[17] A. Levant, "Robust exact differentiation via sliding mode technique," *Automatica*, vol. 34, no. 3, pp. 379–384, 1998.
[18] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Syst. Control Lett.*, vol. 19, no. 6, pp. 467–473, 1992.
[19] Y. X. Su, C. H. Zheng, and D. Sun, "A simple nonlinear velocity stimatiom for high-performance motion control," *IEEE Trans. Ind. Electron.*, vol. 52, no. 4, pp. 1661–1169, Aug. 2005.
[20] X. Wang, Z. Chen, and G. Yang, "Finite-time-convergent differentiator based on singular perturbation technique," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1731–1737, Sep. 2007.
[21] W. Xue, Y. Huang, and X. Yang, "What kinds of system can be used as tracking-differentiator," in *Proc. Chinese Control Conf.*, 2010, pp. 6113–6120.

Low-Frequency Learning and Fast Adaptation in Model Reference Adaptive Control

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Abstract—While adaptive control has been used in numerous applications to achieve system performance without excessive reliance on dynamical system models, the necessity of high-gain learning rates to achieve fast adaptation can be a serious limitation of adaptive controllers. This is due to the fact that fast adaptation using high-gain learning rates can cause high-frequency oscillations in the control response resulting in system instability. In this note, we present a new adaptive control architecture for nonlinear uncertain dynamical systems to address the problem of achieving fast adaptation using high-gain learning rates. The proposed framework involves a new and novel controller architecture involving a modification term in the update law. Specifically, this modification term filters out the high-frequency content contained in the update law while preserving asymptotic stability of the system error dynamics. This key feature of our framework allows for robust, fast adaptation in the face of high-gain learning rates. Furthermore, we show that transient and steady-state system performance is guaranteed with the proposed architecture. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

Index Terms—Adaptive control, command following, fast adaptation, high-gain learning rate, low-frequency learning, nonlinear uncertain dynamical systems, stabilization, transient and steady state performance.

I. INTRODUCTION

While adaptive control has been used in numerous applications to achieve system performance without excessive reliance on system models, the necessity of high-gain learning rates for achieving fast adaptation can be a serious limitation of adaptive controllers [1]. Specifically, in certain applications fast adaptation is required to achieve stringent tracking performance specifications in the face of large system uncertainties and abrupt changes in system dynamics.

This, for example, is the case for high performance aircraft systems that are subjected to system faults or structural damage which can result in major changes in aerodynamic system parameters. In such situations, adaptive control with high-gain learning rates is necessary in order to rapidly reduce and maintain system tracking errors. However, fast adaptation using high-gain learning rates can cause high-frequency oscillations in the control response resulting in system instability [2]–[4]. Hence, there exists a critical trade-off between system stability and adaptation learning rate (i.e., adaptation gain).

In this note, we present a new adaptive control architecture for nonlinear uncertain dynamical systems to address the problem of achieving fast adaptation using high-gain learning rates. The proposed framework involves a new and novel controller architecture involving a modification term in the update law. Specifically, this modification term filters out the high-frequency content contained in the update law while preserving asymptotic stability of the system error dynamics. This key feature of our framework allows for robust, fast adaptation in the face of high-gain learning rates. We further show that transient and steady-state system performance is guaranteed with the proposed architecture. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

The notation used in this technical note is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $\|\cdot\|_2$ denotes the Euclidian norm, and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M and $\mathrm{tr}(\cdot)$ for the trace operator.

II. MODEL REFERENCE ADAPTIVE CONTROL

We begin by presenting a brief review of the model reference adaptive control problem. Specifically, consider the nonlinear uncertain dynamical system given by

$$\dot{x}(t) = Ax(t) + B\Delta(x(t)) + Bu(t), \quad x(0) = x_0, \quad t \geq 0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices such that the pair (A, B) is controllable, and $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ is a matched system uncertainty. We assume that the full state is available for feedback and the control input $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. In addition, we consider the reference system given by

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}}x_{\mathrm{m}}(t) + B_{\mathrm{m}}r(t), \quad x_{\mathrm{m}}(0) = x_{\mathrm{m}0}, \quad t \geq 0 \quad (2)$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$, $t \geq 0$, is the reference state vector, $r(t) \in \mathbb{R}^r$, $t \geq 0$, is a bounded piecewise continuous reference input, $A_{\mathrm{m}} \in \mathbb{R}^{n \times n}$ is Hurwitz, and $B_{\mathrm{m}} \in \mathbb{R}^{n \times r}$.

Assumption 2.1: The matched uncertainty in (1) is linearly parameterized as

$$\Delta(x) = W^{\mathrm{T}}\beta(x), \quad x \in \mathbb{R}^n \quad (3)$$

where $W \in \mathbb{R}^{s \times m}$ is an unknown constant weighting matrix and $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a basis function of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]$.

Here, the aim is to construct a feedback control law $u(t)$, $t \geq 0$, such that the state of the nonlinear uncertain dynamical system given by (1) asymptotically tracks the state of the reference model given by (2) in the presence of matched uncertainty satisfying (3).

Next, consider the feedback control law given by

$$u(t) = u_{\mathrm{n}}(t) + u_{\mathrm{ad}}(t), \quad t \geq 0 \tag{4}$$

where the *nominal* control law $u_{\mathrm{n}}(t)$, $t \geq 0$, is given by

$$u_{\mathrm{n}}(t) = -K_1 x(t) + K_2 r(t), \quad t \geq 0 \tag{5}$$

and the *adaptive* control law $u_{\mathrm{ad}}(t)$, $t \geq 0$, is given by

$$u_{\mathrm{ad}}(t) = -\hat{W}^{\mathrm{T}}(t)\beta\left(x(t)\right), \quad t \geq 0 \tag{6}$$

where $K_1 \in \mathbb{R}^{m \times m}$ and $K_2 \in \mathbb{R}^{m \times r}$ are nominal control gains chosen such that $A_{\mathrm{m}} = A - BK_1$ and $B_{\mathrm{m}} = BK_2$ hold, and $\hat{W}(t) \in \mathbb{R}^{s \times m}$, $t \geq 0$, is an estimate of W satisfying the *standard* update law

