e}$ are bounded, and hence, the positive limit set of (1) is nonempty. Since $\lim_{k\to\infty} \mathrm{dist}(x(k), \Delta f^{-1}(0)) = 0$, it follows that the positive limit set is contained in $\Delta f^{-1}(0)$. Now, since every point in $\Delta f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 3.1 that $\lim_{k\to\infty} x(k) = x^*$, where $x^* \in \Delta f^{-1}(0)$ is Lyapunov stable. If $x^* = x_{\rm e}$, then it follows using similar arguments as mentioned above that there exists a class $\mathcal{L}$ function $\hat\beta(\cdot)$ such that $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \|x(k) - x_{\rm e}\| \le \hat\beta(k)$ for every x_0 satisfying $\|x_0 - x_{\rm e}\| < \delta$ and $k \ge 0$. Hence, $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \sqrt{\|x(k) - x_{\rm e}\|}\sqrt{\hat\beta(k)}$, $k \ge 0$. Next, consider the case where $x^* \ne x_{\rm e}$ and let $\alpha_1(\cdot)$ be a class $\mathcal{K}$ function. In this case, note that $\lim_{k\to\infty} \mathrm{dist}(x(k), \Delta f^{-1}(0))/\alpha_1(\|x(k) - x_{\rm e}\|) = 0$, and hence, it follows using similar arguments as mentioned above that there exists a class $\mathcal{L}$ function $\beta(\cdot)$ such that $\mathrm{dist}(x(k), \Delta f^{-1}(0)) \le \alpha_1(\|x(k) - x_{\rm e}\|)\beta(k)$, $k \ge 0$. Finally, note that $\alpha_1 \circ \alpha$ is of class $\mathcal{K}$, and hence, iii) follows immediately.

Finally, to show that iii) implies i), suppose iii) holds and let $x_{\rm e} \in \Delta f^{-1}(0)$. Then, it follows that $\alpha_1(\|x(k) - x_{\rm e}\|) \le \alpha_2(\|x(0) - x_{\rm e}\|)$, $k \ge 0$, that is, $\|x(k) - x_{\rm e}\| \le \alpha(\|x(0) - x_{\rm e}\|)$, where $k \ge 0$ and $\alpha = \alpha_1^{-1} \circ \alpha_2$ is of class $\mathcal{K}$. Then, $x_{\rm e}$ is Lyapunov stable. Since $x_{\rm e}$ was chosen arbitrarily, it follows that every equilibrium point is Lyapunov stable. Furthermore, $\lim_{k\to\infty} \mathrm{dist}(x(k), \Delta f^{-1}(0)) = 0$. Choosing x_0 sufficiently close to $x_{\rm e}$, it follows from the inequality $\|x(k) - x_{\rm e}\| \le \alpha(\|x_0 - x_{\rm e}\|)$, $k \ge 0$ that trajectories of (1) starting sufficiently close to $x_{\rm e}$ are bounded, and hence, the positive limit set of (1) is nonempty. Since every point in $\Delta f^{-1}(0)$ is Lyapunov stable, it follows from Proposition 3.1 that $\lim_{k\to\infty} x(k) = x^*$, where $x^* \in \Delta f^{-1}(0)$ is Lyapunov stable. Hence, by definition, (1) is semistable. ■

Appendix B
Proof of Theorem 4.3

Define the function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{k\ge 0}\left\{\frac{1 + 2\,k}{1 + k}\mathrm{dist}(s(k, x), \Delta f^{-1}(0))\right\}, \quad x \in \mathcal{D}_0. \tag{64}$$

Note that $V(\cdot)$ is well defined since (1) is semistable. Clearly, i) holds. Furthermore, since $V(x) \ge \mathrm{dist}(x, \Delta f^{-1}(0))$, $x \in \mathcal{D}_0$, it follows that ii) holds.

