Predictive tracking control for a multi-compartment respiratory system with amplitude and rate input constraints

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Accepted author version posted online: 26 Nov 2013. Published online: 05 Feb 2014.
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(Received 19 December 2012; accepted 28 October 2013)

In this article, we develop a predictive tracking controller for a nonlinear multi-compartment lung mechanics model. Specifically, for a given clinically plausible reference volume pattern, we use model predictive control to design a tracking controller that accounts for amplitude and rate input constraints. The predictive control law is derived by minimising a quadratic performance criterion involving a prediction of the system response over a prescribed time step. The proposed tracking control framework is applied to a two-compartment lung mechanics model with nonlinear lung compliance parameters.

Keywords: predictive tracking control; amplitude and rate input constraints; multi-compartment respiratory system; nonlinear lung compliance; switched systems

1. Introduction

One of the most common problems in intensive care units is acute respiratory failure. When this occurs, it is essential to support patients with mechanical ventilation while the fundamental disease process is addressed. For example, a patient with pneumonia may require mechanical ventilation while the pneumonia is being treated with antibiotics, which will eventually effectively cure the disease. The goal of mechanical ventilation is to ensure adequate ventilation, which involves a magnitude of gas exchange that leads to the desired blood level of carbon oxide, and adequate oxygenation, which involves a blood concentration of oxygen that will ensure organ function.

The earliest primary modes of ventilation can be classified, approximately, as volume-controlled and pressure-controlled (Younes, 1994). With the increasing availability of information technology, modern ventilation control algorithms have been used to provide advanced ventilation modes rather than simple volume or pressure control. Examples are proportional-assist ventilation (Younes, Puddy, & Roberts, 1992), adaptive support ventilation (Laubscher, Heinrichs, & Weiler, 1994), and neurally adjusted ventilation (Sinderby, Navalesi, & Beck, 1999). The common theme in modern ventilation control algorithms is the use of pressure-limited ventilation while also guaranteeing adequate minute ventilation (Burns et al., 2011).

In recent research, Chellaboina et al. (2010) and Voliansky, Haddad, and Bailey (2011) developed a model reference direct adaptive control framework for a multi-compartment model of a pressure-limited respiratory and lung mechanics system. More recently, Li and Haddad (2012a) designed a model predictive controller to address control constraints on the sign and range of the input pressure in the respiratory control. This algorithm is based on a time-varying, linear periodic multi-compartment lung model. However, realistic lung models should consider the fact that the lungs, especially diseased lungs, are heterogeneous, both functionally and anatomically, and are comprised of many sub-units, or compartments, that differ in their capacities for gas exchange. This is particularly true for the compliance of the lung units, which are not constant but rather vary with lung volume.

In this article, we develop a model predictive controller based on a nonlinear multi-compartment lung mechanics model with the aim to automatically adjust the pressure generated by mechanical ventilation such that the system output tracks a given clinically plausible breathing pattern. In general, a model predictive control law is computed by an online optimisation problem, and hence, when the system model involves nonlinearities, the model predictive controller requires considerable computational effort. However, for systems with stable zero dynamics, it has been possible to use short prediction horizon times to accurately predict the future system response using a given system model (Singh, Steinberg, & DiGirolamo, 1995; Soroush & Kravaris, 1996). In this article, we formulate a quadratic optimal control problem subject to control input amplitude and rate constraints that minimises the deviation of the multi-compartment respiratory system output from the given reference volume pattern. Then, we derive the
predictive control law by minimising a performance criterion involving the prediction of the future system response over a prescribed time step. The derived optimal control law is given by an explicit form, and thus, avoids an online optimisation.

2. Notation and mathematical preliminaries

The notation used in this article is fairly standard. Specifically, for \( x \in \mathbb{R}^n \) we write \( x \geq 0 \) (respectively, \( x > 0 \)) to indicate that every component of \( x \) is non-negative (respectively, positive). In this case, we say that \( x \) is non-negative or positive, respectively. Likewise, \( A \in \mathbb{R}^{n \times m} \) is non-negative or positive if every entry of \( A \) is non-negative or positive, respectively, which is written as \( A \geq 0 \) or \( A > 0 \), respectively. \( \mathbb{R}_+ \) and \( \mathbb{R}_+^n \) denote the non-negative and positive orthants of \( \mathbb{R}^n \); that is, if \( x \in \mathbb{R}_+^n \), then \( x \in \mathbb{R}_+^n \) and \( x \in \mathbb{R}_+^n \) are equivalent, respectively, to \( x \geq 0 \) and \( x > 0 \). Furthermore, we write \( (\cdot)^T \) to denote the transpose operator, \( (\cdot)^T \) to denote the \( r \)th time derivative of \( (\cdot) \), \( L_f V(x) \) to denote the Lie derivative of a scalar function \( V(x) \) along the vector field of \( f(x) \), \( L_f^0 V(x) \) to denote the zeroth-order Lie derivative, that is, \( L_f^0 V(x) \equiv V(x) \), \( L_f^r V(x) \) to denote the \( r \)th-order Lie derivative, that is, \( L_f^r V(x) \equiv \left. \frac{d^r}{dt^r} \right|_{t=0} \{ L_f L_f^{r-1} V(x) \} \), and \( L_x L_f^r V(x) \) to denote the Lie derivative of a scalar function \( L_f^r V(x) \) with respect to vector field \( g(x) \).