$$\dot{\hat{W}}(t) = \Gamma\beta\left(x(t)\right)e^{\mathrm{T}}(t)PB, \quad \hat{W}(0) = \hat{W}_0, \quad t \geq 0 \tag{7}$$

where $\Gamma \in \mathbb{R}^{s \times s}$ is a positive-definite learning rate matrix, $e(t) \triangleq x(t) - x_{\mathrm{m}}(t)$, $t \geq 0$, is the system error state, and $P \in \mathbb{R}^{n \times n}$ is a positive-definite solution of the Lyapunov equation

$$0 = A_{\mathrm{m}}^{\mathrm{T}}P + PA_{\mathrm{m}} + R \tag{8}$$

where $R \in \mathbb{R}^{n \times n}$ is a given positive-definite matrix. Since A_{m} is Hurwitz, it follows from converse Lyapunov theory [5], [6] that there exists a unique positive-definite $P \in \mathbb{R}^{n \times n}$ satisfying (8) for a given positive definite matrix $R \in \mathbb{R}^{n \times n}$.

Remark 2.1: The feedback control law given by (4)–(6), along with the standard update law given by (7) ensures that $e(t) \to 0$ as $t \to \infty$ and $\tilde{W}(t) \triangleq \hat{W}(t) - W$ remains bounded [7]–[9] for all $t \geq 0$. However, if a high-gain learning rate is used to achieve fast adaptation in the face of large system uncertainty and abrupt changes in the system dynamics, then high-frequency oscillations in the control response can lead to system instability [2]–[4].

III. LOW-FREQUENCY LEARNING IN ADAPTIVE CONTROL

To address the high-frequency oscillations prevalent in standard adaptive control with high-gain feedback, let $\hat{W}_{\mathrm{f}}(t) \in \mathbb{R}^{s \times m}$, $t \geq 0$, be a low-pass filter weight estimate of $\hat{W}(t)$, $t \geq 0$, given by

$$\dot{\hat{W}}_{\mathrm{f}}(t) = \Gamma_{\mathrm{f}}\left[\hat{W}(t) - \hat{W}_{\mathrm{f}}(t)\right], \quad \hat{W}_{\mathrm{f}}(0) = \hat{W}_0, \quad t \geq 0 \tag{9}$$

where $\Gamma_{\mathrm{f}} \in \mathbb{R}^{s \times s}$ is a positive-definite filter gain matrix. Note that since $\hat{W}_{\mathrm{f}}(t)$, $t \geq 0$, is a low-pass filter weight estimate of $\hat{W}(t)$, $t \geq 0$, the filter gain matrix Γ_{f} is chosen such that $\lambda_{\max}(\Gamma_{\mathrm{f}}) \leq \gamma_{\mathrm{f,max}}$, where $\gamma_{\mathrm{f,max}} > 0$ is a design parameter.

Remark 3.1: Let $m = s = 1$, $\hat{W}(t) \in \mathbb{R}$, $t \geq 0$, and $\Gamma_{\mathrm{f}} = \gamma_{\mathrm{f,max}}$. In this case, (9) can be equivalently written as

$$\hat{W}_{\mathrm{f}}(s) = \frac{1}{\tau s + 1}\hat{W}(s) \tag{10}$$

where s is the Laplace variable and $\tau \triangleq \gamma_{\mathrm{f,max}}^{-1}$ is the filter time constant. Hence, since choosing a large time constant τ leads to a low-pass filter, this implies that $\gamma_{\mathrm{f,max}}$ needs to be small enough to cut off the high-frequency content of $\hat{W}(t)$, $t \geq 0$.

Next, we add a modification term to the standard update law given by (7) in order to enforce a distance condition between the trajectories of the weight estimate $\hat{W}(t)$, $t \geq 0$, and the trajectories of its low-pass filtered version $\hat{W}_{\mathrm{f}}(t)$, $t \geq 0$. This leads to a minimization problem

involving an error criterion capturing the distance between $\hat{W}(t)$, $t \geq 0$, and $\hat{W}_{\mathrm{f}}(t)$, $t \geq 0$. Specifically, consider the cost function given by

$$\mathcal{J}(\hat{W}, \hat{W}_{\mathrm{f}}) = \frac{1}{2}\|\hat{W} - \hat{W}_{\mathrm{f}}\|_{\mathrm{F}}^2 \tag{11}$$

and note that the negative gradient of (11) with respect to $\hat{W}$ is given by

$$\frac{\partial\left[-\mathcal{J}\left(\hat{W}(t), \hat{W}_{\mathrm{f}}(t)\right)\right]}{\partial\hat{W}(t)} = -\left(\hat{W}(t) - \hat{W}_{\mathrm{f}}(t)\right), \quad t \geq 0 \tag{12}$$

which gives the structure of the proposed modification term. Using the idea presented in [3], [10]–[12], we now construct the *proposed* update law by adding (12) to (7) to obtain the modified update law

$$\dot{\hat{W}}(t) = \Gamma\left[\beta\left(x(t)\right)e^{\mathrm{T}}(t)PB - \sigma\left(\hat{W}(t) - \hat{W}_{\mathrm{f}}(t)\right)\right],$$
$$\hat{W}(0) = \hat{W}_0, \quad t \geq 0 \tag{13}$$

where $\sigma > 0$ is a modification gain.

Many modification terms to the standard update law given by (7) are reported in the literature; for example, see [12]–[16] and references therein. These modification terms include the σ—modification, which has the form $-\sigma(\hat{W}(t) - W^*)$, where $\sigma > 0$ and W^* is an approximation of the ideal weight. If W^* is not a good approximation of the ideal weight, then the system error can increase [12]. Since W^* is unknown for many practical applications, it is common practice to choose $W^* = 0$. However, a key shortcoming of the σ—modification term with $W^* = 0$ is that it adds pure damping to the update law turning it into a lag filter, which can inhibit the adaptation process. The modification term given by (12) resembles the σ—modification architecture with W^* replaced by $\hat{W}_{\mathrm{f}}(t)$, $t \geq 0$. However, this new modification architecture allows the update law to learn using its low-frequency content, and hence, suppress the undesired high-frequency oscillations possibly contained in the control response. The proposed update law given by (13) significantly differs from the standard update law with a σ—modification. Furthermore, as we see in the next section, the proposed update law does not effect the asymptotic stability of the system error dynamics. A block diagram showing the proposed adaptive control architecture is given in Fig. 1.