To show that $V(\cdot)$ is continuous on $\mathcal{D}_0 \backslash \Delta f^{-1}(0)$, define $K : \mathcal{D}_0 \backslash \Delta f^{-1}(0) \to \mathbb{Z}_+$ by $K(z) \triangleq \inf\{h : \mathrm{dist}(s(k, z), \Delta f^{-1}(0)) < \mathrm{dist}(z, \Delta f^{-1}(0))/2$ for all $k \ge h$

$> 0\}$, and denote

$$\mathcal{W}_\varepsilon \triangleq \{x \in \mathcal{D}_0 : \mathrm{dist}(s(k,x), \Delta f^{-1}(0)) < \varepsilon, k \in \mathbb{Z}_+\}. \tag{65}$$

Note that $\mathcal{W}_\varepsilon \supset \Delta f^{-1}(0)$ is open and positively invariant, and contains an open neighborhood of $\Delta f^{-1}(0)$. Consider $z \in \mathcal{D}_0 \backslash \Delta f^{-1}(0)$ and define $\lambda \triangleq \mathrm{dist}(z, \Delta f^{-1}(0)) > 0$. Then, it follows from semistability of (1) that there exists $h > 0$ such that $s(h,z) \in \mathcal{W}_{\varepsilon/2}$. Consequently, $s(h+k,z) \in \mathcal{W}_{\varepsilon/2}$ for all $k \in \mathbb{Z}_+$, and hence, it follows that $K(z)$ is well defined. Since $\mathcal{W}_{\varepsilon/2}$ is open, there exists a neighborhood $\mathcal{B}_\sigma(s(K(z),z)) \subset \mathcal{W}_{\varepsilon/2}$. Hence, $\mathcal{N} \triangleq s_{-K(z)}(\mathcal{B}_\sigma(s(K(z),z)))$ is a neighborhood of z and $\mathcal{N} \subset \mathcal{D}_0$.

Next, choose $\eta > 0$ such that $\eta < \lambda/2$ and $\mathcal{D}_\eta(z) \subset \mathcal{N}$. Then, for every $k > K(z)$ and $y \in \mathcal{B}_\eta(z)$

$$\frac{1+2\,k}{1+k}\mathrm{dist}(s(k,y), \Delta f^{-1}(0)) \le 2\mathrm{dist}(s(k,y), \Delta f^{-1}(0)) \le \lambda.$$

Therefore, for every $y \in \mathcal{B}_\eta(z)$

$$\begin{aligned} V(z) - V(y) &= \sup_{k\ge 0}\left\{\frac{1+2\,k}{1+k}\mathrm{dist}(s(k,z), \Delta f^{-1}(0))\right\} \\ &\quad - \sup_{k\ge 0}\left\{\frac{1+2\,k}{1+k}\mathrm{dist}(s(k,y), \Delta f^{-1}(0))\right\} \\ &= \sup_{0\le k\le K(z)}\left\{\frac{1+2\,k}{1+k}\mathrm{dist}(s(k,z), \Delta f^{-1}(0))\right\} \\ &\quad - \sup_{0\le k\le K(z)}\left\{\frac{1+2\,k}{1+k}\mathrm{dist}(s(k,y), \Delta f^{-1}(0))\right\}. \end{aligned} \tag{66}$$

Hence,

$$\begin{aligned} |V(z) - V(y)| &\le \sup_{0\le k\le K(z)}\left|\frac{1+2\,k}{1+k}\left(\mathrm{dist}(s(k,z), \Delta f^{-1}(0))\right.\right. \\ &\qquad \left.\left. -\mathrm{dist}(s(k,y), \Delta f^{-1}(0))\right)\right| \\ &\le 2\sup_{0\le k\le K(z)}\left|\mathrm{dist}(s(k,z), \Delta f^{-1}(0))\right. \\ &\qquad \left. -\mathrm{dist}(s(k,y), \Delta f^{-1}(0))\right| \\ &\le 2\sup_{0\le k\le K(z)} \mathrm{dist}(s(k,z), s(k,y)) \\ &\qquad z \in \mathcal{D}_0 \backslash \Delta f^{-1}(0), \quad y \in \mathcal{B}_\eta(z). \end{aligned} \tag{67}$$

Now, it follows from continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions and (67) that $V(\cdot)$ is continuous on $\mathcal{D}_0 \backslash \Delta f^{-1}(0)$.

To show that $V(\cdot)$ is continuous on $\Delta f^{-1}(0)$, consider $x_\mathrm{e} \in \Delta f^{-1}(0)$ and let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}_0 \backslash \Delta f^{-1}(0)$ that converges to x_e. Since x_e is Lyapunov stable, it follows that $x(k) \equiv x_\mathrm{e}$ is the unique solution to (1) with $x_0 = x_\mathrm{e}$. By continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions, $s(k,x_n) \to s(k,x_\mathrm{e}) = x_\mathrm{e}$ as $n \to \infty$, $k \in \mathbb{Z}_+$.