Finally, we write \( \lambda_{\min} (\cdot) \) (respectively, \( \lambda_{\max} (\cdot) \)) to denote the minimum (respectively, maximum) eigenvalue of a Hermitian matrix, \( \sigma_{\min} (\cdot) \) to denote the minimum singular value of a matrix, and \( \| \cdot \| \) for the Euclidean vector norm in \( \mathbb{R}^n \), \( e \equiv [1, \ldots, 1]^T \) to denote the ones vector of order \( n \), and \( \text{mod}(\cdot, \cdot) \) for the modulo operator, that is, \( \text{mod}(t, \tau) \equiv \lfloor t - \lfloor t \rfloor \rfloor \tau \), where \( \lfloor \cdot \rfloor \) denotes the floor function which gives the largest integer less than or equal to the positive number \( \cdot \).

The following definitions introduce the notions of non-negative functions and essentially non-negative vector fields (Haddad, Chellaboina, & Hui, 2010).

**Definition 2.1:** Let \( T > 0 \). A real function \( u : [0, T] \to \mathbb{R}^m \) is a non-negative (resp., positive) function if \( u(t) \geq 0 \) (resp., \( u(t) > 0 \)) on the interval \( [0, T] \).

**Definition 2.2:** Let \( f = [f_1, \ldots, f_n]^T : \mathcal{D} \subset \mathbb{R}_+^n \to \mathbb{R}^n \).
Then \( f \) is essentially non-negative if \( f(x) \geq 0 \) for all \( i = 1, \ldots, n \) and \( x \in \mathbb{R}_+^n \), such that \( x_i = 0, i = 1, \ldots, n \), where \( x_i \) denotes the \( i \)th component of \( x \).

It follows from Definition 2.2 that, if \( f(x) = Ax \), where \( A \in \mathbb{R}^{n \times n} \), then \( f \) is essentially non-negative if and only if \( A \) is essentially non-negative, that is, \( A_{ij} \geq 0, i, j = 1, \ldots, n, i \neq j \), where \( A_{ij} \) denotes the \((i,j)\)th entry of \( A \).

In this article, we consider controlled switched nonlinear dynamical systems \( G_p \) of the form

\[
\dot{x}(t) = f_p(x(t)) + G_p(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \quad (1)
\]

\[
y(t) = h(x(t)), \quad (2)
\]

where \( x(t) \in \mathbb{R}^n, t \geq 0, \) is the state vector, \( u(t) \in \mathbb{R}^m, t \geq 0, \) is the control input, \( y(t) \in \mathbb{R}^n, t \geq 0, \) is the system output, \( p \) is a switching signal taking values in a finite index set \( \mathcal{P} = \{1, \ldots, q\} \), and, for every \( p \in \mathcal{P}, \) \( f_p : \mathbb{R}^n \to \mathbb{R}^n \) and \( G_p : \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are Lipschitz continuous functions, and \( h : \mathbb{R}^n \to \mathbb{R}^m \) is a continuous output function. The family of nonlinear dynamical systems (1) and (2) can be written as the switched dynamical systems \( G_p \) given by

\[
\dot{x}(t) = f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u(t), \quad \sigma(t) \in \Sigma, \quad x(0) = x_0, \quad t \geq 0, \quad (3)
\]

\[
y(t) = h(x(t)), \quad (4)
\]

where \( x(t) \in \mathbb{R}^n, t \geq 0, f_p : \mathbb{R}^n \to \mathbb{R}^n, G_{\sigma} : \mathbb{R}^n \to \mathbb{R}^{n \times m}, \sigma : [0, \infty) \to \mathcal{P} \) is a piecewise constant switching signal, and \( \Sigma \) denotes the set of switching signals. The switching signal \( \sigma \) effectively switches the right-hand side of (3) by selecting different sub-systems from the parameterised family \( \{f_p(x) + G_p(x)u : p \in \mathcal{P}\} \). We denote by \( t_i, i = 1, 2, \ldots, \) the consecutive discontinuities of \( \sigma \) which we call the switching times of (3). Our convention here is that \( \sigma(\cdot) \) is left-continuous, that is, \( \sigma(t^-) = \sigma(t) \), where \( \sigma(t^-) \equiv \lim_{h \rightarrow 0^-} (t + h) \).

The pair \( (x, \sigma) : [0, \infty) \times \Sigma \to \mathbb{R}^n \) is a solution to the switched dynamical system (3) if \( x(\cdot) \) is absolutely continuous and satisfies (3) for almost all \( t \geq 0 \). Here, we assume that if there are infinitely many switching times, then there exists \( \tau > 0 \) such that for every \( T \geq 0 \) there exists a positive integer \( i \) such that \( t_{i+1} - \tau \geq t_i \). Then, \( t \in [t_i, t_{i+1}), \sigma(t) = i_k \), that is, the \( i_k \)th sub-system is active. Hence, the trajectory \( x(t) \) of the switched dynamical system (3) is defined as the trajectory \( x_{ik}(t) \) of the \( i_k \)th sub-system when \( t \in [t_i, t_{i+1}) \).

The following definition and proposition are needed for the main results of the article.

**Definition 2.3:** The switched nonlinear dynamical system given by (1) is non-negative if for every \( x(0) \in \mathbb{R}_+^n \) and \( u(t) \geq 0, t \geq 0, \) the solution \( x(t), t \geq 0, \) to (3) is non-negative.

**Proposition 2.1:** Consider the switched nonlinear dynamical system given by (1). If \( f_p : \mathbb{R}^n \to \mathbb{R}^n, p \in \mathcal{P}, \) is essentially non-negative and \( G_p(x) \geq 0 \) for all \( x \in \mathbb{R}_+^n \) and \( p \in \mathcal{P}, \) then (1) is non-negative.

**Proof:** The proof is similar to the proof of Proposition 4.3 of Haddad et al. (2010). \( \square \)

It follows from Proposition 2.1 that if \( f_p(\cdot), p \in \mathcal{P}, \) is essentially non-negative, then a non-negative input...
signal \( G_p(x)u, p \in \mathcal{P} \), is sufficient to guarantee the non-negativity of the state of (3).

