optimal semistable stabilisation for linear discrete-time dynamical systems
with applications to network consensus
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H₂ optimal semistable stabilisation for linear discrete-time dynamical systems with applications to network consensus

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In this article, we develop H₂ semistability theory for linear discrete-time dynamical systems. Using this theory, we design H₂ optimal semistable controllers for linear dynamical systems. Unlike the standard H₂ optimal control problem, a complicating feature of the H₂ optimal semistable stabilisation problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. An interesting feature of the proposed approach, however, is that a least squares solution over all possible semistabilising solutions corresponds to the H₂ optimal solution. It is shown that this least squares solution can be characterised by a linear matrix inequality minimisation problem. Finally, the proposed framework is used to develop H₂ optimal semistable controllers for addressing the consensus control problem in networks of dynamic agents.

Keywords: semistability; H₂ optimal semistable control; semicontrollability; semiobservability; linear matrix inequalities; multiagent systems; discrete-time systems

1. Introduction

Dynamical network systems cover a very broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations and congestion control in communication networks, to cite as few examples. A unique feature of the closed-loop dynamics under any control algorithm in dynamical networks is the existence of a continuum of equilibria representing a desired state of convergence. Under such dynamics, the desired limiting state is not determined completely by the system dynamics, but depends on the initial system state as well (Hui and Haddad 2007; Hui, Haddad and Bhat 2007).

The dependence of the limiting state on the initial state is not limited to dynamical network systems, it is also seen in the dynamics of compartmental systems (Jacquez and Simon 1993) which arise in chemical kinetics (Bernstein and Bhat 1999), and biomedical (Jacquez 1985), environmental (Odum 1971), economic (Berman and Plemmons 1979), power (Šiljak 1978) and thermodynamic systems (Haddad, Chellaboina and Nersesov 2005a). In all such systems possessing a continuum of equilibria, semistability, and not asymptotic stability, is the relevant notion of stability.

Semistability is the property whereby every trajectory that starts in a neighbourhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability then implies Lyapunov stability, and is implied by asymptotic stability.

Semistability was first introduced in Campbell and Rose (1979) for linear systems, and applied to matrix second-order systems in Bernstein and Bhat (1995). Non-linear extensions were considered in Bhat and Bernstein (2003a, b), which give several stability results for systems having a continuum of equilibria based on non-tangency and arc length of trajectories, respectively. Hui et al. (2007) and Hui and Haddad (2007) build on the results of Bhat and Bernstein (2003a, b) and give semistable stabilisation results for non-linear network dynamical systems. However, optimal semistable stabilisation has never been considered in the literature.

In this article, we use linear matrix inequalities (LMIs) to develop H₂ optimal semistable controllers for linear discrete-time dynamical systems. Linear matrix inequalities provide a powerful design framework for linear control problems (Boyd, Ghaoui, Feron and Balakrishnan 1994). Since LMIs lead to convex or quasiconvex optimisation problems, they can be solved very efficiently using interior-point algorithms. Unlike the standard H₂ optimal control

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problem, a complicating feature of the $H_2$ optimal semistable stabilisation problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. However, an interesting feature of the proposed approach is that a least squares solution over all possible semistabilising solutions corresponds to the $H_2$ optimal solution. It is shown that this least squares solution can be characterised by a linear matrix inequality minimisation problem.

In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Using the results of the first part of this article, we develop an $H_2$ optimal semistable stabilisation framework for addressing the consensus control problem in networks of dynamic agents.

2. Discrete-time $H_2$ semistability theory

In this section, we establish notation along with several key results on discrete-time $H_2$ semistability theory involving the notions of semistability, semicontrollability and semiobservability. The notation we use in this article is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{Z}_+$ (resp., $\mathbb{Z}_-$) denotes the set of non-negative (resp., positive) integers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n \times m}$ (resp., $\mathbb{C}^{n \times m}$) denotes the set of $n \times m$ real (resp., complex) matrices, $(\cdot)^T$ denotes transpose, $(\cdot)^*$ denotes complex conjugate transpose, $(\cdot)^{\#}$ denotes the group generalised inverse, and $I_n$ or $I$ denotes the $n \times n$ identity matrix. Furthermore, we write $\| \cdot \|$ for the Euclidean vector norm, $\| \cdot \|_F$ for the Frobenius matrix norm, $S^\perp$ for the orthogonal complement of a set $S$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix $A$, $\text{spec}(A)$ for the spectrum of square matrix $A$, $\rho(A)$ for the spectral radius of square matrix $A$, $\text{rank}(A)$ for the rank of a matrix $A$, $\text{tr}(\cdot)$ for the trace operator, $\mathbb{E}$ for the expectation operator and $A \succeq 0$ (resp., $A \succ 0$) to denote the fact that the Hermitian matrix $A$ is non-negative (resp., positive) definite. Finally, we write $\Delta V(x(k))$ for $V(x(k+1)) - V(x(k))$, $k \in \mathbb{Z}_+$, $B_i(x)$, $x \in \mathbb{R}^n$, $\epsilon > 0$, for the open ball with radius $\epsilon$ and centre $x$, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum and $\text{vec}(\cdot)$ for the column stacking operator.

The following definition of semistability of a dynamical system is needed. For this definition, consider the non-linear dynamical system given by

$$x(k + 1) = f(x(k)), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad (1)$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$, $k \in \mathbb{Z}_+$ and $f: \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

**Definition 2.1:** Let $\mathcal{D} \subseteq \mathbb{R}^n$ be positively invariant under (1). The equilibrium solution $x(k) \equiv x_c \in \mathcal{D}$ of (1) is Lyapunov stable with respect to $\mathcal{D}$ if, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $x_0 \in B_\epsilon(x_c) \cap \mathcal{D}$, then $x(k) \in B_\epsilon(x_c) \cap \mathcal{D}$, $k \in \mathbb{Z}_+$. The equilibrium solution $x(k) \equiv x_c \in \mathcal{D}$ of (1) is semistable with respect to $\mathcal{D}$ if it is Lyapunov stable with respect to $\mathcal{D}$ and there exists $\delta > 0$ such that if $x_0 \in B_\delta(x_c) \cap \mathcal{D}$, then $\lim_{k \to \infty} x(k)$ exists and corresponds to a Lyapunov stabilizable equilibrium point in $\mathcal{D}$. Finally, the system (1) is said to be semistable with respect to $\mathcal{D}$ if every equilibrium point in $\mathcal{D}$ is semistable with respect to $\mathcal{D}$.

**Proposition 2.1:** Let $\mathcal{D}_c \subseteq \mathbb{R}^n$ be a compact invariant set with respect to (1). Suppose there exists a continuous function $V: \mathcal{D}_c \to \mathbb{R}$ such that $V(f(x)) - V(x) \leq 0$, $x \in \mathcal{D}_c$. Let $\mathcal{R} \triangleq \{ x \in \mathcal{D}_c : V(f(x)) = V(x) \}$ and $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If every element in $\mathcal{M}$ is a Lyapunov stable equilibrium point with respect to $\mathcal{D}_c$, then (1) is semistable with respect to $\mathcal{D}_c$.

**Proof:** The proof is similar to the proof of Theorem 3.3 of Bhat and Bernstein (1999), and hence is omitted. $\square$

Note that if in (1) $f(x) = Ax$, where $A \in \mathbb{R}^{n \times n}$, then (1) is semistable with respect to $\mathbb{R}^n$ if and only if $A$ is semistable, that is, $\text{spec}(A) \subseteq \{ s \in \mathbb{C} : |s| < 1 \} \cup \{ 1 \}$ and, if $1 \in \text{spec}(A)$, then 1 is semisimple. In this case, it can be shown that for every $x_0 \in \mathbb{R}^n$, $\lim_{k \to \infty} x(k)$ exists or, equivalently, $\lim_{k \to \infty} A_k = I_n - (I_n - A)(I_n - A)^{\#}$ (Bernstein 2005; Haddad, Hui, Nersesov and Chellaboina 2005b).

Next, we present the notions of semicon trollability and semiobservability. For these definitions let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{p \times n}$, and consider the linear dynamical system

$$x(k + 1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad (2)$$

$$y(k) = Cx(k), \quad (3)$$

with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}^m$ and output $y(k) \in \mathbb{R}^p$, where $k \in \mathbb{Z}_+$. 


Definition 2.2: Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times r}$. The pair $(A, B)$ is semicontrollable if
\[
\left[\underbrace{\mathcal{N}(B^T(A^T - I_n)^{-1})}_n\right] = \left[\mathcal{N}(A^T - I_n)\right]^{-1},
\]where $(A^T - I_n)^0 \triangleq I_r$.

Definition 2.3: Let $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{r \times n}$. The pair $(A, C)$ is semiobservable if
\[
\left(\bigcap_{i=1}^n \mathcal{N}(C(A - I_n)^{-1})\right) = \mathcal{N}(A - I_n).
\]

Semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to equilibrium controllability, whereas semiobservability is an extension of zero-state observability to equilibrium observability. It is important to note here that since Definitions 2.2 and 2.3 are dual, dual results to the semiobservability results that we establish in this section also hold for semicontrollability.