Let $\varepsilon > 0$ and note that it follows from $ii)$ of Proposition 3.2 that there exists $\delta = \delta(x_\mathrm{e}) > 0$ such that for every solution of (1) in $\mathcal{B}_\delta(x_\mathrm{e})$ there exists $\hat{K} = \hat{K}(x_\mathrm{e}, \varepsilon) > 0$ such that $s_k(\mathcal{B}_\delta(x_\mathrm{e})) \subset \mathcal{W}_\varepsilon$ for all $k \ge \hat{K}$. Next, note that there exists a positive integer N_1 such that $x_n \in \mathcal{B}_\delta(x_\mathrm{e})$ for all $n \ge N_1$. Now, it follows from (64) that

$$V(x_n) \le 2\sup_{0\le k\le \hat{K}} \mathrm{dist}(s(k,x_n), \Delta f^{-1}(0)) + 2\varepsilon, \quad n \ge N_1. \tag{68}$$

Next, it follows from [35, Lemma 3.1 of Ch. I] that $s(\cdot, x_n)$ converges to $s(\cdot, x_\mathrm{e})$ uniformly on $[0, \hat{K}]$. Hence,

$$\begin{aligned} \lim_{n\to\infty} \sup_{0\le k\le \hat{K}} &\mathrm{dist}\left(s(k,x_n), \Delta f^{-1}(0)\right) \\ &= \sup_{0\le k\le \hat{K}} \mathrm{dist}\left(\lim_{n\to\infty} s(k,x_n), \Delta f^{-1}(0)\right) \\ &= \sup_{0\le k\le \hat{K}} \mathrm{dist}(x_\mathrm{e}, \Delta f^{-1}(0)) \\ &= 0 \end{aligned} \tag{69}$$

which implies that there exists a positive integer $N_2 = N_2(x_\mathrm{e}, \varepsilon) \ge N_1$ such that

$$\sup_{0\le k\le \hat{K}} \mathrm{dist}(s(k,x_n), \Delta f^{-1}(0)) < \varepsilon, \quad n \ge N_2. \tag{70}$$

Combining (64) with (70) yields $V(x_n) < 4\varepsilon$ for all $n \ge N_2$, which implies that $\lim_{n\to\infty} V(x_n) = 0 = V(x_\mathrm{e})$.

Finally, we show that $V(x(k))$ is strictly decreasing along the solution of (1) on $\mathcal{D} \backslash \Delta f^{-1}(0)$. Now, note that it follows from the definition of $K(\cdot)$ that the supremum in the definition of $V(s(1,x))$ is reached at some time $\hat{k}$ such that $0 \le \hat{k} < K(x)$. Hence

$$\begin{aligned} V(s(1,x)) &= \mathrm{dist}(s(\hat{k}+1,x), \Delta f^{-1}(0))\frac{1+2\hat{k}}{1+\hat{k}} \\ &= \mathrm{dist}(s(\hat{k}+1,x), \Delta f^{-1}(0))\frac{1+2\hat{k}+2}{1+\hat{k}+1} \\ &\quad \cdot \left[1 - \frac{1}{(1+2\hat{k}+2)(1+\hat{k})}\right] \\ &\le V(x)\left[1 - \frac{1}{2(1+K(x))^2}\right] \end{aligned} \tag{71}$$

which implies

$$\Delta V(x) \le -\frac{1}{2}V(x)(1+K(x))^{-2} < 0, \quad x \in \mathcal{D}_0 \backslash \Delta f^{-1}(0) \tag{72}$$

and hence, iii) holds. ■

Appendix C
Proof of Theorem 7.1

For the proof of Theorem 7.1, we write $x(k) \to \mathcal{M}$ as $k \to \infty$ to denote that $x(k)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $K > 0$ such that $\mathrm{dist}(x(k), \mathcal{M}) < \varepsilon$ for all $k > K$. First, we show $\mathcal{M} = \{x \in \mathbb{R}^q : x = \alpha\mathbf{e}, \alpha \in \mathbb{R}\}$ is the set of equilibria of (57). To see this, note that it follows from Assumption 7.1 that for $\mathcal{C}_{(i,j)} = 1, i \neq j, \phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j$, and hence, $\alpha\mathbf{e}$ is an equilibrium state of (57) for every $\alpha \in \mathbb{R}$.