3. Predictive output tracking control problem

In this section, we consider the problem of characterising a predictive constrained output feedback control law for nonlinear essentially non-negative dynamical systems to track a given output reference trajectory. Specifically, we consider a controlled switched nonlinear dynamical system \( G_p \) given by (1) and (2), where, for \( p \in \mathcal{P} \), \( f_p(\cdot) \) is essentially non-negative, \( G_p(\cdot) \) is non-negative, and \( h(\cdot) \) is non-negative. Moreover, we assume that \( f_p(\cdot), G_p(\cdot), \) and \( h(\cdot) \) are smooth (at least \( C^2 \)) mappings and the control input \( u(\cdot) \) is restricted to a class of admissible controls consisting of absolutely continuous functions such that \( u(t) \in \mathcal{U}, t \geq 0 \), where \( \mathcal{U} \) is defined by

\[
\mathcal{U} \triangleq \{ u(t) \in \mathbb{R}^m : 0_m \leq u(t) \leq \bar{u}, \, \bar{u} \leq \tilde{u}(t) \leq u^T(t) - u^T(0) \leq \bar{u}^T, \, \text{a.e.} \, t \geq 0 \},
\]

where \( u, \bar{u}, \) and \( \tilde{u} \) are given input amplitude and rate constraint bounds. Note that since the control input \( u(\cdot) \) is restricted to be non-negative, it follows from Proposition 2.1 that \( x(t) \geq 0 \) for all \( x(0) \in \mathbb{R}^n \) and \( t \geq 0 \). For a mechanical ventilation problem, the input rate constraint \( \bar{u} \leq \tilde{u}(t) \leq u^T \) for almost every \( t > 0 \) is critical since rapid changes in the driving input pressure may cause discomfort and inefficacy of muscular lung contraction and control.

Defining \( v(t) \triangleq \tilde{u}(t) \) for almost every \( t \geq 0 \) and \( z(t) \triangleq [x^T(t), u^T(t)]^T, t \geq 0 \), it follows that the augmented nonlinear dynamical system \( \hat{G}_p \), given by

\[
\begin{align*}
\dot{z}(t) &= \hat{f}_p(z(t)) + \sum_{i=1}^{m} \hat{g}_i v(t), \\
&= [f_p(x) + G_p(x)u]^T, \quad p \in \mathcal{P}, \quad \text{a.e.} \, t \geq 0, \\
y(t) &= h(z(t)),
\end{align*}
\]

where \( \hat{f}_p(z) = [(f_p(x) + G_p(x)u)^T, 0_m^T]^T, \hat{g}_i \in \mathbb{R}^{n+m}, \) \( i = 1, \ldots, m \), is such that the \( (n+i) \)th component is 1 and zero elsewhere and \( h(z(t)) = h(x(t)) \), subsumes (3) and (4). Note that it follows from (5) that \( \bar{u} \leq v(t) \leq e_T, \, t \geq 0 \). Furthermore, for a sufficiently small time \( \delta > 0 \), it follows using a first-order Taylor series expansion that

\[
u_i(t + \delta) \approx u(t) + \delta v_i(t), \quad i = 1, \ldots, m, \quad t \geq 0,
\]

where \( u_i(t), t \geq 0 \), denotes the \( i \)th component of \( u(t), t \geq 0 \). Since \( u(t + \delta) \in \mathcal{U}, t \geq 0 \), it follows from (5) that \( v_i(t) \) satisfies

\[-\frac{u_i(t)}{\delta} \leq v_i(t) \leq \frac{e_T - e_i(t)}{\delta}, \quad i = 1, \ldots, m. \] Hence,

\[
v(t) \in \mathcal{V}, t \geq 0,
\]

where

\[
\mathcal{V} \triangleq \{ v(t) \in \mathbb{R}^m : v_{i_{\min}}(t) \leq v_i(t) \leq v_{i_{\max}}(t), \, t \geq 0, \quad i = 1, \ldots, m \},
\]

where \( v_{i_{\min}}(t) \triangleq \max\{v_i - \frac{u_i(t)}{\delta} + \varepsilon, v_{i_{\max}}(t) \triangleq \min\{v_i \frac{e_T - e_i(t)}{\delta} - \varepsilon, \varepsilon > 0 \) is a small positive scalar.

Next, we assume that \( \hat{G}_p, \) for every \( p \in \mathcal{P}, \) has a (vector) relative degree \( r \triangleq \{r_1, \ldots, r_m\} \), where \( r_i \) denotes the relative degree of \( \hat{G}_p \) with respect to the output \( y_i, i = 1, \ldots, m \). Thus, the \( r \)th derivative of \( y(t), t \geq 0 \), is given by

\[
y^{(r)}(t) = a_p(z(t)) + D_p(z(t))v(t), \quad p \in \mathcal{P}, \quad \text{a.e.} \, t \geq 0,
\]

where \( a_p(z) = [L_{\hat{h}_1}^{(r_1-1)}(z), \ldots, L_{\hat{h}_m}^{(r_m-1)}(z)]^T \) and \( D_p(z) \in \mathbb{R}^{m \times m} \) is a matrix function whose \( i \)th row is given by \( D_i(z) = [L_{\hat{h}_i}^{(r_i-1)}(z), \ldots, L_{\hat{h}_i}^{(r_i-1)}(z)], i = 1, \ldots, m \). The following two assumptions are needed for the main results of this section.

**Assumption 3.1:** For \( p \in \mathcal{P}, \) (i) \( D_p(z) \) is non-singular for all \( z \in \mathbb{R}^m \times \mathcal{U} \) and (ii) the zero dynamics of \( \hat{G}_p \) are uniformly asymptotically stable.