Remark 3.2: To further elucidate the mechanism behind the proposed adaptive control architecture given by (9) and (13), let $m = s = 1$ and $z(t) \triangleq \beta(x(t))e^{\mathrm{T}}(t)PB$ with $\beta(x) = 1$, $\Gamma = \gamma$, $\Gamma_{\mathrm{f}} = \gamma_{\mathrm{f}}$, $\hat{W}(0) = 0$, and $\hat{W}_{\mathrm{f}}(0) = 0$. In this case, $\hat{W}(s)/z(s) = (\gamma/s)((s + \gamma_{\mathrm{f}})/(s + \gamma_{\mathrm{f}} + \gamma\sigma))$, whereas for the standard model reference adaptive controller (7) we obtain $\hat{W}(s)/z(s) = \gamma/s$. For the proposed adaptive control architecture, since γ/s is multiplied by a lead compensator, it can improve the phase margin of the closed-loop system. Furthermore, for a high-gain adaptation rate, that is, $\gamma \to \infty$, it follows that $\hat{W}(s)/z(s) = (1/\sigma) + (\gamma_{\mathrm{f}}/\sigma s)$, and hence, the addition of the filter modifies the integral-type standard model reference adaptive controller into a proportional-integral-type model reference adaptive controller. In the case where $\beta(x) \neq 1$, the nonlinearity in the adaptation law (13) can be isolated by defining $z(t)$ as above. In this case, the adaptation law is linear with respect to $z(t)$ and a similar analysis as above holds. As shown in Section VI, this modified architecture can improve robustness as compared to a pure integral-type controller.

IV. TRANSIENT AND STEADY-STATE PERFORMANCE GUARANTEES

In this section, we establish transient and steady-state performance properties for the proposed adaptive control architecture. Define $e(t) \triangleq x(t) - x_{\mathrm{m}}(t)$, $t \geq 0$, $\tilde{W}(t) \triangleq \hat{W}(t) - W$, $t \geq 0$, and $\tilde{W}_{\mathrm{f}}(t) \triangleq \hat{W}_{\mathrm{f}}(t) - W$, $t \geq 0$. Then, the system error, weight update error, and
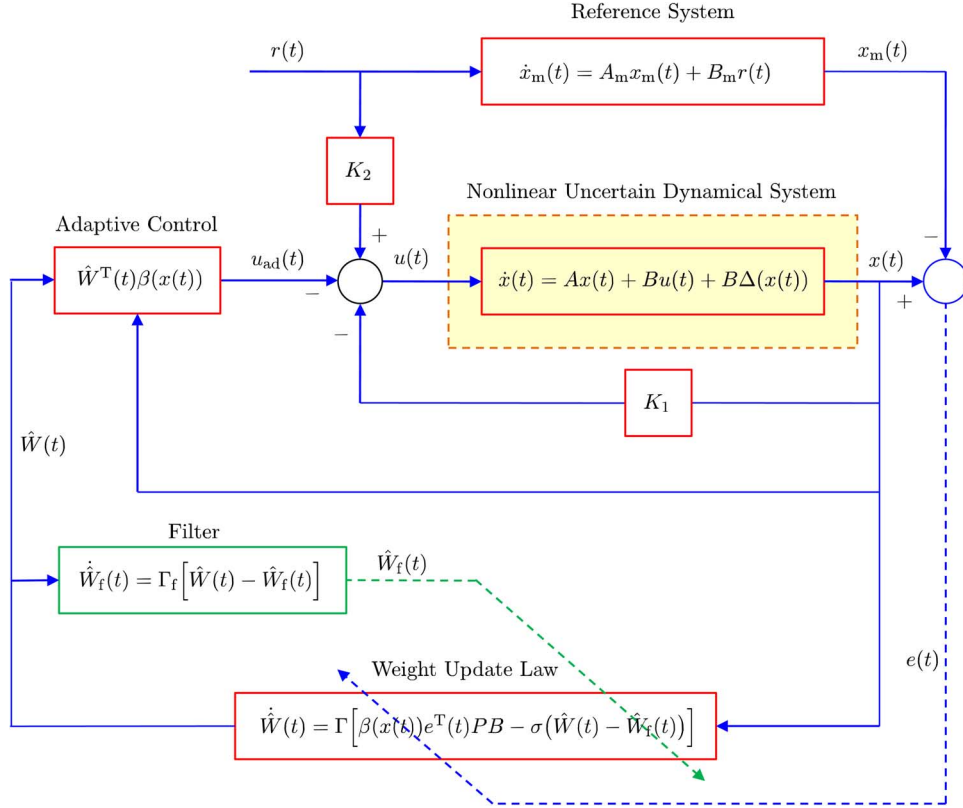
Fig. 1. Visualization of proposed adaptive control architecture.

filtered weight update error dynamics are, respectively, given by

$$\dot{e}(t) = A_\mathrm{m} e(t) - B\tilde{W}^\mathrm{T}(t)\beta\left(x(t)\right), \quad e(0) = e_0, \quad t \geq 0 \tag{14}$$

$$\dot{\tilde{W}}(t) = \Gamma\left[\beta\left(x(t)\right)e^\mathrm{T}(t)PB - \sigma\left(\tilde{W}(t) - \tilde{W}_\mathrm{f}(t)\right)\right],$$
$$\tilde{W}(0) = \tilde{W}_0, \quad t \geq 0 \tag{15}$$

$$\dot{\tilde{W}}_\mathrm{f}(t) = \Gamma_\mathrm{f}\left[\tilde{W}(t) - \tilde{W}_\mathrm{f}(t)\right], \quad \tilde{W}_\mathrm{f}(0) = \tilde{W}_\mathrm{f,0}, \quad t \geq 0. \tag{16}$$

The next theorem presents the main result of this note.