To show Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$, consider the Lyapunov function candidate $V(x) = \frac{1}{2}(x - \alpha\mathbf{e})^{\mathrm{T}}(x - \alpha\mathbf{e})$. Note that $V(x) > 0$, $x \in \mathbb{R}^q$, $x \neq \alpha\mathbf{e}$, and $V(x) = 0$ if and only if $x = \alpha\mathbf{e}$. Furthermore, $V(\cdot)$ is radially unbounded. Now, since $\phi_{ij}(x_i, x_j) = -\phi_{ji}(x_j, x_i)$ for all $i, j = 1, \ldots, q$, $i \neq j$, and $\mathbf{e}^{\mathrm{T}}x(k+1) = \mathbf{e}^{\mathrm{T}}x(k)$, $k \in \overline{\mathbb{Z}}_+$, it follows from Assumptions 7.2 and 7.3 that

$$\begin{aligned}
\Delta V(x(k))
&= V(x(k+1)) - V(x(k)) \\
&= \frac{1}{2}[x(k+1) - \alpha\mathbf{e}]^{\mathrm{T}}[x(k+1) - \alpha\mathbf{e}] \\
&\quad - \frac{1}{2}[x(k) - \alpha\mathbf{e}]^{\mathrm{T}}[x(k) - \alpha\mathbf{e})] \\
&= \frac{1}{2}x^{\mathrm{T}}(k+1)f(x(k)) - \frac{1}{2}x^{\mathrm{T}}(k)x(k) \\
&= \sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} x_i(k+1)\phi_{ij}(x_i(k), x_j(k)) \\
&\quad - \frac{1}{2}\sum_{i=1}^{q}\left[\sum_{j=1, j\neq i}^{q}\phi_{ij}(x_i(k), x_j(k))\right]^2 \\
&= \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}[x_i(k+1) - x_j(k+1)]\,\phi_{ij}(x_i(k), x_j(k)) \\
&\quad - \frac{1}{2}\sum_{i=1}^{q}\left[\sum_{j=1, j\neq i}^{q}\phi_{ij}(x_i(k), x_j(k))\right]^2 \\
&\leq 0, \quad x(k) \in \mathbb{R}^q, \quad k \in \overline{\mathbb{Z}}_+
\end{aligned} \tag{73}$$

which, using [3, Th. 13.2], establishes Lyapunov stability of the equilibrium state $\alpha\mathbf{e}$.

To show that $\alpha\mathbf{e}$ is semistable, note that

$$\begin{aligned}
\Delta V(x(k)) &= \frac{1}{2}x^{\mathrm{T}}(k+1)x(k+1) - \frac{1}{2}x^{\mathrm{T}}(k)x(k) \\
&= \sum_{i=1}^{q}\sum_{j=1, j\neq i}^{q} x_i(k)\phi_{ij}(x_i(k), x_j(k)) \\
&\quad + \frac{1}{2}\sum_{i=1}^{q}\left[\sum_{j=1, j\neq i}^{q}\phi_{ij}(x_i(k), x_j(k))\right]^2 \\
&\geq \sum_{i=1}^{q-1}\sum_{j=i+1}^{q}[x_i(k) - x_j(k)]\,\phi_{ij}(x_i(k), x_j(k)) \\
&= \sum_{i=1}^{q-1}\sum_{j\in\mathcal{K}_i}[x_i(k) - x_j(k)]\,\phi_{ij}(x_i(k), x_j(k))
\end{aligned} \tag{74}$$

where $\mathcal{K}_i \triangleq \mathcal{N}_i \setminus \{1, \ldots, i-1\}$ and $\mathcal{N}_i \triangleq \{j \in \{1, \ldots, q\} : \phi_{ij}(x_i, x_j) = 0$ if and only if $x_i = x_j\}$, $i = 1, \ldots, q$. Now, note that $\Delta V(x) = 0$ if and only if $(x_i - x_j)\phi_{ij}(x_i, x_j) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$.