Part (i) of Assumption 3.1 guarantees that the system \( \hat{G}_p \) is input–output feedback linearisable for every \( p \in \mathcal{P}, \) whereas (ii) ensures that the internal dynamics of \( \hat{G}_p \) remain asymptotically stable for every \( p \in \mathcal{P} \) when the system output \( y(t), t \geq 0 \), is set to reference signal \( y_i(t), t \geq 0 \).

Prediction tracking control is a challenging problem when the system dynamics undergo fast variations. To address this problem, Li and Haddad (2012a) merged model predictive control with repetitive control by transforming a periodic time-varying linear system into a run-to-run invariant system. However, their results were limited to linear systems. For nonlinear systems with stable zero dynamics, Sorouh and Kravaris (1996) showed that model predictive controllers can be computed using geometric control methods over a small prediction horizon. A key advantage of this is that the derived control law has a closed-form solution, which greatly reduces the computational effort.

**Assumption 3.2:** For a given bounded reference input \( y_i(t), t \geq 0, \quad y_i^{(r_i)}(t), i = 1, \ldots, m, \) are bounded, where \( y_i(t) \) is the \( i \)th element of \( y(t) \), and there exists \( y_i(t) \in \mathbb{R}^m \) and \( u_i(t) \in \mathbb{R}^m, t \geq 0, \) satisfying (3) and (4) with \( y_i(t) = h(x_i(t)) \).

To achieve asymptotic tracking, we design a control law such that the system error \( e(t) \triangleq y(t) - y_i(t), t \geq 0 \), is bounded and converges to zero asymptotically. Specifically, using the approach given by Singh et al. (1995), we define a vector function \( \phi_p(t) \triangleq [\phi_1(t), \ldots, \phi_m(t)]^T, p \in \mathcal{P} \) for
almost every $t \geq 0$, where
\[
\phi_p(t) = e_p^{(r-1)}(t) + \alpha_{i,r-1} e_p^{(r-2)}(t) + \cdots + \alpha_{i1} e_i(t) \\
+ \alpha_{i0} \int_0^t e_i(\tau) d\tau, \quad i = 1, \ldots, m, \tag{11}
\]
e_i(t) = y_i(t) - y_{\hat{t},i}(t), and the coefficients $\alpha_{i,j} > 0$, $j = 0, \ldots, r - 1$, are chosen such that the polynomial
\[
s^r + \alpha_{i,r-1} s^{r-1} + \cdots + \alpha_{i1}s + \alpha_{i0} = 0, \quad i = 1, \ldots, m, \tag{12}
\]
is Hurwitz. Differentiating (11) with respect to time yields, for almost every $t \geq 0$,
\[
\dot{\phi}_p(t) = e_p^{(r)}(t) + \alpha_{i,r-1} e_p^{(r-1)}(t) + \cdots + \alpha_{i1} e_i(t) \\
+ \alpha_{i0} e_i(t), \quad p \in \mathcal{P}, \quad i = 1, \ldots, m. \tag{13}
\]
Thus, it follows from (10) and (13) that
\[
\dot{\phi}_p(t) = y_p^{(r)}(t) - y_i^{(r)}(t) + \psi_p(t) \\
= a_p(z(t)) + D_p(z(t)) v(t) - y_i^{(r)}(t) + \psi_p(t), \quad p \in \mathcal{P}, \quad a.e. \quad t \geq 0, \tag{14}
\]
where
\[
\psi_p(t) = [\psi_{1p}^{(T)}(t), \ldots, \psi_{mp}^{(r)}(t)]^T \quad \text{with} \quad \psi_{ip}(t) = \alpha_{i,r-1} e_i^{(r-1)}(t) + \cdots + \alpha_{i1} e_i(t) + \alpha_{i0} e_i(t), \quad i = 1, \ldots, m, \quad \text{and}
\]
y_i^{(r)}(t) = [y_i^{(r-1)}(t), \ldots, y_i^{(0)}(t)]^T. \quad \text{Now, for sufficiently small } \tau > 0, \text{ it follows from (14), using a first-order Taylor series expansion, that}
\[
\phi_p(t + \tau) \approx \phi_p(t) + \tau \dot{\phi}_p(t) = \phi_p(t) + \tau[a_p(z(t)) \\
+ D_p(z(t)) v(t) - y_i^{(r)}(t) + \psi_p(t)], \quad p \in \mathcal{P}, \quad a.e. \quad t \geq 0. \tag{15}
\]

Next, we use model predictive control to design a tracking controller for the dynamical system $\hat{G}_p$. As discussed in Soroush and Kravaris (1996), model predictive control involves the prediction of the future system response using a given system dynamics model and the calculation of a sequence of controller actions obtained by minimising a given performance index. In the model predictive control literature (Chen & Allgöwer, 1998; Mayne, Rawlings, Rao, & Scokaert, 2000), a large prediction horizon has been used to address stability and unstable zero dynamics. However, large prediction horizons degrade system robustness and require significant online computational effort. For systems with stable zero dynamics, it has been shown (Lu, 1996; Singh et al., 1995; Soroush & Kravaris, 1996) that it is possible to use short prediction horizons to accurately predict the future system response using a given system dynamics model. As shown below, such a prediction equation with an appropriate reference trajectory yields a model predictive control law whose implementation does not require an online optimisation.