Theorem 4.1: Consider the nonlinear uncertain dynamical system given by (1), the reference system given by (2), and the feedback control law given by (4)–(6), and assume that Assumption 2.1 holds. Furthermore, let the update law be given by (13). Then, the solution $(e(t), \tilde{W}(t), \tilde{W}_\mathrm{f}(t))$ of the closed-loop system given by (14)–(16) is Lyapunov stable for all $(e_0, \tilde{W}_0, \tilde{W}_\mathrm{f,0}) \in \mathbb{R}^n \times \mathbb{R}^{s \times m} \times \mathbb{R}^{s \times m}$ and $t \geq 0$, and $x(t) \to x_\mathrm{m}(t)$ as $t \to \infty$. In addition, for all $t \geq 0$, the system error, weight update error, and filtered weight update error satisfy the transient performance bounds given by

$$\|e(t)\|_2 \leq \left[\frac{1}{\lambda_{\min}(P)}\left(\lambda_{\max}(P)\|e_0\|_2^2 + \|\Gamma^{-1}\|_\mathrm{F}\right.\right.$$
$$\left.\left.\cdot\|\tilde{W}_0\|_\mathrm{F}^2 + \sigma\left\|\Gamma_\mathrm{f}^{-1}\right\|_\mathrm{F}\|\tilde{W}_\mathrm{f,0}\|_\mathrm{F}^2\right)\right]^{\frac{1}{2}} \tag{17}$$

$$\left\|\tilde{W}(t)\right\|_\mathrm{F} \leq \left[\frac{1}{\lambda_{\min}(\Gamma^{-1})}\left(\lambda_{\max}(P)\|e_0\|_2^2 + \|\Gamma^{-1}\|_\mathrm{F}\right.\right.$$
$$\left.\left.\cdot\|\tilde{W}_0\|_\mathrm{F}^2 + \sigma\left\|\Gamma_\mathrm{f}^{-1}\right\|_\mathrm{F}\|\tilde{W}_\mathrm{f,0}\|_\mathrm{F}^2\right)\right]^{\frac{1}{2}} \tag{18}$$

$$\left\|\tilde{W}_\mathrm{f}(t)\right\|_\mathrm{F} \leq \left[\frac{1}{\lambda_{\min}\left(\Gamma_\mathrm{f}^{-1}\right)}\left(\lambda_{\max}(P)\|e_0\|_2^2 + \|\Gamma^{-1}\|_\mathrm{F}\|\tilde{W}_0\|_\mathrm{F}^2\right.\right.$$
$$\left.\left.+ \sigma\left\|\Gamma_\mathrm{f}^{-1}\right\|_\mathrm{F}\|\tilde{W}_\mathrm{f,0}\|_\mathrm{F}^2\right)\right]^{\frac{1}{2}}. \tag{19}$$

Proof: To show Lyapunov stability of the closed-loop system (14)–(16), consider the Lyapunov function candidate

$$V(e, \tilde{W}, \tilde{W}_f) = e^\mathrm{T} Pe + \operatorname{tr}\tilde{W}^\mathrm{T}\Gamma^{-1}\tilde{W} + \sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}\Gamma_\mathrm{f}^{-1}\tilde{W}_\mathrm{f} \tag{20}$$

where $P > 0$ satisfies (8), and note that $V(0, 0, 0) = 0$. Since $P, \Gamma, \Gamma_\mathrm{f}$ are positive-definite, $V(e, \tilde{W}, \tilde{W}_\mathrm{f}) > 0$ for all $(e, \tilde{W}, \tilde{W}_\mathrm{f}) \neq (0, 0, 0)$. In addition, $V(e, \tilde{W}, \tilde{W}_\mathrm{f})$ is radially unbounded. Differentiating (20) along the closed-loop system trajectories of (14)–(16) yields

$$\dot{V}\left(e(t), \tilde{W}(t), \tilde{W}_f(t)\right)$$
$$= 2e^\mathrm{T}(t)P\left[A_\mathrm{m}e(t) - B\tilde{W}^\mathrm{T}(t)\beta\left(x(t)\right)\right] + 2\operatorname{tr}\tilde{W}^\mathrm{T}(t)$$
$$\cdot\left[\beta\left(x(t)\right)e^\mathrm{T}(t)PB - \sigma\left(\tilde{W}(t) - \tilde{W}_\mathrm{f}(t)\right)\right]$$
$$+ \sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}(t)\left[\tilde{W}(t) - \tilde{W}_\mathrm{f}(t)\right]$$
$$= e^\mathrm{T}(t)\left[A_\mathrm{m}^\mathrm{T}P + PA_\mathrm{m}\right]e(t) - 2\sigma\operatorname{tr}\tilde{W}^\mathrm{T}(t)\tilde{W}(t)$$
$$- 2\sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}(t)\tilde{W}_\mathrm{f}(t) + 2\sigma\operatorname{tr}\tilde{W}^\mathrm{T}(t)\tilde{W}_\mathrm{f}(t)$$
$$+ 2\sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}(t)\tilde{W}(t)$$
$$\leq -e^\mathrm{T}(t)Re(t) - 2\sigma\operatorname{tr}\tilde{W}^\mathrm{T}(t)\tilde{W}(t) - 2\sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}(t)$$
$$\cdot\tilde{W}_\mathrm{f}(t) + 2\left[\sigma\operatorname{tr}\tilde{W}^\mathrm{T}(t)\tilde{W}(t) + \sigma\operatorname{tr}\tilde{W}_\mathrm{f}^\mathrm{T}(t)\tilde{W}_\mathrm{f}(t)\right]$$
$$= -e^\mathrm{T}(t)Re(t), \quad t \geq 0. \tag{21}$$

Hence, the closed-loop system given by (14)–(16) is Lyapunov stable for all $(e_0, \tilde{W}_0, \tilde{W}_\mathrm{f,0}) \in \mathbb{R}^n \times \mathbb{R}^{s \times m} \times \mathbb{R}^{s \times m}$ and $t \geq 0$. Now, by

the LaSalle-Yoshizawa theorem [6], $\lim_{t\to\infty}\|e(t)\|_2^2 = 0$, and hence, $x(t) \to x_{\mathrm{m}}(t)$ as $t \to \infty$.