To see this, first assume that $(x_i - x_j)\phi_{ij}(x_i, x_j) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$. Then, it follows from (74) that $\Delta V(x) \geq 0$, $x \in \mathbb{R}^q$. However, it follows from (73) that $\Delta V(x) \leq 0$, $x \in \mathbb{R}^q$, and hence, $\Delta V(x) = 0$. Conversely, assume $\Delta V(x) = 0$. In this case, it follows from Assumption 7.2 and (73) that $[x_i(k+1) - x_j(k+1)]\phi_{ij}(x_i(k), x_j(k)) = 0$ and $\sum_{j=1, j\neq i}^{q}\phi_{ij}(x_i(k), x_j(k)) = 0$, $k \in \overline{\mathbb{Z}}_+$, $i, j = 1, \ldots, q$, $i \neq j$. Now, since

$$\begin{aligned}
&[x_i(k+1) - x_j(k+1)]\phi_{ij}(x_i(k), x_j(k)) \\
&\quad = [x_i(k) - x_j(k)]\,\phi_{ij}(x_i(k), x_j(k)) \\
&\qquad + \left[\sum_{h=1, h\neq i}^{q}\phi_{ih}(x_i(k), x_h(k)) - \sum_{l=1, l\neq j}^{q}\phi_{jl}(x_j(k), x_l(k))\right]\phi_{ij}(x_i(k), x_j(k)) \\
&\quad = [x_i(k) - x_j(k)]\,\phi_{ij}(x_i(k), x_j(k)) \\
&\qquad k \in \overline{\mathbb{Z}}_+, \quad i, j = 1, \ldots, q, \quad i \neq j
\end{aligned} \tag{75}$$

it follows that $(x_i - x_j)\phi_{ij}(x_i, x_j) = 0$, $i = 1, \ldots, q$, $j \in \mathcal{K}_i$.

Finally, to show $x(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ as $k \to \infty$ and $\frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a globally semistable equilibrium state, let

$$\begin{aligned}
\mathcal{R} &\triangleq \{x \in \mathbb{R}^q : \Delta V(x) = 0\} \\
&= \{x \in \mathbb{R}^q : [x_i(k) - x_j(k)]\,\phi_{ij}(x_i(k), x_j(k)) = 0 \\
&\qquad\qquad i = 1, \ldots, q, \quad j \in \mathcal{K}_i\}.
\end{aligned}$$

Now, it follows from Assumption 7.1 that the communication graph topology of $\mathcal{G}$ is strongly connected, which implies that $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = x_2 = \cdots = x_q\}$. Since $\mathcal{R}$ consists of the equilibrium states of (57), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, it follows from Theorem 4.2 that for every initial condition $x(0) \in \mathbb{R}^q$, $x(k) \to \mathcal{M}$ as $k \to \infty$, and hence, $\alpha\mathbf{e}$ is a semistable equilibrium state of (57). Moreover, since $\mathbf{e}^{\mathrm{T}}x(k) = \mathbf{e}^{\mathrm{T}}x(0)$ for all $k \in \overline{\mathbb{Z}}_+$, it follows that $x(k) \to \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ as $k \to \infty$. Hence, with $\alpha = \frac{1}{q}\mathbf{e}^{\mathrm{T}}x(0)$, $\alpha\mathbf{e} = \frac{1}{q}\mathbf{e}\mathbf{e}^{\mathrm{T}}x(0)$ is a globally semistable equilibrium state of (57). ■

References

[1] S. P. Bhat and D. S. Bernstein, "Nontangency-based Lyapunov tests for convergence and stability in systems having a continuum of equilibra," *SIAM J. Control Optim.*, vol. 42, pp. 1745–1775, 2003.

[2] S. P. Bhat and D. S. Bernstein, "Arc-length-based Lyapunov tests for convergence and stability with applications to systems having a continuum of equilibria," *Math. Control, Signals, Syst.*, vol. 22, pp. 155–184, 2010.

[3] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton, NJ, USA: Princeton Univ. Press, 2008.

[4] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-time semistability and consensus for nonlinear dynamical networks," *IEEE Trans. Autom. Control*, vol. 53, pp. 1887–1900, Sep. 2008.

[5] P. Ramazi, H. Jardón-Kojakhmetov, and M. Cao, "Limit sets within curves where trajectories converge to," *Appl. Math. Lett.*, vol. 68, pp. 94–100, 2017.

[6] J. D. Wolfe, D. F. Chichka, and J. L. Speyer, "Decentralized controllers for unmanned aerial vehicle formation flight," in *Proc. AIAA Conf. Guid., Navig., Control*, San Diego, CA, USA, 1996.