Since the switching signal $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is piecewise constant, there exists $p \in \mathcal{P}$ such that $\sigma(t) = p$ for a given time $t \geq 0$. To develop a model predictive controller for (3) and (4) at a given $t \geq 0$, consider the minimisation problem
\[
\min_{v(t) \in \mathcal{V}_i} J_p(v(t)) = \frac{1}{2} \phi_p^T(t + \tau) Q \phi_p(t + \tau) + \frac{1}{2} v^T(t) R v(t), \tag{16}
\]
where $Q > 0$, $Q \in \mathbb{R}^{m \times m}$, and $R \geq 0$, $R \in \mathbb{R}^{m \times m}$. Note that the first quadratic term in the performance criterion (16) captures a weighted least squares measure of the predicted tracking errors, as well as their derivatives and integrals, whereas the second quadratic term in (16) penalises the control rate. Next, note that $v(t) \in \mathcal{V}_i$ can be re-written as
\[
A v(t) - b(t) \leq 0, \quad t \geq 0, \tag{17}
\]
where
\[
A = [1 \quad 0 \ldots 0] \quad \text{and} \quad b(t) = \begin{bmatrix} v_{1,\text{max}}(t) \\
-v_{1,\text{min}}(t) \\
\vdots \\
v_{m,\text{max}}(t) \\
-v_{m,\text{min}}(t) \end{bmatrix}.
\]
The constrained optimisation problem given by (16) and (17) can be solved using Lagrange multiplier methods. Specifically, introducing the Lagrange multiplier $\lambda = [\lambda_1, \dot{\lambda}_1, \ldots, \lambda_m, \dot{\lambda}_m]^T \in \mathbb{R}^{2m}$ and forming the Lagrangian
\[
\mathcal{L}(J_p(v), \lambda) = J_p(v) + \lambda^T(A v - b), \tag{18}
\]
it follows from the Kuhn–Tucker necessary conditions for optimality that, for $t \geq 0$,
\[
\frac{\partial}{\partial v}(J_p(v(t)) + \lambda^T(A v(t) - b(t))) = 0, \quad p \in \mathcal{P}, \tag{19}
\]
\[
\lambda^T(A v(t) - b(t)) = 0, \tag{20}
\]
\[
\lambda_i = 0, \quad v_i(t) < v_{i,\text{max}}(t), \quad \dot{\lambda}_i = 0, \quad v_i(t) > v_{i,\text{min}}(t), \quad i = 1, \ldots, m, \tag{21}
\]
\[
\lambda_i \geq 0, \quad v_i(t) = v_{i,\text{max}}(t), \quad \dot{\lambda}_i \geq 0, \quad v_i(t) = v_{i,\text{min}}(t), \quad i = 1, \ldots, m. \tag{22}
\]
Next, using (15) and (16), (19) can be re-written as

\[
(\tau^2 D_p^T(\hat{z}(t))QD_p(\hat{z}(t)) + R)v(t) + \tau D_p^T(\hat{z}(t))Q(\phi_p(t) + \tau[a_p(\hat{z}(t)) - y'(t))] + A^T\lambda = 0, \quad p \in \mathcal{P}, \quad \text{a.e. } t \geq 0,
\]

where \(\hat{z}(t)\) denotes the prediction of \(z(t)\) at time \(t \geq 0\). In order to solve (19)–(22), we use the numerical iterative approach developed by Lu (1996). First, however, we define the saturation map \(S : \mathbb{R}^m \to \mathbb{R}^m\) as \(S(v) \triangleq [s_1(v_1), \ldots, s_m(v_m)]^T\), where

\[
s_i(v_i) = \begin{cases} 
  v_{i,max}, & v_i \geq v_{i,max}, \\
  v_i, & v_{i,min} < v_i < v_{i,max}, \\
  v_{i,min}, & v_i \leq v_{i,min}, 
\end{cases} \quad i = 1, \ldots, m. 
\]

The optimal controller \(v^*(t), \; t \geq 0\), satisfying the necessary conditions (19)–(22) is given by the following theorem. For the statement of this theorem define \(\Gamma_p(\hat{z}) \triangleq \tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R\) and \(\beta_p \triangleq \left(\sum_{i=1}^m \sum_{j=1}^m \Gamma_{i,j}(\hat{z})\right)^{-1/2}\), where \(\Gamma_{i,j}(\hat{z})\) denotes the \((i,j)\)th entry of \(\Gamma_p(\hat{z})\).

**Theorem 3.1:** For \(t \geq 0\) such that \(\mu(t) \in [0, \infty) : v(t) = \hat{u}(t)) \neq 0\), where \(\mu(\cdot)\) denotes the Lebesgue measure in \(\mathbb{R}_+\), and every \(v_0 \in \mathcal{V}\), consider the unbounded sequence \(\{v_k\}_{k=0}^\infty\) generated by

\[
v_{k+1} = S(\beta_p(\tau D_p^T(\hat{z})Q(\tau[V_p^T - a_p(\hat{z}) - \psi_p]) - \phi_p))) - \beta_p(\tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R) - I_m) v_k\]

\(\triangleq T(v_k).\) (25)

Then, for sufficiently small \(\tau > 0\), there exists a unique optimal controller \(v^*(t)\) such that \(T(v^*(t)) = v^*(t)\) and, for each \(v_0 \in \mathcal{V}\), the sequence \(\{v_k\}_{k=0}^\infty\) converges to \(v^*(t)\).