Definition 2.4: Let $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{r \times n}$ and $K \in \mathbb{R}^{m \times n}$. The pair $(A, C)$ is semiobservable with respect to $K$ if
\[
\mathcal{N}(K) \cap \left(\bigcap_{i=1}^n \mathcal{N}(C(A - I_n)^{-1})\right) = \mathcal{N}(K) \cap \mathcal{N}(A - I_n).
\]
The following result shows that semiobservability is unchanged by full state feedback.

Proposition 2.2: Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{r \times n}$ and $K \in \mathbb{R}^{m \times n}$, where $R$ is positive definite. If the pair $(A, C)$ is semiobservable, then the pair $(A + BK, C^T + K^T R K)$ is semiobservable with respect to $K$.

Proof: Note that $\mathcal{N}(C^T + K^T R K) = \mathcal{N}(C) \cap \mathcal{N}(K)$. Hence,
\[
\mathcal{N}(K) \cap \left(\bigcap_{i=1}^n \mathcal{N}(C(A - I_n))^T(A + BK)^{-1})\right) = \mathcal{N}(K) \cap \left(\bigcap_{i=1}^n \mathcal{N}(C(A - I_n))^{-1}\right) = \mathcal{N}(K) \cap \mathcal{N}(A - I_n + BK)
\]
which implies that the pair $(A + BK, C^T + K^T R K)$ is semiobservable with respect to $K$.

Next, we connect semistability with Lyapunov theory and semiobservability to arrive at a characterisation of the $\mathcal{H}_2$ norm of semistable systems. For this result, we consider the linear dynamical system
\[
x(k + 1) = Ax(k), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \quad (8)
\]
where $A \in \mathbb{R}^{n \times n}$, with output Equation (3). Furthermore, for a given semistable system define the $\mathcal{H}_2$ norm of
\[
G(z) \sim \begin{bmatrix} \begin{array}{c} A \\ C \end{array} & \begin{array}{c} x_0 \\ 0 \end{array} \end{bmatrix}
\]
by
\[
\|G\|_2 = \left[\sum_{k=0}^\infty \|G(k)\|_F^2\right]^{1/2} = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \|G(e^{j\theta})\|_F^2 \, d\theta\right]^{1/2}.
\]
The following proposition presents necessary and sufficient conditions for well-posedness of the $\mathcal{H}_2$ norm of a semistable system.

Proposition 2.3: Consider the linear dynamical system (8) with output (3) and assume that $A$ is semistable. Then the following statements are equivalent:

(i) For every $x_0 \in \mathbb{R}^n$, $\|G\|_2 < \infty$.
(ii) $\sum_{k=0}^\infty (A^k)^T R A^k < \infty$, where $R = C^T C$.
(iii) $\mathcal{N}(A - I_n) \subset \mathcal{N}(C)$.

Proof: The equivalence of (i) and (ii) follows from the fact
\[
\|G\|_2^2 = \sum_{k=0}^\infty x_0^T (A^k)^T R A^k x_0.
\]
To show (ii) implies (iii) note that since $A$ is semistable it follows that either $\rho(A) < 1$ or there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ such that $A = S [\begin{array}{c} J \end{array} 0 \quad \begin{array}{c} 0 \end{array}] S^{-1}$, where $J \in \mathbb{R}^{r \times r}$, $r = \text{rank } A$, and $\rho(J) < 1$. Now, if $\rho(A) < 1$, then (iii) holds trivially since $\mathcal{N}(A - I_n) = \{0\} \subset \mathcal{N}(C)$.

Alternatively, if $1 \in \text{spec}(A)$, then
\[
\mathcal{N}(A - I_n) = \{x \in \mathbb{R}^n : x = S [\begin{array}{c} 0 \end{array} \begin{array}{c} y \end{array}]^T, y \in \mathbb{R}^{n-r} \}.
\]
Now,
\[
\sum_{k=0}^\infty (A^k)^T R A^k = S^{-T} \sum_{k=0}^\infty \left[\begin{array}{cc} J^k \end{array} 0 \\ 0 \end{array} \right] \hat{R} \left[\begin{array}{cc} J^k \end{array} 0 \\ 0 \end{array} \right] S = S^{-T} \sum_{k=0}^\infty \left[\begin{array}{cc} J^k \end{array} \hat{R}_1 J^k \end{array} \right] S.
\]
where
\[
\hat{R} = S^T RS = \begin{bmatrix} \hat{R}_1 & \hat{R}_{12} \\ \hat{R}'_{12} & \hat{R}_2 \end{bmatrix}.
\] (13)

Next, it follows from (12) that
\[
\sum_{k=0}^{\infty} (A^k)^T R A^k < \infty
\] (14)
if and only if \( \hat{R}_2 = 0 \) or, equivalently,
\[
[0_{1, x}, y] \hat{R} [0_{1, x}, y]^T = 0, \quad y \in \mathbb{R}^{n-r},
\] (15)
which is further equivalent to \( x^T Rx = 0, \quad x \in \mathcal{N}(A - I_n) \).
Hence, \( \mathcal{N}(A - I_n) \subseteq \mathcal{N}(C) \).

Finally, the proof of (iii) implies (ii) by reversing the steps of the proof given above. \( \square \)

**Lemma 2.1:** Let \( A \in \mathbb{R}^{n \times n} \). If there exists an \( n \times n \) matrix \( P \succeq 0 \) and an \( I \times n \) matrix \( C \) such that \( (A, C) \) is semiobservable and
\[
P = A^T PA + R,
\] (16)

where \( R \triangleq C^T C \), then (i) \( \mathcal{N}(P) \subseteq \mathcal{N}(A - I_n) \subseteq \mathcal{N}(R) \) and (ii) \( \mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n) = \{0\} \).

**Proof:** (i) If \( (A - I_n)x = 0 \), then (16) implies \( x^T Rx = x^T (P - A^T PA)x = 0 \), and hence \( Rx = 0 \). Thus, \( \mathcal{N}(A - I_n) \subseteq \mathcal{N}(R) \). If \( Px = 0 \), then
\[
0 \leq x^T Rx = x^T (PA + R)x = -x^T A^T PAX \leq 0,
\] (17)
and hence \( x^T Rx = 0 \) or, equivalently, \( Rx = 0 \). Thus, \( \mathcal{N}(P) \subseteq \mathcal{N}(R) \).

Next, let \( x \in \mathcal{N}(P) \subseteq \mathcal{N}(R) \). If \( (A - I_n)x \in \mathcal{N}(P) \subseteq \mathcal{N}(R) \) for some \( k \geq 0 \), then
\[
0 = x^T (A^T - I_n)^k R(A - I_n)^k x
= x^T (A^T - I_n)^k (P - A^T PA)(A - I_n)^k x
= -x^T (A^T - I_n)^k A^T P A (A - I_n)^k x
= -x^T (A^T - I_n)^{k+1} P (A - I_n)^{k+1} x,
\] (18)
and hence \( P(x - I_n)x \in \mathcal{N}(R) \), which implies that \( (A - I_n)x \in \mathcal{N}(P) \subseteq \mathcal{N}(R) \). Since \( (A - I_n)x \in \mathcal{N}(P) \subseteq \mathcal{N}(R) \) for \( k = 0 \), it follows by induction that \( x \) is contained in the null space of the left-hand side of (5). Equation (5) now implies that \( x \in \mathcal{N}(A - I_n) \). Thus, \( \mathcal{N}(P) \subseteq \mathcal{N}(A - I_n) \subseteq \mathcal{N}(R) \).

(ii) Consider \( x \in \mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n) \). Then \( (A - I_n)x = 0 \) and there exists \( z \in \mathbb{R}^n \) such that \( x = (A - I_n)z \). Now, it follows from (i) that \( Rx = R(A - I_n)z = 0 \). Thus,
\[
0 = z^T Rx = z^T (P - A^T PA)x = -z^T (A - I_n)^T P x
= -x^T P x,
\] (19)
and hence \( Px = 0 \). Finally,
\[
z^T Rx = z^T (P - A^T PA)z = z^T P z - (x+z)^T P (x+z)
= -x^T P x - z^T P z - z^T P x = 0,
\]
and hence \( Rx = 0 \). This implies that \( x \) is contained in the null space of the left-hand side of (5). Hence, by (5), \( (A - I_n)z = x = 0 \) as required. \( \square \)

**Theorem 2.1:** Consider the linear dynamical system (8). Suppose there exists an \( n \times n \) matrix \( P \succeq 0 \) and a matrix \( C \in \mathbb{R}^{k \times n} \) such that \( (A, C) \) is semiobservable and (16) holds. Then (8) is semistable with respect to \( \mathbb{R}^n \). Furthermore, \( \|G(z)\|^2 = (z_0 - x_0)^T P (z_0 - x_0) \), where \( x_0 \triangleq x_0 - (A - I_n)(A - I_n)^T x_0 \).