[7] D. Swaroop and J. K. Hedrick, "Constant spacing strategies for platooning in automated highway systems," *ASME J. Dyn. Syst., Meas., Control*, vol. 121, pp. 462–470, 1999.

[8] F. Paganini, J. C. Doyle, and S. H. Low, "Scalable laws for stable network congestion control," in *Proc. IEEE Conf. Decis. Control*, Orlando, FL, USA, 2001, pp. 185–190.

[9] R. Vidal, O. Shakernia, and S. Sastry, "Formation control of nonholonomic mobile robots with omnidirectional visual servoing and motion segmentation," in *Proc. IEEE Int. Conf. Robot. Autom.*, 2003, pp. 584–589.

[10] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, pp. 1465–1476, Sep. 2004.

[11] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[12] J. Cortés and F. Bullo, "Coordination and geometric optimization via distributed dynamical systems," *SIAM J. Control Optim.*, vol. 44, pp. 1543–1574, 2005.

[13] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[14] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[15] F. Bullo, J. Cortes, and S. Martiez, *Distributed Control of Robotic Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2009.

[16] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods for Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[17] L. Fang and P. J. Antsaklis, "Information consensus of asynchronous discrete-time multi-agent systems," in *Proc. Amer. Control Conf.*, Portland, OR, USA, 2005, pp. 1883–1888.

[18] D. B. Kingston and R. W. Beard, "Discrete-time average-consensus under switching network topologies," in *Proc. Amer. Control Conf.*, Minneapolis, MN, USA, 2006, pp. 3551–3556.

[19] P. Lin and Y. Jia, "Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies," *Automatica*, vol. 45, no. 9, pp. 2154–2158, 2009.

[20] M. Zhu and S. Martínez, "Discrete-time dynamic average consensus," *Automatica*, vol. 46, no. 2, pp. 322–329, 2010.

[21] K. You and L. Xie, "Network topology and communication data rate for consensusability of discrete-time multi-agent systems," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2262–2275, Oct. 2011.

[22] Y. Su and J. Huang, "Two consensus problems for discrete-time multi-agent systems with switching network topology," *Automatica*, vol. 48, no. 9, pp. 1988–1997, 2012.

[23] T. Yang, Z. Meng, D. V. Dimarogonas, and K. H. Johansson, "Global consensus for discrete-time multi-agent systems with input saturation constraints," *Automatica*, vol. 50, no. 2, pp. 499–506, 2014.

[24] Q. Hui and W. M. Haddad, "Distributed nonlinear control algorithms for network consensus," *Automatica*, vol. 44, no. 9, pp. 2375–2381, 2008.

[25] J. M. Berg, D. H. S. Maithripala, Q. Hui, and W. M. Haddad, "Thermodynamics-based control for network systems," *ASME J. Dyn. Syst. Meas. Control*, vol. 135, no. 5, pp. 1–12, 2013.

[26] W. M. Haddad, T. Rajpurohit, and X. Jin, "Stochastic semistability for nonlinear dynamical systems with application to consensus on networks with communication uncertainty," *IEEE Trans. Autom. Control*, vol. 65, no. 7, pp. 2826–2841, Jul. 2020.

[27] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Thermodynamics: A Dynamical Systems Approach*. Princeton, NJ, USA: Princeton Univ. Press, 2005.

[28] W. M. Haddad, *A Dynamical Systems Theory of Thermodynamics*. Princeton, NJ, USA: Princeton Univ. Press, 2019.

[29] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton, NJ, USA: Princeton Univ. Press, 2006.

[30] W. M. Haddad, V. Chellaboina, Q. Hui, and S. G. Nersesov, "Energy and entropy based stabilization for lossless dynamical systems via hybrid controllers," *IEEE Trans. Autom. Control*, vol. 52, pp. 1604–1614, Sep. 2007.

[31] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, "Hybrid decentralized maximum entropy control for large-scale dynamical systems," *Nonlinear Anal.: Hybrid Syst.*, vol. 1, pp. 244–263, 2007.

[32] W. M. Haddad and J. Lee, "Finite-time stability of discrete autonomous systems," *Automatica*, vol. 122, no. 109282, pp. 1–8, 2020.

[33] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[34] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York, NY, USA: Academic, 1979.

[35] J. K. Hale, *Ordinary Differential Equations*, 2nd ed. New York, NY, USA: Wiley, 1980.

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