**Proof:** First, we show that for a fixed time \(t \geq 0\) such that \(\mu(t) \in [0, \infty) : v(t) = \hat{u}(t)) \neq 0\), the optimal control \(v^*(t)\) satisfying (19)–(22) is a fixed point of (25). If \(v_{i,min}(t) < v_{i}(t) < v_{i,max}(t)\) for a fixed \(t \geq 0\) and \(i = 1, \ldots, m\), then it follows from (21)–(23) that

\[
(\tau^2 D_p^T(\hat{z}(t))QD_p(\hat{z}(t)) + R)v(t) + \tau D_p^T(\hat{z}(t))Q(\phi_p(t) + \tau[a_p(\hat{z}(t)) - y'(t)] + A^T\lambda = 0, \quad p \in \mathcal{P}, \quad \text{a.e. } t \geq 0,
\]

In this case, (25) becomes \(T(v^*(t)) = v^*(t)\). If, alternatively, \(v_{i}(t) = v_{i,max}(t)\), for a fixed \(t \geq 0\) and every \(i \in \{1, \ldots, m\}\), then, by (21) and (22), \(\lambda_i \geq 0\) and \(\lambda_i = 0\), which implies that \((A^T\lambda_i) = \lambda_i - \lambda_i = \lambda_i\). Thus, the \(i\)th component of (23) satisfies

\[
((\tau^2 D_p^T(\hat{z}(t))QD_p(\hat{z}(t)) + R)v(t) + \tau D_p^T(\hat{z}(t))Q(\phi_p(t) + \tau[a_p(\hat{z}(t)) - y'(t)] + \psi_p(t)))]_i = -\lambda_i,
\]

and hence, since \(\lambda_i \geq 0\), the \(i\)th component of the right-hand side of (25) becomes

\[
s_i(\beta_p(\hat{\lambda}_i) + v_i^*(t)) = s_i(\beta_p(\hat{\lambda}_i) + v_{i,max}(t)) = v_{i,max}(t) = v_i^*(t).
\]

Analogously, if \(v_i^*(t) = v_{i,min}(t)\) for a fixed \(t \geq 0\) and every \(i \in \{1, \ldots, m\}\), then a similar argument as given above yields

\[
s_i(-\beta_p(\hat{\lambda}_i) + v_i^*(t)) = s_i(-\beta_p(\hat{\lambda}_i) + v_{i,min}(t)) = v_{i,min}(t) = v_i^*(t),\]

since \(\hat{\lambda}_i \geq 0\). Hence, \(v^*(t)\) is a fixed point of (25).

Next, we show that \(T(\cdot)\) is a contraction mapping. To show this, define

\[
\eta_p(v) \triangleq \beta_p(\tau D_p^T(\hat{z})Q(\tau[V_p^T - a_p(\hat{z}) - \psi_p]) - \phi_p))) - \beta_p(\tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R) - I_m) v.
\]

Then, for \(p \in \mathcal{P}\) and every \(v, r \in \mathbb{R}^m\),

\[
\|T(v) - T(r)\| \leq \|\eta_p(v) - \eta_p(r)\|
\]

\[
= -\left[\beta_p(\tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R) - I_m) v - r\right]
\]

\[
\leq \left\| I_m - \beta_p(\tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R) \right\| \|v - r\|
\]

\[
\leq \lambda_{max} \left( I_m - \beta_p(\tau^2 D_p^T(\hat{z})QD_p(\hat{z}) + R) \right) v - r \|
\]

\[
= \alpha \|v - r\|
\]

(27)

Now, since \(\beta_p = \left(\sum_{i=1}^m \sum_{j=1}^m \Gamma_{i,j}(\hat{z})\right)^{-1/2}\), where \(\|\cdot\|_F\) is the Frobenius matrix norm, it follows that

\[
\alpha = 1 - \beta_p \lambda_{min}(\Gamma_p(\hat{z})) = 1 - \beta_p \sigma_{min}(\Gamma_p(\hat{z})) < 1.
\]

Hence, \(T : \mathbb{R}^m \to \mathbb{R}^m\) is a contraction mapping. Now, since \(\mathbb{R}^m\) with spacial norm \(\|\cdot\|_q, q \in [1, \infty]\), is a complete space, it follows from the Banach fixed point theorem (Haddad & Chellaboina, 2008, p. 68) that there exists a unique \(v^* \in \mathbb{R}^m\) such that \(T(v^*) = v^*\), and the sequence \(\{v_k\}_{k=0}^\infty \subseteq \mathcal{V}\) converges to \(v^*\). Furthermore, since \(\lambda_1\) is closed, it follows
from Proposition 2.9 of Haddad and Chellaboina (2008, p. 29) that \( \bar{v}^* \in V_i \). Since \( \bar{v}^* \) is unique, \( \bar{v}^* = v^*(t) \).

If for almost every \( t \geq 0 \), \( v^*(t) \) satisfying (25) is such that \( v_{i,\text{min}}(t) < v^*_i(t) < v_{i,\text{max}}(t), \) \( i = 1, \ldots, m \), then the optimal control law \( v^*(t) \) collapses to

\[
v^*(t) = \left( t^2 D_p^T(\hat{z}(t))Q D_p(\hat{z}(t)) + R \right)^{-1} \tau D_p^T(\hat{z}(t)) \times Q[\tau(y_{p,0}^\tau(t) - a_p(\hat{z}(t)) - \psi_p(t) - \phi_p(t))],
\]

\[ p \in P, \quad \text{a.e. } t \geq 0. \tag{29} \]

If, in addition, in this case the weighting matrix \( R = 0_{m \times m} \), then substituting (29) into (14) yields

\[
\hat{v}_p(t) = -\frac{1}{\tau} \phi_p(t), \quad p \in P, \quad \text{a.e. } t \geq 0. \tag{30}
\]

Thus, \( \phi_p(t) \to 0 \) as \( t \to \infty \) almost everywhere. Now, (30) implies \( \hat{v}_p(t) \to 0 \) as \( t \to \infty \). Furthermore, in this case substituting (30) into (13) yields

\[
\dot{\epsilon}_i(t) + \left( \alpha_i, r_i - 1 + \frac{1}{\tau} \right) \epsilon_i(t) + \cdots + \left( \alpha_i, 0 + \frac{1}{\tau} \alpha_i, 1 \right) \epsilon_i(t) + \frac{1}{\tau} \int_0^t \epsilon_i(\tau)d\tau = 0,
\]

\[ i = 1, \ldots, m, \quad p \in P, \quad \text{a.e. } t \geq 0. \tag{31} \]

Thus, since \( \alpha_{i,j}, j = 0, \ldots, r_i - 1 \), are chosen such that (12) is Hurwitz, \( e_i(t) \to 0 \) as \( t \to \infty \), \( i \in \{1, \ldots, m\} \).