**Proof:** Since, by Lemma 2.1, \( \mathcal{N}(A - I_n) \cap \mathcal{R}(A - I_n) = \{0\} \), it follows from Lemma 4.14 of Berman and Plemmons (1979) that \( A - I_n \) is group invertible. Let \( L \triangleq I_n - (A - I_n)(A - I_n)^T \) and note that \( L^2 = L \). Hence, \( L \) is the unique \( n \times n \) matrix satisfying \( \mathcal{N}(L) = \mathcal{R}(A - I_n) \), \( \mathcal{R}(L) = \mathcal{N}(A - I_n) \) and \( L x = x \) for all \( x \in \mathcal{N}(A - I_n) \).

Consider the non-negative function
\[
V(x) = x^T P x + x^T L^T L x.
\] (20)
If \( V(x) = 0 \) for some \( x \in \mathbb{R}^n \), then \( Px = 0 \) and \( Lx = 0 \). It follows from (i) of Lemma 2.1 that \( x \in \mathcal{N}(A - I_n) \), while \( Lx = 0 \) implies \( x \in \mathcal{R}(A - I_n) \). Now, it follows from (ii) of Lemma 2.1 that \( x = 0 \). Hence, \( V(\cdot) \) is positive definite. Next, since \( L(A - I_n) = A - I_n - (A - I_n)(A - I_n)^T (A - I_n) = 0 \), it follows that
\[
\Delta V(x) = -x^T Rx + x^T (A - I_n)^T L L^T (A - I_n)x
+ x^T (A - I_n)^T L^T L x + x^T L^T L (A - I_n) x
\leq 0.
\] (21)
Note that \( \Delta V^{-1}(0) = \mathcal{N}(R) \).

To find the largest invariant subset \( \mathcal{M} \) of \( \mathcal{N}(R) \), consider a solution \( y \) of (8) such that \( Cx(k) = 0 \) for all \( k \in \mathbb{Z}_+ \). Then, \( Cx(k+1) - Cx(k) = 0 \), that is, \( C(A - I_n)x(k) = 0 \). Similarly, \( C(A - I_n)x(k+1) - C(A - I_n)x(k) = C(A - I_n)x(k) = 0 \), and so on. This implies \( C(A - I_n)x(k) = 0 \) for all \( k \in \mathbb{Z}_+ \) and \( i = 1, 2, \ldots \). Equation (5) now implies that \( x(k) \in \mathcal{N}(A - I_n) \) for all \( k \in \mathbb{Z}_+ \). Thus, \( \mathcal{M} \subseteq \mathcal{N}(A - I_n) \).

However, \( \mathcal{N}(A - I_n) \) consists of only equilibrium points, and hence is invariant. Hence, \( \mathcal{M} = \mathcal{N}(A - I_n) \).

Now, let \( x_0 \in \mathcal{N}(A - I_n) \) be an equilibrium point of (8) and consider the function \( U(x) = V(x - x_0) \), which is positive definite with respect to \( x_0 \). Then it follows that \( \Delta U(x) = -(x - x_0)^T R (x - x_0) \leq 0 \), \( x \in \mathbb{R}^n \). Thus, it follows that \( x_0 \) is Lyapunov stable, and hence, by Proposition 2.1, (8) is semistable.
Next, since \( A \) is semistable, it follows from (vi) of Proposition 11.9.2 of Bernstein (2005) that 
\[
\lim_{k \to \infty} A^k = I_n - (A - I_n)(A - I_n)^T.
\]
Now, noting that \( Ax_{\infty} = x_{\infty} \), (8) can be equivalently written as
\[
\begin{align*}
    x(k + 1) - x_{\infty} &= A(x(k) - x_{\infty}), \quad x(0) = x_{\infty}, \quad k \in \mathbb{Z}_+.
\end{align*}
\]
Hence,
\[
\begin{align*}
    \sum_{k=0}^{N} (x(k) - x_{\infty})^T R(x(k) - x_{\infty}) \\
    &= -(x(N) - x_{\infty})^T P(x(N) - x_{\infty}) + (x_0 - x_{\infty})^T P(x_0 - x_{\infty}).
\end{align*}
\]
(23)
Now, it follows from the semiobservability of \((A, C)\) that \( R x_{\infty} = 0 \). Hence, letting \( N \to \infty \) and noting that \( x(k) \to x_{\infty} \) as \( t \to \infty \) it follows from (23) that
\[
\sum_{k=0}^{\infty} x^T(k) R x(k) = (x_0 - x_{\infty})^T P(x_0 - x_{\infty}).
\]
Finally, defining the free response of (8) by \( z(k) = C x(k) = C A^k x_0 \), \( k \in \mathbb{Z}_+ \) and noting that \( R = C^T C \), it follows from Parseval’s theorem that
\[
\begin{align*}
    (x_0 - x_{\infty})^T P(x_0 - x_{\infty}) &= \sum_{k=0}^{\infty} z^T(k) z(k) \\
    &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \| G(e^{j\theta}) \|^2_{F} \, d\theta.
\end{align*}
\]
(25)
This completes the proof. \( \square \)

Next, we give a necessary and sufficient condition for characterising semistability using the Lyapunov Equation (16). Before we state this result, the following lemmas are needed.

Lemma 2.2: Consider the linear dynamical system (8). If (8) is semistable, then for every \( n \times n \) non-negative definite matrix \( R \),
\[
\sum_{k=0}^{\infty} (x(k) - x_{\infty})^T R (x(k) - x_{\infty}) < \infty,
\]
(26)
where \( x_{\infty} = [I_n - (A - I_n)(A - I_n)^T] x_0 \).

Proof: Since \( A \) is semistable, it follows from the Jordan decomposition that there exists an invertible matrix \( S \in \mathbb{C}^{n \times n} \) such that \( A = S J S^{-1} \), where \( J \in \mathbb{C}^{r \times r} \), \( r = \text{rank } A \), and \( \rho(J) < 1 \). Let \( z(k) = S^{-1} x(k) \) and \( z_{\infty} = S^{-1} x_{\infty} \). \( k \in \mathbb{Z}_+ \). Then (8) becomes
\[
\begin{align*}
    z(k + 1) &= \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} z(k), \quad z(0) = S^{-1} x_0, \quad k \in \mathbb{Z}_+,
\end{align*}
\]
(27)
which implies that \( \lim_{k \to \infty} z(k) = 0 \), \( i = 1, \ldots, r \), and \( z_j(k) = z_j(0) \), \( j = r + 1, \ldots, n \), that is, \( z_{\infty} = [0, \ldots, 0, \bar{z}_{r+1}(0), \ldots, \bar{z}_n(0)]^T \). Now,
\[
\begin{align*}
    \sum_{k=0}^{\infty} (x(k) - x_{\infty})^T R (x(k) - x_{\infty}) \\
    &= \sum_{k=0}^{\infty} (z(k) - z_{\infty})^* S^* R S (z(k) - z_{\infty}) \\
    &= \sum_{k=0}^{\infty} \bar{z}_j^* S^* R \bar{z}_j, \quad \text{for } j = 1, \ldots, r.
\end{align*}
\]
where \( \bar{z}_j(k) \triangleq [z_1(k), \ldots, z_j(k), 0, \ldots, 0]^T \). Since
\[
\bar{z}(k + 1) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \bar{z}(k)
\]
(29)
and \( \rho(J) < 1 \), it follows that
\[
\sum_{k=0}^{\infty} \bar{z}_j^* S^* R \bar{z}_j < \infty,
\]
(30)
which proves the result. \( \square \)

Lemma 2.3: Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times m} \). If \( A \) and \( B \) are semistable, then \( A \otimes B \) is semistable.

Proof: Let \( \lambda \in \text{spec}(A) \) and \( \mu \in \text{spec}(B) \). Since \( A \) and \( B \) are both semistable, it follows that \( |\lambda| < 1 \) or \( \lambda = 1 \) and \( \text{am}_\mu(1) = \text{gm}_\mu(1) \), and \( |\mu| < 1 \) or \( \mu = 1 \) and \( \text{am}_\lambda(1) = \text{gm}_\lambda(1) \), where \( \text{am}_\lambda(1) \) and \( \text{gm}_\lambda(1) \) denote algebraic multiplicity of \( \lambda \in \text{spec}(X) \) and geometric multiplicity of \( \lambda \in \text{spec}(X) \), respectively. Then it follows from the fact that \( \lambda \mu \in \text{spec}(A \otimes B) \), where \( \text{spec}(A \otimes B) \subset \{ z \in \mathbb{C} : |z| < 1 \} \cup \{ 1 \} \). Next, it follows from Fact 7.4.12 of Bernstein (2005) that \( \text{gm}_\lambda(1) \leq \text{gm}_{A \otimes B}(1) \leq \text{gm}_{A \otimes B}(1) \leq \text{gm}_\lambda(1) \). Since \( \text{am}_\lambda(1) = \text{gm}_\lambda(1) \) and \( \text{gm}_\lambda(1) = \text{gm}_{A \otimes B}(1) \), it follows that \( \text{am}_{A \otimes B}(1) = \text{am}_{A \otimes B}(1) \), and hence \( A \otimes B \) is semistable. \( \square \)

Lemma 2.4: Let \( x \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \), and assume that \( A \) is semistable. Then \( \sum_{k=0}^{\infty} A^k x \) exists if and only if \( x \in \mathbb{R}(A - I_n) \). In this case, \( \sum_{k=0}^{\infty} A^k x = -(A - I_n)^{-1} x \).