Finally, since $\dot{V}(e(t), \tilde{W}(t), \tilde{W}_{\mathrm{f}}(t)) \leq 0, t \geq 0$, it follows that:

$$V(e, \tilde{W}, \tilde{W}_f) \leq V(e_0, \tilde{W}_0, \tilde{W}_{\mathrm{f},0}). \tag{22}$$

Now, using $V(e_0, \tilde{W}_0, \tilde{W}_{\mathrm{f},0}) \leq \lambda_{\max}(P)\|e_0\|_2^2 + \|\Gamma^{-1}\|_{\mathrm{F}}\|\tilde{W}_0\|_{\mathrm{F}}^2 + \sigma\|\Gamma_{\mathrm{f}}^{-1}\|_{\mathrm{F}}\|\tilde{W}_{\mathrm{f},0}\|_{\mathrm{F}}^2$, with the inequalities $\lambda_{\min}(P)\|e\|_2^2 \leq V(e, \tilde{W}, \tilde{W}_f)$, $\lambda_{\min}(\Gamma^{-1})\|e\|_2^2 \leq V(e, \tilde{W}, \tilde{W}_f)$, and $\lambda_{\min}(\Gamma_{\mathrm{f}}^{-1})\|e\|_2^2 \leq V(e, \tilde{W}, \tilde{W}_f)$, (22) gives (17)–(19), respectively. This completes the proof. $\square$

Remark 4.1: Theorem 4.1 highlights the stability as well as the transient and steady-state performance guarantees of the closed-loop system given by (14)–(16). In particular, it follows from Theorem 4.1 that $e(\cdot) \in \mathcal{L}_\infty$, $\tilde{W}(\cdot) \in \mathcal{L}_\infty$, and $\tilde{W}_{\mathrm{f}}(\cdot) \in \mathcal{L}_\infty$, as well as $x(t) \to x_{\mathrm{m}}(t)$ as $t \to \infty$.

V. OBSERVATIONS AND DISCUSSION

The proposed update law given by (13) along with (9) can be extended to include p multiple low-pass filters in order to shape (i.e., decrease) the negative slope of the filter after the cut off frequency. This can be done by considering the update law given by

$$\dot{\hat{W}}(t) = \Gamma\left[\beta(x(t))e^{\mathrm{T}}(t)PB - \sigma\left(\hat{W}(t) - \hat{W}_{\mathrm{f},1}(t)\right)\right],$$
$$\hat{W}(0) = \hat{W}_0, \quad t \geq 0 \tag{23}$$

$$\dot{\hat{W}}_{\mathrm{f},1}(t) = \Gamma_{\mathrm{f}}\left[\hat{W}_{\mathrm{f},2}(t) - \hat{W}_{\mathrm{f},1}(t)\right], \ \hat{W}_{\mathrm{f},1}(0) = \hat{W}_0 \tag{24}$$

$$\dot{\hat{W}}_{\mathrm{f},2}(t) = \Gamma_{\mathrm{f}}\left[\hat{W}_{\mathrm{f},3}(t) - \hat{W}_{\mathrm{f},2}(t)\right], \ \hat{W}_{\mathrm{f},2}(0) = \hat{W}_0 \tag{25}$$

$$\vdots$$

$$\dot{\hat{W}}_{\mathrm{f},p}(t) = \Gamma_{\mathrm{f}}\left[\hat{W}(t) - \hat{W}_{\mathrm{f},p}(t)\right], \ \hat{W}_{\mathrm{f},p}(0) = \hat{W}_0. \tag{26}$$

In this case, identical results to Theorem 4.1 hold with $x(t) \to x_{\mathrm{m}}(t)$ as $t \to \infty$ by considering the Lyapunov function candidate $V(e, \tilde{W}, \tilde{W}_f) = e^{\mathrm{T}}Pe + \mathrm{tr}\,\tilde{W}^{\mathrm{T}}\Gamma^{-1}\tilde{W} + \sigma\sum_{i=1}^{p}\mathrm{tr}\,\tilde{W}_{\mathrm{f}i}^{\mathrm{T}}\Gamma_{\mathrm{f}}^{-1}\tilde{W}_{\mathrm{f},i}$, where $\tilde{W}_{\mathrm{f},i}(t) \triangleq \hat{W}_{\mathrm{f},i}(t) - W$.

Next, consider a case involving uncertainty in the control effectiveness. Specifically, replace B in (1) with $B = B_0\Lambda$, where $B_0 \in \mathbb{R}^{n \times m}$ is a known matrix and $\Lambda \in \mathbb{R}^{m \times m}$ is an unknown diagonal matrix with diagonal entires $\lambda_{(i,i)} > 0$, $i = 1, \ldots, m$. Let the nominal control gains $K_1 \in \mathbb{R}^{m \times m}$ and $K_2 \in \mathbb{R}^{m \times r}$ be such that $A_{\mathrm{m}} = A - B_0 K_1$ and $B_{\mathrm{m}} = B_0 K_2$ hold. Furthermore, let the adaptive control law $u_{\mathrm{ad}}(t), t \geq 0$, be given by

$$u_{\mathrm{ad}}(t) = -\hat{W}_{\mathrm{a}}^{\mathrm{T}}(t)\beta_{\mathrm{a}}(x(t), u_n(t)), \quad t \geq 0 \tag{27}$$

where $\beta_{\mathrm{a}} : \mathbb{R}^n \to \mathbb{R}^{s+m}$ is a known basis function of the form $\beta_{\mathrm{a}}(x, u_n) = [\beta^{\mathrm{T}}(x), u_n^{\mathrm{T}}]^{\mathrm{T}}$ and $\hat{W}_{\mathrm{a}}(t) \in \mathbb{R}^{(s+m) \times m}, t \geq 0$, is an estimate of $W_{\mathrm{a}} \triangleq [W^{\mathrm{T}}, I_m - \Lambda^{-1}]^{\mathrm{T}} \in \mathbb{R}^{(s+m) \times m}$ satisfying the update law

$$\dot{\hat{W}}_{\mathrm{a}}(t) = \Gamma\left[\beta_{\mathrm{a}}(x(t), u_n(t))e^{\mathrm{T}}(t)PB_0 - \sigma\left(\hat{W}_{\mathrm{a}}(t) - \hat{W}_{\mathrm{f}}(t)\right)\right],$$
$$\hat{W}_{\mathrm{a}}(0) = \hat{W}_{\mathrm{a},0}, \quad t \geq 0 \tag{28}$$

$$\dot{\hat{W}}_{\mathrm{f}}(t) = \Gamma_{\mathrm{f}}\left[\hat{W}_{\mathrm{a}}(t) - \hat{W}_{\mathrm{f}}(t)\right], \ \hat{W}_{\mathrm{f}}(0) = \hat{W}_0. \tag{29}$$