**Proposition 3.1:** If \( 0 \leq u(0) \leq \epsilon \bar{v} \) and \( v^*(t) \) satisfying (25) is such that \( T(v^*(t)) = v^*(t) \), then \( u^*(t) \in U \) for all \( t \geq 0 \).

**Proof:** The control rate constraint \( \epsilon \bar{v} \leq v^*(t) \leq \epsilon \bar{w} \), \( t \geq 0 \), is automatically satisfied since (17) and (24) hold. Now, to show that the amplitude constraint \( 0 \leq u_i^*(t) \leq \epsilon \bar{w} \), \( t \geq 0 \), holds, suppose that at some time \( t_1 \geq 0 \), \( u_i^*(t_1) = 0 \) for \( i \in \{1, \ldots, m\} \). Then, it follows from the definition of \( v_{i,\text{min}}(t) \) that \( v_i^*(t) \geq v_{i,\text{min}}(t_1) > -\frac{\epsilon \bar{v}}{2} = 0 \).

Thus, \( u_i^*(t) \) is strictly increasing, and hence, \( u_i^*(t) > 0 \) for \( t \geq t_1 \). Similarly, suppose that for some \( t_2 \geq 0 \), \( u_i^*(t_2) = \pi \) for \( i \in \{1, \ldots, m\} \). Then, \( v_i^*(t_2) \leq v_{i,\text{max}}(t_2) < \frac{\epsilon \bar{w} - u_i^*(t_2)}{\delta} = 0 \).

Thus, \( u_i^*(t) \) is strictly decreasing, and hence, \( u_i^*(t_2) < \pi \) for \( t \geq t_2 \). Hence, \( u_i^*(t) \in U \) for all \( t \geq 0 \).

A block diagram of the constrained tracking control architecture given in Theorem 3.1 is shown in Figure 1.

4. **Nonlinear multi-compartment lung model**

In this section, we present a nonlinear model for the dynamic behavior of a multi-compartment respiratory system in response to an arbitrary applied inspiratory pressure. This model was first developed by Volyanskyy et al. (2011).

Here, we assume that the bronchial tree has a dichotomy architecture (Weibel, 1963); that is, in every generation each airway unit branches into two airways of the subsequent generation. In addition, we assume that the lung compliance is a nonlinear function of lung volume.

In this model, the lungs are represented as \( 2^n \) lung units which are connected to the pressure source by \( n \) generations of airway units, where each airway is divided into two airways of the subsequent generation leading to \( 2^n \) compartments (see Figure 2 for a four-compartment model). Let \( x_i, i = 1, 2, \ldots, 2^n \), denote the lung volume in the \( i \)th compartment, \( c_i^m(x_i) \) (respectively, \( c_i^{\text{ex}}(x_i) \)), \( i = 1, 2, \ldots, 2^n \), denote the compliance at inspiration (respectively, expiration) of each compartment as a nonlinear function of the volume of the \( i \)th compartment, and let \( R_{ij}^m \) (respectively, \( R_{ij}^{\text{ex}} \)), \( i = 1, 2, \ldots, 2^n \), \( j = 0, \ldots, n \), denote the resistance (to air flow) of the \( i \)th airway in the \( j \)th generation during the inspiration (respectively, expiration) period with \( R_{ij}^m \) and \( R_{ij}^{\text{ex}} \).
(respectively, $R_{\text{ex}}^i$) denoting the inspiration (respectively, expiration) of the parent (i.e., 0th generation) airway.

Here, we assume that a pressure of $p_{\text{in}}(t)$, $t \geq 0$, is generated (by the inspiratory muscles) or applied (by a mechanical ventilator) during inspiration.

Now, the state equations for inspiration are given by Volyanskyy et al. (2011) as

$$R_{\text{in},i} x_i(t) + \frac{1}{c_{\text{in}}^i(x_i(t))} x_i(t) + \sum_{j=0}^{n-1} R_{\text{ex},j,kj} \sum_{l=(k_j-1)2^{j+1}+1}^{k_j 2^{j+1}} \dot{x}_l(t) = p_{\text{in}}(t),$$

$$x_i(0) = x_{i0}, \quad 0 \leq t \leq T_{\text{in}}, \quad i = 1, 2, \ldots, 2^n,$$

(32)

where $c_{\text{in}}^i(x_i)$, $i = 1, 2, \ldots, 2^n$, are nonlinear functions of $x_i$, $i = 1, 2, \ldots, 2^n$, given by Crooke, Marini, and Hotchkiss (2002) as

$$c_{\text{in}}^i(x_i) \triangleq \begin{cases} a_{i1}^i + b_{i1}^i x_i, & \text{if } 0 \leq x_i \leq x_{i1}^i, \\ a_{i2}^i, & \text{if } x_{i1}^i \leq x_i \leq x_{i2}^i, \\ a_{i3}^i + b_{i3}^i x_i, & \text{if } x_{i2}^i \leq x_i \leq V_T, \end{cases}$$

(33)

where $a_{ij}^i$, $j = 1, 2, 3$, and $b_{ij}^i$, $j = 1, 3$, are model parameters with $b_{i1}^i > 0$ and $b_{i3}^i < 0$, $x_{i1}^i, j = 1, 2$, are volume ranges wherein the compliance is constant, $V_T$ denotes tidal volume, and

$$k_j = \left\lfloor \frac{k_{j+1} - 1}{2} \right\rfloor + 1, \quad j = 0, \ldots, n - 1, \quad k_n = i,$$

(34)

where $\lfloor q \rfloor$ denotes the floor function. Figure 3 shows a typical piecewise linear compliance function for inspiration. A similar compliance representation holds for expiration and is also shown in Figure 3.