Proof: The proof is similar to the proofs of (viii) and (ix) of Lemma 5.2 of Haddad et al. (2005b), and hence is omitted. \( \square \)

Theorem 2.2: Consider the linear dynamical system (8). Then (8) is semistable if and only if for every semiobservable pair \((A, C)\) there exists an \( n \times n \) matrix \( P \geq 0 \) such that (16) holds. Furthermore, if \((A, C)\) is semiobservable and \( P \) satisfies (16), then
\[
P = \sum_{k=0}^{\infty} (A^k)^T R A^k + P_0
\]
(31)
for some $P_0 = P_0^T \in \mathbb{R}^{n \times n}$ satisfying
\[ A^T P_0 A = 0 \] (32)
and
\[ P_0 \geq -\sum_{k=0}^{\infty} (A^k)^T R A^k. \] (33)
In addition, $\min_{P \in \mathcal{P}} \|P\|_F$ has a unique solution $P$
given by
\[ P = \sum_{k=0}^{\infty} (A^k)^T R A^k, \] (34)
where $\mathcal{P}$ denotes the set of all $P$ satisfying (16).
Finally, (8) is semistable if and only if for every
semiobservable pair $(A, C)$ there exists an $n \times n$ matrix $P > 0$
such that (16) holds.

**Proof:** Sufficiency for the first implication follows from Theorem 2.1. To show necessity, assume that (8) is semistable. Then, $\lim_{k \to \infty} x(k) = x_e$, where $x_e = [I_n - (A - I_0)(A - I_0)^\dagger] x_0$. For a semi-
observable pair $(A, C)$, let
\[ P = \sum_{k=0}^{\infty} ((I_n - A)(I_n - A)^\dagger)^T (A^k)^T R A^k (I_n - A)(I_n - A)^\dagger. \] (35)
Then, for $x_0 \in \mathbb{R}^n$,
\[ x_0^T P x_0 = \sum_{k=0}^{\infty} x_0^T ((I_n - A)(I_n - A)^\dagger)^T (A^k)^T R A^k (I_n - A)(I_n - A)^\dagger x_0 \]
\[ = \sum_{k=0}^{\infty} (x_0 - x_e)^T (A^k)^T R A^k (x_0 - x_e) \]
\[ = \sum_{k=0}^{\infty} (x(k) - x_e)^T R (x(k) - x_e), \] (36)
where we used the fact that $x(k) - x_e = A^k(x_0 - x_e)$.
It follows from Lemma 2.2 that $P$ is well defined.
Since $x_e \in \mathcal{N}(A - I_0)$, it follows from (5) that $Rx_e = 0$,
and hence
\[ x_0^T P x_0 = \sum_{k=0}^{\infty} x_0^T R x(k) = \sum_{k=0}^{\infty} x_0^T (A^k)^T R A^k x_0, \quad x_0 \in \mathbb{R}^n, \] (37)
which implies that
\[ P = \sum_{k=0}^{\infty} (A^k)^T R A^k. \] (38)
Now, (16) is immediate using the fact that $Rx_e = 0$.

Next, since $A$ is semistable, it follows from
the above result that there exists an $n \times n$ non-
negative-definite matrix $P$ such that (16) holds or,
equivalently, $vec P = (A \otimes A)^T vec P + vec R$, that is,
$(I_p - (A \otimes A)^T) vec P = vec R$. Hence, $vec R \in \mathcal{R}(I_p - (A \otimes A)^T)$ and $\mathcal{P} = \{ P \in \mathbb{R}^{n \times n} : vec^{-1}((I_p - (A \otimes A)^T) vec R) + vec^{-1}(z) \}$ for some $z \in \mathcal{N}(I_p - (A \otimes A)^T)$. Next, it follows from Lemma 2.3 that
$A \otimes A$ is semistable, and hence by Lemma 2.4,
\[ vec^{-1}((I_p - (A \otimes A)^T)^T vec R) \]
\[ = \sum_{k=0}^{\infty} vec^{-1}((A^k)^T (A^k)^T vec R) \]
\[ = \sum_{k=0}^{\infty} vec^{-1}((A^k)^T \otimes (A^k)^T vec R) \]
\[ = \sum_{k=0}^{\infty} (A^k)^T R A^k, \] (39)
where we used the facts that $(X \otimes Y)^T = X^T \otimes Y^T$, $(X \otimes Y)(Z \otimes W) = XZ \otimes YW$, and vec($XYZ$) = $(Z^T \otimes X)$vec $Y$ (Bernstein 2005, chapter 7). Hence,
\[ P = \sum_{k=0}^{\infty} (A^k)^T R A^k + vec^{-1}(z), \] (40)
where vec$^{-1}(z)$ satisfies vec$^{-1}(z) = (vec^{-1}(z))^T$, $A^T vec^{-1}(z) A = 0$, and vec$^{-1}(z) \geq -\sum_{k=0}^{\infty} (A^k)^T R A^k$. If $P$ is such that $\min_{P \in \mathcal{P}} \|P\|_F$ holds, then it follows that $P$ is the unique solution of a least squares minimisation problem and is given by
\[ P = vec^{-1}((I_p - (A \otimes A)^T)^T vec R) = \sum_{k=0}^{\infty} (A^k)^T R A^k. \] (41)

Finally, suppose $(A, C)$ is semiobservable. Then, it follows from the first part of the theorem that there exists an $n \times n$ matrix $P \succeq 0$ such that (16) holds. Let $\hat{P} \triangleq P + L^T L$, where $L = I_n - (A - I_0)(A - I_0)^\dagger$. Then using similar arguments as in the proof of Theorem 2.1, it can be shown that $\hat{P} > 0$ and satisfies (16). Conversely, if there exists $P > 0$ such that (16) holds, consider the function $V(x) = x^T P x$. Using similar arguments as in the proof of Theorem 2.1, it can be shown that the largest invariant subset $\mathcal{M}$ of $\mathcal{N}(R)$ is given by $\mathcal{M} = \mathcal{N}(A - I_0)$. For $x_e \in \mathcal{N}(A - I_0)$, Lyapunov stability of $x_e$ now follows by considering the Lyapunov function $V(x - x_e)$.
Theorem 2.3: Consider the linear dynamical system (8) with output (3). Assume that $A$ is semistable and $(A, C)$ is semiobservavable. Let $P_{\text{min}}$ be the solution to the linear matrix inequality minimisation problem

$$\min \{ \text{tr} PV : P \geq 0 \text{ and } A^T PA + R - P \leq 0 \},$$

where $V \in \mathbb{R}^{n \times n}$, $V \geq 0$. Then

$$\text{tr} P_{\text{min}} V = \text{tr} \sum_{k=0}^{\infty} (A^k)^T R A^k V.$$  \hspace{1cm} (43)

Proof: Let $\hat{P} = \sum_{k=0}^{\infty} (A^k)^T R A^k$ and $P \geq 0$ be such that

$$A^T P A + R - P \leq 0.$$  \hspace{1cm} (44)

(Note that $A^T \hat{P} A + R = \hat{P}$, which implies that a $P \geq 0$ satisfying (44) exists.) Now, let $W \in \mathbb{R}^{n \times n}$, $W \geq 0$, be such that

$$P = A^T P A + R + W.$$  \hspace{1cm} (45)

Next, since $(A, C)$ is semiobservavable it follows that if $x_c \in \mathcal{N}(A - L)$, then $R x_c = 0$, and hence it follows from (45) that $W x_c = 0$. Now, using identical arguments as in the proof of Theorem 2.2 it follows that

$$P = \sum_{k=0}^{\infty} (A^k)^T (R + W) A^k \geq \sum_{k=0}^{\infty} (A^k)^T R A^k = \hat{P}. \hspace{1cm} (46)$$

Finally, since $\hat{P}$ is an element of the feasible set of the optimisation problem (42), $\text{tr} P_{\text{min}} V = \text{tr} PV$. \hfill \square

Finally, we provide a dual result to Theorem 2.3 which is necessary for developing feedback controllers guaranteeing closed-loop semistability.

Theorem 2.4: Consider the linear dynamical system (8) with output (3). Assume that $A$ is semistable and let $V \in \mathbb{R}^{n \times n}$, $V \geq 0$, be such that $(A, V)$ is semioorthonable. Let $Q_{\text{min}}$ be the solution to the LMI minimisation problem

$$\min \{ \text{tr} QR : Q \geq 0 \text{ and } AQA^T + V - Q \leq 0 \}.$$  \hspace{1cm} (47)

Then

$$\text{tr} Q_{\text{min}} R = \text{tr} \sum_{k=0}^{\infty} (A^k)^T R A^k V = \text{tr} P_{\text{min}} V,$$  \hspace{1cm} (48)

where $P_{\text{min}}$ is the solution to the LMI minimisation problem given by (42).