Then, using similar arguments as in the proof of Theorem 4.1 and assuming that $r(\cdot)$ is uniformly continuous, it can be shown that identical

results to Theorem 4.1 hold with $x(t) \to x_{\mathrm{m}}(t)$ as $t \to \infty$ by considering the Lyapunov function candidate given by $V(e, \tilde{W}, \tilde{W}_f) = e^{\mathrm{T}}Pe + \mathrm{tr}(\tilde{W}\Lambda^{1/2})^{\mathrm{T}}\Gamma^{-1}(\tilde{W}\Lambda^{1/2}) + \sigma\,\mathrm{tr}(\tilde{W}_{\mathrm{f}}\Lambda^{1/2})^{\mathrm{T}}\Gamma_{\mathrm{f}}^{-1}(\tilde{W}_{\mathrm{f}}\Lambda^{1/2})$.

Finally, consider the case where the nonlinear uncertain dynamical system given by (1) includes bounded exogenous disturbances and/or the matched uncertainty in (1) cannot be perfectly parameterized. Then, Assumption 2.1 can be relaxed by considering

$$\Delta(t, x) = W^{\mathrm{T}}(t)\beta(x) + \varepsilon(x), \quad x \in \mathcal{D}_x \tag{30}$$

where $W(t) \in \mathbb{R}^{s \times m}, t \geq 0$, is an unknown time-varying matrix satisfying $\|W(t)\|_{\mathrm{F}} \leq w, w > 0$, and $\|\dot{W}(t)\|_{\mathrm{F}} \leq \dot{w}$, $\dot{w} > 0$, $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a known basis function of the form $\beta(x) = [1, \beta_1(x), \beta_2(x), \ldots, \beta_{s-1}(x)]$, $\varepsilon : \mathcal{D}_x \to \mathbb{R}^m$ is the system modeling error satisfying $\|\varepsilon(x)\|_2 \leq \varepsilon_{\max}, \varepsilon_{\max} > 0$, and $\mathcal{D}_x$ is a compact subset of $\mathbb{R}^n$. In this case, the proposed update law given by (13) along with (9) can be replaced by

$$\dot{\hat{W}}(t) = \Gamma\,\mathrm{Proj}\left(\hat{W}(t), \left[\beta(x(t))e^{\mathrm{T}}(t)PB\right.\right.$$
$$\left.\left. - \sigma\left(\hat{W}(t) - \hat{W}_{\mathrm{f}}(t)\right)\right]\right) \tag{31}$$
$$\hat{W}(0) = \hat{W}_0, \quad t \geq 0,$$

$$\dot{\hat{W}}_{\mathrm{f}}(t) = \Gamma_{\mathrm{f}}\,\mathrm{Proj}\left(\hat{W}_{\mathrm{f}}(t), \left[\hat{W}(t) - \hat{W}_{\mathrm{f}}(t)\right]\right), \ \hat{W}_{\mathrm{f}}(0) = \hat{W}_0 \tag{32}$$

with $\Gamma = \gamma I_s, \gamma > 0$, and $\Gamma_{\mathrm{f}} = \gamma_{\mathrm{f}} I_s, \gamma_{\mathrm{f}} > 0$, where $\mathrm{Proj} : \mathbb{R}^{s \times m} \times \mathbb{R}^{s \times m} \to \mathbb{R}^{s \times m}$ denotes the *projection operator* [17], to guarantee uniform boundedness of the system error $e(t), t \geq 0$, weight update error $\tilde{W}(t), t \geq 0$, and the filtered weight update error $\tilde{W}_{\mathrm{f}}(t), t \geq 0$.

VI. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the utility and efficacy of the proposed adaptive control architecture.

A. Disturbance Rejection

To illustrate the key ideas of the proposed adaptive control architecture, we first consider a linear dynamical system with an external disturbance. Since for this problem the system loop transfer function (broken at the control input) can be equivalently written as a linear time-invariant dynamical system, we resort to classical control theory tools to analyze the closed-loop system and compare the proposed architecture with the standard adaptive controller architecture. Specifically, consider the linear dynamical system representing a disturbed aircraft rolling dynamics model given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}w(t), \quad t \geq 0 \tag{33}$$

$x_1(0) = 0, x_2(0) = 0$, where x_1 represents the roll angle in radians, x_2 represents the roll rate in radians per second, and $w(\cdot), \dot{w}(\cdot) \in \mathcal{L}_\infty$ is an external disturbance. In this case, the basis function $\beta(x)$ can be chosen as $\beta(x) = 1$.

For our simulations, we choose $K_1 = [0.16, 0.57]$ and $K_2 = 0.16$, a second-order reference system corresponding to a natural frequency $\omega_{\mathrm{n}} = 0.40 \, \mathrm{rad/s}$ and a damping ratio $\zeta = 0.707$, and we let $R = I_2$. Figs. 2 and 3 compare the standard adaptive control architecture given by (7) and the proposed adaptive control architecture given by (13) for different learning rates. Note that for higher learning rates, both adaptive control architectures achieve a desired level of disturbance rejection. However, as is well known, the system phase margin diminishes
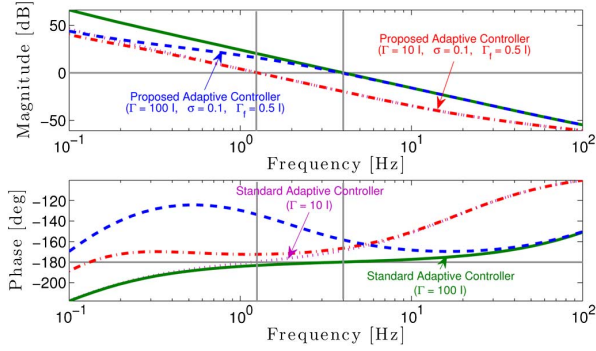
Fig. 2. Bode plots of the loop gain transfer function for the disturbed aircraft rolling dynamics model with the standard adaptive controller (7) and the proposed adaptive controller (13) for different adaptation gains.
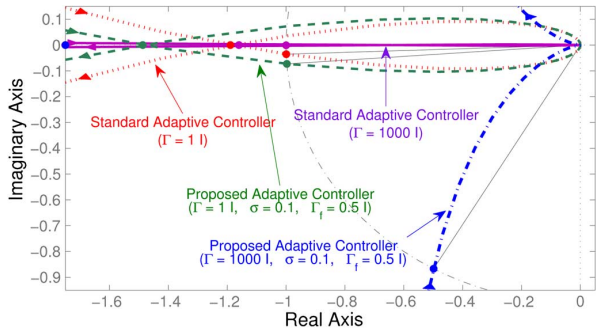