Proof: The proof is a direct consequence of Theorem 2.3 by noting that $(A, V)$ is semioorthonable if and only if $(A^T, V)$ is semiobservavable. Now, replacing $A$ with $A^T$ and $R$ with $V$ in Theorem 2.3 it follows that

$$\text{tr} Q_{\text{min}} R = \text{tr} \sum_{k=0}^{\infty} (A^T)^k V A^k R = \text{tr} \sum_{k=0}^{\infty} (A^k)^T R A^k V = \text{tr} P_{\text{min}} V.$$  \hspace{1cm} (49)

This completes the proof. \hfill \square

3. Optimal semistable stabilisation

In this section, we consider the problem of optimal state feedback control for semistable stabilisation of linear dynamical systems. Specifically, we consider the discrete-time controlled linear system given by

$$x(k + 1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \hspace{1cm} (50)$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathbb{Z}_+$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathbb{Z}_+$, is the control input, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$, with the state feedback controller $u(k) = Kx(k)$, where $K \in \mathbb{R}^{m \times n}$ is such that the closed-loop system given by

$$x(k + 1) = (A + BK)x(k), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+ \hspace{1cm} (51)$$

is semistable and the performance criterion

$$J(K) \triangleq \sum_{k=0}^{\infty} \left[ (x(k) - x_c)^T R_1 (x(k) - x_c) + (u(k) - u_0)^T R_2 (u(k) - u_0) \right]$$  \hspace{1cm} (52)

is minimised, where $R_1 \triangleq E_1^T E_1$, $R_2 \triangleq E_2^T E_2 > 0$, $R_{12} \triangleq E_1^T E_2 = 0$, $u_c = K x_c$ and $x_c = \lim_{k \to -\infty} x(k)$.

Note that it follows from Lemma 2.2 that if the closed-loop system is semistable, then $J(K)$ is well defined. To develop necessary conditions for the optimal semistable control problem, we assume that $(A, B)$ is semioorthonable, $(A, E_1)$ is semiobservavable and $x_c \in \mathcal{N}(K)$. In this case, it follows from Proposition 2.2 that $(A + BK, R_1 + K^T R_2 K)$ is semiobservavable with respect to $K$, and hence $(R_1 + K^T R_2 K)x_c = 0$. Thus,

$$J(K) = \sum_{k=0}^{\infty} x_c^T (\hat{A}^k)^T (R_1 + K^T R_2 K) \hat{A}^k x_0 = \text{tr} \sum_{k=0}^{\infty} (\hat{A}^k)^T \hat{R} \hat{A}^k V = \text{tr} P_{LS} V,$$  \hspace{1cm} (53)

where we assume that the initial state $x_0 \in \mathbb{R}^n$ is a random variable such that $E[x_0] = 0$ and $E[x_0 x_0^T] = V$, $\hat{A} \triangleq A + BK$, $\hat{R} \triangleq R_1 + K^T R_2 K$, and $P_{LS} \triangleq \text{tr} \sum_{k=0}^{\infty} (\hat{A}^k)^T \hat{R} \hat{A}^k$ denotes the least squares solution to

$$P = \hat{A}^T P \hat{A} + \hat{R}. \hspace{1cm} (54)$$
Unlike the standard $H_2$ optimal control problem, $P_{LS} \geq 0$ is not a unique solution to (54).

The following theorem presents an LMI solution to the $H_2$ optimal semistable control problem.

**Theorem 3.1:** Consider the linear dynamical system (50) and assume that $(A, E)$ is semis observable and $(A, V)$ is semicontrollable. Let $Q \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{m \times n}$ be the solution to the LMI minimisation problem

$$
\min_{Q \in \mathbb{R}^{n \times n}, X \in \mathbb{R}^{m \times n}, W \in \mathbb{R}^{n \times n}} \text{tr } W,
$$

subject to

$$
\begin{bmatrix}
Q & (E_1 Q + E_2 X)^T \\
E_1 Q + E_2 X & W
\end{bmatrix} > 0, \tag{56}
$$

$$
\begin{bmatrix}
V - Q & (AQ + BX)^T \\
AQ + BX & -Q
\end{bmatrix} \leq 0. \tag{57}
$$

Then $K = XQ^{-1}$ is a semistabilising controller for (50), that is, $A + BK$ is semistable. Furthermore, $K$ minimises the $H_2$ performance criterion $J(K)$ given by (52).

**Proof:** Since $K = XQ^{-1}$ it follows from (57) using Schur compliments that

$$(A + BK)Q(A + BK)^T + V - Q \leq 0, \tag{58}$$

which, since $(A, V)$ is semicontrollable, implies that $A + BK$ is semistable. Next, note that (56) holds if and only if

$$W > (E_1 Q + E_2 X)Q^{-1}(E_1 Q + E_2 X)^T, \tag{59}$$

which implies that the minimisation problem (55)–(57) is equivalent to

$$\min \text{tr}(E_1 Q + E_2 X)Q^{-1}(E_1 Q + E_2 X)^T, \tag{60}$$

subject to

$$AQ^T + AX^T B^T + BX^T A^T + BXQ^{-1} X^T B^T + V - Q \leq 0, \tag{61}$$

$$Q > 0. \tag{62}$$

Hence, noting that (60)–(62) is equivalent to

$$\min \text{tr } Q \tilde{R}, \tag{63}$$

subject to

$$\tilde{A} Q \tilde{A}^T + V - Q \leq 0, \tag{64}$$

$$Q > 0. \tag{65}$$

the result follows as a direct consequence of Theorems 2.4 and 2.2.

4. Information flow models

In the remainder of this article, we use the optimal control framework developed in §3 to design optimal controllers for multiagent network dynamical systems. Specifically, we use undirected and directed graphs to represent a dynamical network and present solutions to the consensus problem for networks with both graph topologies (or information flow) (Olfati-Saber and Murray). Specifically, let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted directed graph (or digraph) denoting the dynamical network (or dynamic graph) with the set of nodes (or vertices) $\mathcal{V} = \{1, \ldots, n\}$ involving a finite nonempty set denoting the agents, the set of directed edges (or information flow) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and an adjacency matrix $A \in \mathbb{R}^{n \times n}$ such that $A_{ij} = 1$, $i, j = 1, \ldots, n$, if $(j, i) \in \mathcal{E}$, and 0 otherwise. The edge $(i, j) \in \mathcal{E}$ denotes that agent $G_i$ can obtain information from agent $G_j$, but not necessarily vice versa. Moreover, we assume that $A_{ii} = 0$ for all $i \in \mathcal{V}$. A graph or undirected graph $\mathcal{G}$ associated with the adjacency matrix $A \in \mathbb{R}^{n \times n}$ is a directed graph for which the arc set is symmetric, that is, $A = A^T$. A graph $\mathcal{G}$ is balanced if

$$\sum_{i=1}^{n} A_{i,j} = \sum_{j=1}^{n} A_{i,j} \quad \text{for all} \quad i, j = 1, \ldots, n. \quad \text{Finally, we denote the value of the node} \quad i, \quad i = 1, \ldots, n, \quad \text{at time} \quad k \quad \text{by} \quad x_i(k) \in \mathbb{R}. \quad \text{The consensus problem involves the design of a dynamic algorithm that guarantees information state equipartition, that is,} \quad \lim_{k \to \infty} x_i(k) = x_\infty \quad \text{for} \quad i = 1, \ldots, n.$$

The information flow model is a network dynamical system involving the trajectories of the dynamical network characterised by the multiagent dynamical system $\mathcal{G}$ given by

$$x_i(k+1) = x_i(k) + \sum_{j=1, j \neq i}^{q} \phi_j(x(k)), \quad x_i(0) = x_0, \quad k \in \mathbb{Z}_+, \quad \text{for} \quad i = 1, \ldots, q, \tag{66}$$

where $q \geq 2$, or, in vector form

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \mathbb{Z}_+, \tag{67}$$

where $x(k) \triangleq [x_1(k), \ldots, x_q(k)]^T \in \mathbb{R}^q$, $k \in \mathbb{Z}_+$, represents the information state vector, $\phi_j: \mathbb{R}^q \to \mathbb{R}$ is continuous, $i, j = 1, \ldots, q$, $i \neq j$, and represents the information flow from the $j$-th agent to the $i$-th agent, and $f = [f_1, \ldots, f_q]^T: \mathbb{R}^q \to \mathbb{R}^q$ is such that $f(x) = x_i + I(x)$, where for each $i \in \{1, \ldots, q\}$, $I(x) \triangleq \sum_{j=1, j \neq i}^{q} \phi_j(x)$. This non-linear model is proposed in Haddad et al. (2005a, b) and is called a power...
balance equation. Here, however, we address a slightly more general model in which \( \phi_A(x) \) has no special structure and \( x \) need not be constrained to the non-negative orthant.

**Definition 4.1** (Berman and Plemmons 1979): A directed graph \( \mathbf{G} \) is strongly connected if for any ordered pair of vertices \((i, j), i \neq j\), there exists a path (i.e. sequence of arcs) leading from \( i \) to \( j \).

Recall that \( \mathcal{A} \in \mathbb{R}^{q \times q} \) is irreducible, that is, there does not exist a permutation matrix such that \( \mathcal{A} \) is congreident to a lower-block triangular matrix, if and only if \( \mathbf{G} \) is strongly connected (see Theorem 2.7 of Berman and Plemmons (1979)). Furthermore, note that for an undirected graph, \( \mathcal{A} = \mathcal{A}^T \), and hence every undirected graph is balanced.