Fig. 3. Nyquist plots of the loop gain transfer function for the disturbed aircraft rolling dynamics model with the standard adaptive controller (7) and the proposed adaptive controller (13) for different adaptation gains.

as the learning rate is increased with the standard adaptive control architecture. In contrast, the phase margin increases as the learning rate is increased with the proposed adaptive control architecture. This demonstrates the fast and robust adaptation property of the proposed architecture in the face of high-gain learning rates. For additional insights, see Remark 3.2.

B. Uncertainty Suppression

Next, consider the nonlinear dynamical system representing a controlled wing rock aircraft dynamics model given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta(x(t)) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ t \geq 0 \quad (34)$$

$x_1(0) = 0$, $x_2(0) = 0$, with $\Delta(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 |x_1| x_2 + \alpha_4 |x_2| x_2 + \alpha_5 x_1^3$, where α_i, $i = 1, \ldots, 5$, are unknown parameters that are derived from the aircraft aerodynamic coefficients. For our simulation, we set $\alpha_1 = 0.2314$, $\alpha_2 = 0.7848$, $\alpha_3 = -0.0624$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0215$ [18]. In this case, the basis function $\beta(x)$ can be chosen as $\beta(x) = [x_1, x_2, |x_1| x_2, |x_2| x_2, x_1^3]^T$. We use the same reference system as given in the previous example. Furthermore, we set $R = I_2$, $\Gamma = 100 I_5$, $\sigma = 0.1$, and $\Gamma_f = 0.5 I_5$ for the proposed adaptive control architecture given by (13). Here, our aim is to track a given filtered square-wave roll angle reference command $r(t)$, $t \geq 0$.

Fig. 4 shows the closed-loop system performance of the standard adaptive controller given by (7) with a learning rate of $\Gamma = 100 I_5$ and Fig. 5 shows the closed-loop system performance of the proposed adaptive controller. Note that the control response of the proposed adaptive controller is clearly superior as compared to the standard adap-
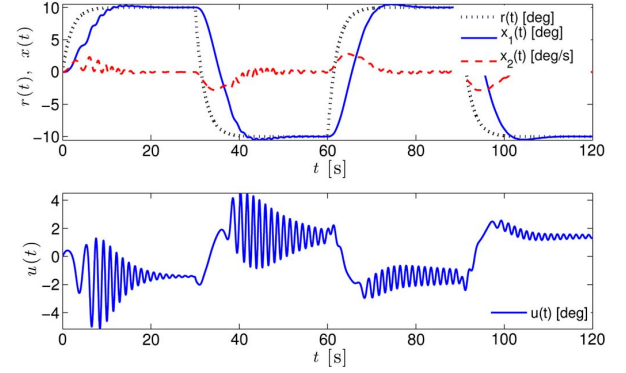

Fig. 4. Closed-loop system response of wing rock aircraft dynamics model with the standard adaptive controller ((7) with $\Gamma = 100 I_5$).
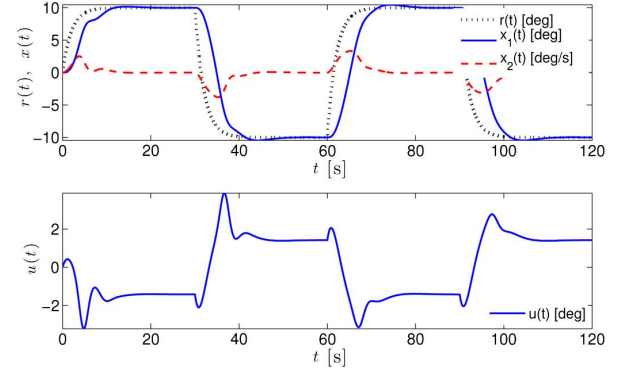

Fig. 5. Closed-loop system response of wing rock aircraft dynamics model with the proposed adaptive controller ((13) with $\Gamma = 100 I_5$, $\sigma = 0.1$, $\Gamma_f = 0.5 I_5$).
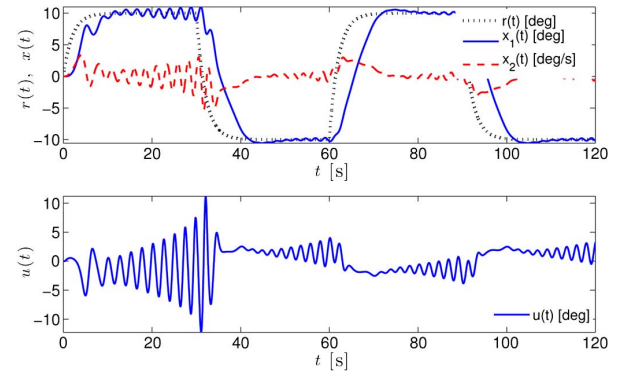

Fig. 6. Closed-loop system response of wing rock aircraft dynamics model with the proposed adaptive controller ((13) with $\Gamma = 100 I_5$, $\sigma = 0.1$, $\Gamma_f = 0.5 I_5$) under an input time-delay of 0.362 s.

tive controller. This is excepted since fast and robust adaptation can be achieved with the proposed controller without incurring high-frequency oscillations in the control response.

To illustrate the point that a high-gain learning rate within the proposed adaptive controller architecture does not hinder system performance, we inserted an input time-delay of 0.362 s to the wing rock dynamics model given by (34). Fig. 6 shows that the closed-loop system performance of the proposed adaptive controller is oscillatory. Then, we increased the learning rate of the proposed controller from $\Gamma = 100 I_5$ to $\Gamma = 10000 I_5$. Note that with the standard adaptive controllers (7), it is well known that increasing the learning rate decreases the time-delay margin of the controlled system. In contrast, Fig. 7 shows
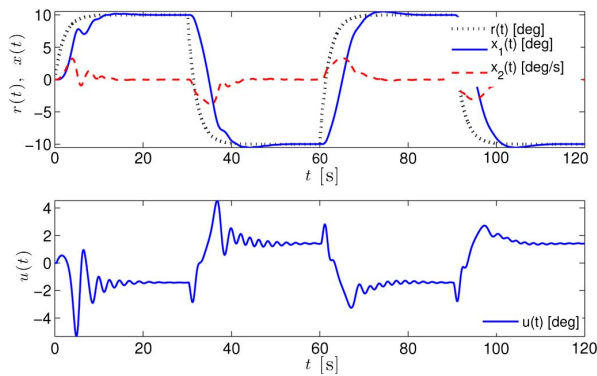
Fig. 7. Closed-loop system response of wing rock aircraft dynamics model with the proposed adaptive controller ((13) with $\Gamma = 10000I_5$, $\sigma = 0.1$, $\Gamma_f = 0.5I_5$) under an input time-delay of 0.362 s.

that the closed-loop system performance of the proposed adaptive controller is improved as the learning rate is increased. In future research, we will investigate the guaranteed robustness properties of the proposed adaptive controller against input time-delays.

VII. CONCLUSION

It is well known that standard model reference adaptive control methods employ high-gain learning rates to achieve fast adaptation in order to rapidly reduce system tracking errors in the face of large system uncertainties. High-gain learning rates, however, lead to increased controller effort, reduced stability margins, and can cause high-frequency oscillations in the control response resulting in system instability. In this note, we presented a new robust adaptive control architecture that allows for fast adaptation of nonlinear uncertain dynamical systems while guaranteeing transient and steady-state performance bounds. Specifically, the proposed architecture filters out the high-frequency content contained in the update law without hindering asymptotic stability of the system error dynamics. Future research will include analyzing stability gain and time-delay margins of the proposed framework as well as extending the proposed framework to non-model reference adaptive control architectures.

REFERENCES

[1] Z. T. Dydek, A. M. Annaswamy, and E. Lavretsky, "Adaptive control and the NASA X-15-3 flight revisited," *IEEE Control Sys. Mag.*, vol. 30, pp. 32–48, 2010.

[2] C. E. Rohrs, L. S. Valavani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics," *IEEE Trans. Autom. Control*, vol. AC-30, no. 9, pp. 881–889, Sep. 1985.

[3] T. Yucelen and A. J. Calise, "Kalman filter modification in adaptive control," *AIAA J. Guid., Control, Dyn.*, vol. 33, pp. 426–439, 2010.

[4] N. Hovakimyan, C. Cao, E. Kharisov, E. Xargay, and I. M. Gregory, " $\mathcal{L}_1$ adaptive control for safety-critical systems," *IEEE Control Sys. Mag.*, vol. 31, pp. 54–104, 2011.

[5] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[6] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control. A Lyapunov-Based Approach*. Princeton, NJ: Princeton Univ. Press, 2008.

[7] K. J. Astrom and B. Wittenmark, *Adaptive Control*. Reading, MA: Addison-Wesley, 1989.

[8] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[9] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Upper Saddle River, NJ: Prentice-Hall, 1989.

[10] P. V. Osburn, H. P. Whitaker, and A. Kezer, *New Developments in the Design of Adaptive Control Systems*. Washington, DC: Institute of Aeronautical Sciences, 1961, Paper No 61-39.

[11] R. L. Butchart and B. Shackcloth, "Synthesis of model reference adaptive control systems by Lyapunov's second method," in *Proc. IFAC Symp. on Adaptive Control*, Teddington, UK, 1965.

[12] K. Y. Volyanskyy, W. M. Haddad, and A. J. Calise, "A new neuroadaptive control architecture for nonlinear uncertain dynamical systems: Beyond σ- and e-modifications," *IEEE Trans. Neural Networks*, vol. 20, no. 11, pp. 1707–1723, Nov. 2010.

[13] P. Ioannou and P. Kokotovic, "Instability analysis and improvement of robustness of adaptive control," *Automatica*, vol. 20, no. 5, pp. 583–594, 1984.

[14] K. S. Narendra and A. M. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Trans. Autom. Control*, vol. AC-32, no. 2, pp. 134–145, Feb. 1987.

[15] G. Chowdhary and E. N. Johnson, "Theory and flight test validation of a concurrent learning adaptive controller," *AIAA J. Guid. Control Dyn.*, vol. 34, pp. 592–607, 2010.

[16] A. J. Calise and T. Yucelen, "Adaptive loop transfer recovery," *AIAA J. Guid., Control, Dyn.*, vol. 35, no. 3, pp. 807–815, May/Jun. 2012.

[17] J. B. Pomet and L. Praly, "Adaptive nonlinear regulation: Estimation from Lyapunov equation," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 729–740, Jun. 1992.

[18] S. N. Singh, W. Yim, and W. R. Wells, "Direct adaptive and neural control of wing-rock motion of slender delta wings," *AIAA J. Guid. Control Dyn.*, vol. 18, pp. 25–30, 1995.

Continuous-Discrete Observer for State Affine Systems With Sampled and Delayed Measurements

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Abstract—The observation of a class of multi-input multi-output (MIMO) state affine systems with both sampled and delayed output measurements is addressed. These two constraints disturb simultaneously the convergence of the observer. Assuming some persistent excitation conditions to hold, and by using Lyapunov tools adapted to impulsive systems, two classes of global exponential observers are proposed. Some explicit relations between maximum allowable delay and maximum allowable sampling period are given. An extension to some classes of nonlinear systems is also given.

Index Terms—Continuous-discrete observer, delayed measurements, impulsive systems.

I. INTRODUCTION

This work is devoted to observer design for a class of uniformly observable systems with sampled and delayed measurements. In the last decades, the design of nonlinear observers for continuous systems with sampled measurements has received a great attention. This interest is motivated by many engineering applications, such as network control systems (NCSs) in which the output is transmitted over a shared digital communication network, and is only available at discrete-time instants. For linear systems it is usually possible to design observers by using the discrete time model of the continuous time system. This is