**Assumption 1:** For the connectivity matrix \( \mathcal{C} \in \mathbb{R}^{q \times q} \) associated with the multiagent dynamical system \( \mathcal{G} \) defined by

\[
\mathcal{C}_{(i,j)} = \begin{cases} 
0 & \text{if } \phi_j(x) \equiv 0, \ i \neq j, \ i,j = 1, \ldots, q, \\
1, & \text{otherwise,}
\end{cases}
\]

and \( \mathcal{C}_{(i,i)} = -\sum_{k=1, k \neq i}^{q} \mathcal{C}_{(i,k)}, \ i = j, \ i,j = 1, \ldots, q, \) rank \( \mathcal{C} = q - 1 \), and for \( \mathcal{C}_{(i,i)} = 1, \ i \neq j, \) \( \phi_j(x) \equiv 0 \) if and only if \( x_i = x_j \).

**Assumption 2:** For \( i, j = 1, \ldots, q \), \( (x_i - x_j)\phi_j(x) \leq 0 \), \( x \in \mathbb{R}^q \).

**Assumption 3:** For \( i, j = 1, \ldots, q \), \( |\phi_j(x)| \leq \lambda_j |x_i - x_j| \), \( \lambda_j > 0 \), \( x \in \mathbb{R}^q \).

The negative of the connectivity matrix, that is, \(-\mathcal{C}\), is known in the literature as the Laplacian of the graph \( \mathbf{G} \). Furthermore, note that \( \mathcal{C}_{(i,j)} = -\mathcal{A}_{(i,j)} \) for all \( i, j = 1, \ldots, q, i \neq j \). The fact that \( \phi_j(x) \equiv 0 \) only if and only if \( x_i = x_j, i \neq j \), implies that agents \( G_i \) and \( G_j \) are connected, and hence can share information; alternatively, \( \phi_j(x) \equiv 0 \) implies that agents \( G_i \) and \( G_j \) are disconnected, and hence cannot share information. Assumption 1 implies that if the information in the connected agents \( G_i \) and \( G_j \) is equal, then information exchange between these agents is not possible. This statement is reminiscent of the zeroth law of thermodynamics, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if \( \mathcal{C} = -\mathcal{A}^T \) and rank \( \mathcal{C} = q - 1 \), then it follows that the connectivity matrix \( \mathcal{C} \) and the adjacency matrix \( \mathcal{A} \) are irreducible, which implies that for any pair of agents \( G_i \) and \( G_j, i \neq j \), of \( \mathcal{G} \) there exists a sequence of information connectors (information arcs) of \( \mathcal{G} \) that connect \( G_i \) and \( G_j \). Assumption 2 implies that information flows from information rich agents to information poor agents and is reminiscent of the second law of thermodynamics, which states that heat (energy) must flow in the direction of lower temperatures. Assumption 3 implies that the amount of transferred information flow does not exceed an amount which is proportional to the information difference between agents \( G_i \) and \( G_j \). This assumption is an information flow constraint similar to the capacity limitation constraint of communication channels in information theory. In particular, this constraint implies that the communication channel for information exchange between two agents cannot exceed an amount of information which is proportional to the available transferred information between two agents. For further details on Assumptions 1–3, see Haddad et al. 2005a,b).

5. Semistability of information flow models

As noted in §1, a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state is not determined completely by the system dynamics, but on the initial system state as well. For such a system possessing a continuum of equilibria, semistability, and not asymptotic stability is the relevant notion of stability. For the statement of the next result, let \( \mathbf{e} \in \mathbb{R}^q \) denote the ones vector of order \( q \), that is, \( \mathbf{e} = [1, \ldots, 1]^T \).

**Proposition 5.1:** Consider the information flow model (67) and assume that Assumptions 1 and 2 hold. Then \( I(x) = 0 \) for all \( i = 1, \ldots, q \) if and only if \( x_1 = \cdots = x_q \).

**Proof:** The proof of the first conclusion is similar to the proof of Theorem 1 of Olfati-Saber and Murray (2004), and hence is omitted. The second conclusion is a direct consequence of the first conclusion.

The following lemmas involving graph–theoretic notions are needed for the main result of this section. For the statement of the next result, let \( |\mathcal{V}| \) denote the cardinality of the set \( \mathcal{V} \).

**Lemma 5.1:** Assume that \( \mathbf{G} \) is an undirected strongly connected graph with \( n \) nodes and value \( z_i \in \mathbb{R} \) for \( i = 1, \ldots, n \). Furthermore, assume that for each node \( i \), the set of nodes of its neighbours is given by \( \mathcal{V}_n = \{i_1, \ldots, i_{n_i}\} \), where \( n_i = |\mathcal{V}_n| \). If for each node \( i \), \( z_{i_1} = \cdots = z_{i_{n_i}} \) and, for some \( m \in \{1, \ldots, n\} \) and some \( m_j \in \mathcal{V}_n \), \( z_m = z_{m_j} \), then \( z_1 = \cdots = z_n \).
Proof: The result is trivial for the cases where \( n = 2 \) and \( n = 3 \). Consider the case where \( n \geq 4 \). Let \( m, m' \in \{1, \ldots, n\} \), be the nodes satisfying \( z_m = z_{m'} \) for some \( m_1 \in \mathcal{V}_n \). If \( |\mathcal{V}_n| = 1 \), then we consider the node \( m_1 \). Since \( \mathcal{G} \) is strongly connected and \( n \geq 4 \), it follows that \( \mathcal{V}_n \neq \emptyset \). Hence, for every neighbour \( s \in \mathcal{V}_m \), \( z_s = z_{m'} = z_m \). Choose a neighbour \( s \in \mathcal{V}_m \) such that \( |\mathcal{V}_s| \geq 2 \) (this is possible since \( \mathcal{G} \) is strongly connected). Then, by connectivity, it follows that for every node \( k \in \mathcal{V}\{s, m, m'\} \), \( z_k = z_{m'} = z_m \) or \( z_k = z_s = z_{m'} \), and hence the conclusion follows.

Otherwise, if \( |\mathcal{V}_n| \geq 2 \), then choose a neighbour \( m_2 \in \mathcal{V}_n \) such that \( |\mathcal{V}_m| \geq 2 \) (this is possible since \( \mathcal{G} \) is strongly connected). Then, by connectivity, it follows that for every node \( k \in \mathcal{V}\{m, m_2\} \), \( z_k = z_{m_2} \) or \( z_k = z_{m} \), and hence the conclusion follows.

For the next result, recall that a cycle of the graph \( \mathcal{G} \) is a connected graph where every vertex has exactly two neighbours (Godsil and Royle 2001) and an odd cycle of the graph \( \mathcal{G} \) is a cycle of \( \mathcal{G} \) with an odd number of edges (Diesel 1997, p. 14).

**Lemma 5.2:** Assume that \( \mathcal{G} \) is an undirected strongly connected graph with \( n \) nodes and value \( z_i \in \mathbb{R} \) for \( i = 1, \ldots, n \). Furthermore, assume that for each node \( i \), the set of nodes of its neighbours is given by \( \mathcal{V}_n = \{i_1, \ldots, i_m\} \), where \( n_i = |\mathcal{V}_n| \). If \( \mathcal{G} \) contains an odd cycle and for each \( i \), \( z_{i_1} = \cdots = z_{i_{m_i}} \), then \( z_1 = \cdots = z_n \).

**Proof:** Since \( \mathcal{G} \) contains a cycle of length \( m \), where \( 3 \leq m \leq n \) is odd, without loss of generality, let \( 1, \ldots, m \) be the nodes of the cycle. Then, by connectivity, \( z_1 = z_2 = \cdots = z_m = z_{m+1} = \cdots = z_n \), which implies that there exists a node \( i \) such that \( z_i = z_{i_{m_i}} \), where \( i_{m_i} \in \mathcal{V}_n \). Thus, it follows from Lemma 5.1 that \( z_1 = \cdots = z_n \).

Next, we present the main stability result of this section for information flow models. Note that although general stability results have been developed in Moreau (2005) and Angeli and Bliman (2006), the conditions of those results are restrictive. Specifically, in Moreau (2005) it is always required that for each \( i \in \{1, \ldots, q\} \), the right-hand side \( f(x) \) of (67) is contained in the relative interior of the convex hull of \( x_i \) and its neighbours \( x_j \). Although Angeli and Bliman (2006) extended the results of Moreau (2005) to the case where the linear combination of \( x_i \) and its neighbours \( x_j \) is not necessarily convex, the results still need several technical assumptions. In the following result, we present improved results for semistability of (67). For this result, we define an in-neighbour of the \( i \)-th agent to be those agents whose information can be received by the \( i \)-th agent.

**Theorem 5.1:** Consider the information flow model (67) and assume that Assumptions 1–3 hold. For \( i = 1, \ldots, q_2 \geq 2 \), let \( n_i \geq 1 \) be the number of neighbours of the \( i \)-th agent in the case where \( \mathcal{G} \) is a graph and \( n_i \geq 1 \) be the number of in-neighbours of the \( i \)-th agent in the case where \( \mathcal{G} \) is a digraph. Then the following statements hold:

(i) If \( p_i \phi_j(x) = -p_j \phi_i(x) \) and \( \lambda_{ij} < \frac{2p_i}{(n_i p_j + n_j p_i)} \) for all \( i, j = 1, \ldots, q \), \( i \neq j \), \( p_i > 0 \), then for every \( \alpha \in \mathbb{R} \), \( \alpha \) is a semistable equilibrium state of (67). Furthermore, \( \lim_{k \to \infty} x(k) = \alpha \cdot \mathbf{e} \) as \( k \to \infty \), where \( \alpha_i = \frac{\sum_{j=1}^{q} p_j x_j(0)}{\sum_{j=1}^{q} p_j} \).

(ii) If \( p_i \phi_j(x) = -p_j \phi_i(x) \), \( (n_i p_j + n_j p_i) \), \( \lambda_{ij} \leq \frac{2p_i}{(n_i p_j + n_j p_i)} \) for all \( i, j = 1, \ldots, q \), \( i \neq j \), \( p_i > 0 \), and \( \lambda_{ij} < \frac{2p_i}{(n_i p_j + n_j p_i)} \) for some \( i, j = 1, \ldots, q \), then for every \( \alpha \in \mathbb{R} \), \( \alpha \) is a semistable equilibrium state of (67). Furthermore, \( \lim_{k \to \infty} x(k) = \alpha \cdot \mathbf{e} \) as \( k \to \infty \).

(iii) If \( \mathcal{G} \) contains an odd cycle, \( p_i \phi_j(x) = -p_j \phi_i(x) \), \( (n_i p_j + n_j p_i) \), \( \lambda_{ij} \leq \frac{2p_i}{(n_i p_j + n_j p_i)} \) for all \( i, j = 1, \ldots, q \), \( i \neq j \), \( p_i > 0 \), then for every \( \alpha \in \mathbb{R} \), \( \alpha \) is a semistable equilibrium state of (67). Furthermore, \( \lim_{k \to \infty} x(k) = \alpha \cdot \mathbf{e} \) as \( k \to \infty \).

(iv) Let \( \phi_i(x) = \phi_i(x, x_i) = (1/p_i) A_{i,i}(x_i - x_i) \) for all \( i, j = 1, \ldots, q \), \( i \neq j \). Assume that \( C^T \mathbf{e} = 0 \) and \( p_i \geq n_i^2 \), \( i = 1, \ldots, q \). Furthermore, assume that \( p_i > n_i^2 \) for some \( r \in \{1, \ldots, q\} \) such that \( A_{r,r} = 1 \). Then for every \( \alpha \in \mathbb{R} \), \( \alpha \) is a semistable equilibrium state of (67). Furthermore, \( \lim_{k \to \infty} x(k) = \alpha \cdot \mathbf{e} \) as \( k \to \infty \).

**Proof:** First, note that it follows from Lemma 5.1 that \( \alpha \cdot \mathbf{e} \) is an equilibrium state of (67).

(i) To show Lyapunov stability of the equilibrium state \( \alpha \cdot \mathbf{e} \), consider the Lyapunov function candidate given by

\[
V(x) = (x - \alpha \cdot \mathbf{e})^T P(x - \alpha \cdot \mathbf{e}),
\]

where \( P \triangleq \text{diag}(p_1, \ldots, p_q) \). Now, since \( p_i \phi_j(x) = -p_j \phi_i(x) \), \( x \in \mathbb{R}^q \), \( i \neq j \), \( i, j = 1, \ldots, q \), and \( \mathbf{e}^T P x(k+1) = \mathbf{e}^T P x(k) \), \( k \in \mathbb{Z}_+ \), it follows from Assumptions 2 and 3 that

\[
\Delta V(x(k)) = 2 \sum_{i=1}^{q} \sum_{j=1, j \neq i}^{q} p_i \left[ \sum_{j=1, j \neq i}^{q} \phi_j(x(k)) \right]^2
\]
Let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : \Delta V(x) = 0\} = \{x \in \mathbb{R}^q : (x_i - x_j) \times \phi_j(x) = 0, \quad i = 1, \ldots, q, \ j \in K_i\}$. Now, by Assumption 1 the directed graph associated with the connectivity matrix $\mathcal{C}$ for the multiagent dynamical system (67) is strongly connected, which implies that $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Since the set $\mathcal{R}$ consists of the equilibrium states of (67), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is given by $\mathcal{M} = \mathcal{R}$. Hence, it follows from Proposition 2.1 that $\alpha\varepsilon$ is a semistable equilibrium state of (67). To show that $x(k) \to \alpha\varepsilon$ as $k \to \infty$, note that since $p^T(x(k) = p^T x(0)$ and $x(k) \to \mathcal{M}$ as $k \to \infty$, where $p \triangleq [p_1, \ldots, p_q]^T \in \mathbb{R}^q$, it follows that $x(k) \to \alpha\varepsilon$ as $k \to \infty$.

(ii) Using similar arguments as (i), it can be shown that $\alpha\varepsilon$ is Lyapunov stable. To show semistability of $\alpha\varepsilon$, let $\mathcal{R} \triangleq \{x \in \mathbb{R}^q : \Delta V(x) = 0\}$, where $V()$ is given by (69). In this case, it follows from (70) that

$$
\mathcal{R} = (\mathcal{R}_1 \cup \mathcal{R}_2) \cap \mathcal{R}_3 = (\mathcal{R}_1 \cap \mathcal{R}_3) \cup (\mathcal{R}_2 \cap \mathcal{R}_3),
$$

where $\mathcal{R}_1 \triangleq \{x \in \mathbb{R}^q : \phi_j(x) = 0, i = 1, \ldots, q, j \in K_i\}$, $\mathcal{R}_2 \triangleq \{x \in \mathbb{R}^q : ((n_i / p_i) + (n_j / p_j))\phi_j(x) = 2(x_i - x_j), i = 1, \ldots, q, j \in K_i\}$, and $\mathcal{R}_3 \triangleq \{x \in \mathbb{R}^q : \phi_j(x) = \phi_k(x), i = 1, \ldots, q, j \in N_i, k \in N_i \setminus \{j\}\}$. If $\phi_j(x) = 0$, then $x_i = x_j$, $i = 1, \ldots, q$, $j \in K_i$. Now, by Assumption 1 the directed graph associated with the connectivity matrix $\mathcal{C}$ for the multiagent dynamical system (67) is strongly connected, which implies that $x_1 = \cdots = x_q$. Hence, $\mathcal{R}_1 \cap \mathcal{R}_3 = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Next, we consider the case where $((n_i / p_i) + (n_j / p_j))\phi_j(x) = 2(x_i - x_j)$ and $x \in \mathcal{R}_3$, $i = 1, \ldots, q, j \in K_i$. Since $p_i\phi_j(x) = -p_j\phi_j(x)$, it follows that $((n_i / p_i) + (n_j / p_j))\phi_j(x) = 2(x_i - x_j)$ and $x \in \mathcal{R}_3$, $i = 1, \ldots, q, j \in K_i$. Hence, $((n_i / p_i) + (n_j / p_j))\phi_j(x) = 2(x_i - x_j)$ and $x \in \mathcal{R}_3$, $i = 1, \ldots, q, j \in K_i$. Since $((n_i / p_i) + (n_j / p_j))\phi_j(x) = 2(x_i - x_j)$ and $x \in \mathcal{R}_3$, $i = 1, \ldots, q, j \in K_i$. Furthermore, since $\phi_j(x) = \phi_k(x)$, it follows that $x_j = x_k$, $i = 1, \ldots, q, j \in K_i$, $j \neq k$. Note that since $p_i\phi_j(x) = -p_j\phi_j(x)$, $\mathcal{G}$ is an undirected graph. Thus, $\mathcal{A} = \mathcal{A}_1^T$, and hence, $\mathcal{G}$ is strongly connected.

Now, it follows from (70) that for $x \in \mathcal{R}_2 \cap \mathcal{R}_3$, $(x_i - x_j)\phi_{in}(x) = 0$, which implies that $x_j = x_m$. Hence, it follows from Lemma 5.1 that $x_i = \cdots = x_q$, $\mathcal{R}_2 \cap \mathcal{R}_3 = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Therefore, $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. Now, the set $\mathcal{R}$ consists of the equilibrium states of (67), it follows that the largest invariant set $\mathcal{M}$ contained in $\mathcal{R}$ is the set of equilibria of (67). Hence, it follows from Proposition 2.1 that $\alpha\varepsilon$ is a semistable equilibrium state of (67). To show that $x(k) \to \alpha\varepsilon$ as $k \to \infty$, note that since $p^T(x(k) = p^T x(0)$ and $x(k) \to \mathcal{M}$ as $k \to \infty$, it follows that $x(k) \to \alpha\varepsilon$ as $k \to \infty$.

(iii) Using similar arguments as (i), it can be shown that $\alpha\varepsilon$ is Lyapunov stable. Furthermore,
using similar arguments as (ii), it follows that for $x \in \mathcal{R}_2 \cap \mathcal{R}_3$, $x_j = x_k$, $j, k \in \mathcal{N}_j$, $i = 1, \ldots, q$, $j \neq k$. Now, it follows from Lemma 5.2 that $x_1 = \cdots = x_q$. Hence, $\mathcal{R} = \{x \in \mathbb{R}^q : x_1 = \cdots = x_q\}$. The rest of the proof follows as the proof of (i).

(vi) Let $W = I_q + P^{-1}A$. First, we show that $W$ is irreducible. Note that $W$ is a stochastic matrix (Horn and Johnson 1985, p. 526). Furthermore, since

$$W - I_q = \begin{bmatrix} \frac{1}{p_1} A_{(1,1)} & \frac{1}{p_1} A_{(1,2)} & \cdots & \frac{1}{p_1} A_{(1,q)} \\ \frac{1}{p_2} A_{(2,1)} & \frac{1}{p_2} A_{(2,2)} & \cdots & \frac{1}{p_2} A_{(2,q)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_q} A_{(q,1)} & \frac{1}{p_q} A_{(q,2)} & \cdots & \frac{1}{p_q} A_{(q,q)} \end{bmatrix}$$

(74)

it follows that $\text{rank}(W - I_q) = \text{rank} C = q - 1$. Since $C^T e = 0$, it follows that $(W^T - I_q) p = 0$. Now, it follows from Berman and Plemmons (1979, p. 52) that $W$ is irreducible.

Next, note that $|\lambda_i| \leq \|W\| = 1$, $i = 1, \ldots, q$, $\lambda_i \in \text{spec}(W)$, and $\|W\|$ is an induced norm of $W$. Then, $\rho(W) = 1$. It follows from Theorem 1.4 of Berman and Plemmons (1979) that $\rho(W) = 1$ is a simple eigenvalue. Next, we show that $W$ is a primitive matrix (Horn and Johnson 1985, p. 516). Since $p_i \geq n_i^+$ for all $i \in \{1, \ldots, q\}$ and $p_i > n_i^+$ for some $r \in \{1, \ldots, q\}$, it follows that $\text{tr} W = \sum_{i=1}^q (1 + (1/p_i) A_{(i,i)} \geq 1 + (1/p_i) A_{(r,r)} > 0$. Then it follows from Corollary 2.28 of Berman and Plemmons (1979) that $W$ is primitive. Now, it follows from Theorem 2 of Haddad, Chellaboina and August (2003) that $W$ is semistable, and hence $\lim_{k \to \infty} W^k = I_q - (W - I_q)^R$. Next, it follows from (vi) of Lemma 5.2 of Haddad et al. (2005b) that $\mathcal{N}(W - I_q) = \mathcal{R}(I_q - (W - I_q)^R)$. Since $(W - I_q) e = 0$ and $\text{rank}(W - I_q) = q - 1$, it follows that $\mathcal{N}(W - I_q) = \{e\}$, where $e \in \mathbb{R}^q$, and hence $\mathcal{R}(I_q - (W - I_q)^R) = \{e\}$, which implies that $\lim_{k \to \infty} x(k) = \lim_{k \to \infty} W^k x(0) = e$. To show that $x(k) \to e$ as $k \to \infty$, note that since $p^T x(k) = p^T x(0)$ and $x(k) \to M$ as $k \to \infty$, it follows that $x(k) \to e$ as $k \to \infty$. \hfill \Box

To illustrate some of the results of Theorem 5.1, consider the linear dynamical system

$$x_1(k + 1) = \frac{1}{2}(x_2(k) + x_3(k)), \quad x_1(0) = x_{10}, \quad k \in \mathbb{Z}_+,$$

(75)

$$x_2(k + 1) = \frac{1}{2}(x_3(k) + x_1(k)), \quad x_2(0) = x_{20},$$

(76)

$$x_3(k + 1) = \frac{1}{2}(x_1(k) + x_2(k)), \quad x_3(0) = x_{30}.$$

(77)

Note that the systems (75)-(77) is an information flow model of the form given by (67) and it follows from (iii) of Theorem 5.1 that consensus and semistability of (75)-(77) are guaranteed. Figure 1 shows the trajectories of (75)-(77) versus time. Note that it is not easy to use the methods in Moreau (2005) and Angeli and Bihan (2006) to prove semistability and consensus for (75)-(77). However, using Theorem 5.1 this is straightforward.

6. Optimal fixed-structure control of network consensus

In multiagent coordination (Jadbabaie, Lin and Morse 2003; Olfati-Saber and Murray 2004) and distributed network averaging (Xiao and Boyd 2004) with a fixed communication topology, we require that $x_k \in \text{span}\{e\}$. In this section, we consider the design of a fixed-structure consensus protocol for (67) such that the closed-loop system is semistable, kernel $(f) = \text{span}\{e\}$, and (52) is minimised. Here, we consider the consensus protocol (67) given by

$$x_i(k + 1) = u_i(k), \quad x_i(0) = x_{i0}, \quad k \in \mathbb{Z}_+,$$

(78)

$$u_i(k) = x_i(k) + \sum_{j=1, j \neq i}^q \phi_{ij}(x(k)),$$

(79)

$$\phi_{ij}(x(k)) = \frac{1}{k_i} A_{ij}(x_j(k) - x_i(k)),$$

(80)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Figure1}
\caption{Trajectories versus time for (75)-(77).}
\end{figure}
where \( k_i > n_i^\tau, \ i = 1, \ldots, q \), \( C \) satisfies Assumption 1 and the conditions of Theorem 5.1. Note that for (78)–(80) Assumptions 2 and 3 are automatically satisfied. Since, by Theorem 5.1, the closed-loop system given by (67) is semistable, the optimal fixed-structure control problem involves seeking \( k_i, k_i > n_i^\tau, \ i = 1, \ldots, q \), such that the cost functional

\[
J(K) = \sum_{k=0}^{\infty} \left[ (x(k) - \alpha^e e)^T R_1 (x(k) - \alpha^e e) + (u(k) - u_k)^T R_3 (u(k) - u_k) \right],
\]

is minimised, where \( u_k = \alpha_k K^{-1} A e \), \( R_1 = E_1^T E_1 \geq 0, \ R_2 = E_2^T E_2 > 0 \) and \( E_1^T E_2 = 0 \).

The following theorem presents a bilinear matrix inequality (BMI) solution to the fixed-structure optimal semistable control problem for network consensus. For this result, define \( \mathcal{L} \triangleq \{ L \in \mathbb{R}^{p \times q} : L = \text{diag}(\ell_1, \ldots, \ell_q) \in \mathbb{R}^{q \times q}, \ell_i > n_i^\tau, \ i = 1, \ldots, q \} \).

**Theorem 6.1:** Consider the consensus protocol (78)–(80) and assume that \( (I_q + A, E_1) \) is semiobservable and \( (I_q + A, V) \) is semicontrollable. Let \( Q \in \mathbb{R}^{p \times q} \) and \( L \in \mathcal{L} \) be the solution to the BMI minimisation problem

\[
\min_{Q \in \mathbb{R}^{p \times q}, L \in \mathcal{L}, W \in \mathbb{R}^{p \times p}} \text{tr } W,
\]

subject to

\[
\begin{bmatrix}
Q & (E_1 Q + E_2 Q + E_2 LAQ)^T \\
E_1 Q + E_2 Q + E_2 LAQ & W
\end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
V - Q & (E_1 Q + E_2 Q + E_2 LAQ)^T \\
E_1 Q + E_2 Q + E_2 LAQ & -Q
\end{bmatrix} \leq 0.
\]

Then \( u = (I_q + K^{-1} A)x \) is a semistabilising controller for (78) and \( x(k) \to \alpha^e e \) as \( k \to \infty \), where \( K^{-1} = L \) and \( \alpha^e = \sum_{i=1}^{q} k_i x_i(0)/\left(\sum_{i=1}^{q} k_i\right) \). Furthermore, \( K \) minimises the \( H_2 \) performance criterion \( J(K) \) given by (81).

**Proof:** Convergence to the consensus state \( \alpha^e e \) is a direct consequence of Theorem 5.1. The optimality proof is similar to the proof of Theorem 3.1, and hence is omitted.

**Remark 6.1:** Due to the diagonal structure on \( K \), the optimisation problem given in Theorem 6.1 is a bilinear matrix inequality. A suboptimal solution to this problem can be obtained by using a two-stage optimisation process. Specifically, by fixing \( Q \) one can design the controller \( K \). Then, with \( K \) fixed, \( Q \) can be obtained. This process continues until convergence or an acceptable controller is found.

### 7. Conclusion

In this article, we extended \( H_2 \) theory to include semistable systems. Using this framework along with linear matrix inequalities we developed an \( H_2 \) optimal semistable stabilisation framework for linear discrete-time dynamical systems. The proposed framework was used to design \( H_2 \) optimal consensus control algorithms for multiagent dynamical networks. Future extensions will concentrate on the development of mixed-norm \( H_2/H_\infty, H_2/L_1 \) and \( H_\infty/L_1 \) semistable stabilisation problems.

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### References