Next, we consider the state equation for the expiration process. Here, we assume that the expiration process is passive and the external pressure applied is $p_{\text{ex}}(t)$, $t \geq 0$. Following an identical procedure as in the inspiration case, we obtain the state equation for expiration as

$$R_{\text{ex},i} x_i(t) + \sum_{j=0}^{n-1} R_{\text{ex},j,kj} \sum_{l=(k_j-1)2^{j+1}+1}^{k_j 2^{j+1}} \dot{x}_l(t) + \frac{1}{c_{\text{ex}}^i(x_i(t))} x_i(t) = p_{\text{ex}}(t),$$

$$x_i(T_{\text{ex}}) = x_{i0}^{\text{ex}}, \quad T_{\text{in}} \leq t \leq T_{\text{ex}} + T_{\text{in}}, \quad i = 1, 2, \ldots, 2^n,$$

(35)

Figure 3. Typical inspiration and expiration compliance functions as function of compartmental volumes.

where

$$c_{\text{ex}}^i(x_i) \triangleq \begin{cases} a_{i1}^{\text{ex}} + b_{i1}^{\text{ex}} x_i, & \text{if } 0 \leq x_i \leq x_{i1}^{\text{ex}}, \\ a_{i2}^{\text{ex}}, & \text{if } x_{i1}^{\text{ex}} \leq x_i \leq x_{i2}^{\text{ex}}, \quad i = 1, \ldots, 2^n, \\ a_{i3}^{\text{ex}} + b_{i3}^{\text{ex}} x_i, & \text{if } x_{i2}^{\text{ex}} \leq x_i \leq V_T, \end{cases}$$

(36)

$a_{ij}^{\text{ex}}$, $j = 1, 2, 3$, and $b_{ij}^{\text{ex}}$, $j = 1, 3$, are model parameters with $b_{i1}^{\text{ex}} > 0$ and $b_{i3}^{\text{ex}} < 0$, $x_{i1}^{\text{ex}}, j = 1, 2$, are volume ranges wherein the compliance is constant, and $k_j$ is given by (34).

Finally, we provide a smooth characterisation of the nonlinear compliance using the cubic spline data interpolation method of Boor (1978). Figure 4 shows the smoothed approximation of the piecewise linear compliance function $c_{\text{ex}}^i(x_i)$. A similar approximation holds for $c_{\text{ex}}^i(x_i)$ which is also shown in Figure 4.

5. Tracking control for pressure-limited mechanical ventilation

In this section, we use the constrained tracking control framework developed in Section 3 to design a predictive output tracking controller for the nonlinear multi-compartmental lung mechanics model given in Section 4. The goal of this controller is to track a given clinically plausible volume pattern while satisfying a given set of amplitude and rate input constraints. First, however, we re-write the state Equations (32) and (35) for inspiration and expiration, respectively, into vector–matrix state space form. Specifically, define the state vector $x \triangleq [x_1, x_2, \ldots, x_2]^T$, where $x_i$ denotes the lung volume of the $i$th compartment.
Now, the state Equation (32) for inspiration can be re-written as
\[
R_{in} \dot{x}(t) + C_{in}(x(t))x(t) = p_{app}(t)e,
\]
\[
x(0) = x_0^{in}, \quad 0 \leq t \leq T_{in},
\]
where \(C_{in}(x)\) is a diagonal matrix function given by
\[
C_{in}(x) \triangleq \text{diag} \left[ \frac{1}{c_1^{in}(x_1)}, \ldots, \frac{1}{c_n^{in}(x_2^n)} \right]
\] (38)
and
\[
R_{in} \triangleq \sum_{j=0}^{2^n} \sum_{k=1}^{2^n} R_{j,k}^{in} Z_{j,k} Z_{j,k}^T,
\] (39)
where \(Z_{j,k} \in \mathbb{R}^{2^n}\) is such that the \(l\)th component of \(Z_{j,k}\) is 1 for all \(l = (k-1)2^{n-j} + 1, (k-1)2^{n-j} + 2, \ldots, k2^{n-j}, \)
\(k = 1, \ldots, 2^j, j = 0, 1, \ldots, n\), and zero elsewhere.

Similarly, the state Equation (35) for expiration can be re-written as
\[
R_{ex} \dot{x}(t) + C_{ex}(x(t))x(t) = p_{app}(t)e,
\]
\[
x(T_{in}) = x_0^{ex}, \quad T_{in} \leq t \leq T_{ex} + T_{in}
\] (40)
where
\[
C_{ex}(x) \triangleq \text{diag} \left[ \frac{1}{c_1^{ex}(x_1)}, \ldots, \frac{1}{c_n^{ex}(x_2^n)} \right].
\] (41)
and
\[
R_{ex} \triangleq \sum_{j=0}^{n} \sum_{k=1}^{2^j} R_{j,k}^{ex} Z_{j,k} Z_{j,k}^T
\] (42)
Furthermore, it follows from Proposition 4.1 of Chellaboina et al. (2010) that \(R_{in}\) and \(R_{ex}\) are positive-definite and, hence, \(R_{in}\) and \(R_{ex}\) are invertible matrices. Hence, (37) and (40) can be re-written as
\[
\dot{x}(t) = A_{in}(x(t))x(t) + B_{in}u(t), \quad x(0) = x_0^{in}, 0 \leq t \leq T_{in},
\] (43)
\[
\dot{x}(t) = A_{ex}(x(t))x(t) + B_{ex}u(t), \quad x(T_{in}) = x_0^{ex}, T_{in} \leq t \leq T_{ex} + T_{in},
\] (44)
where \(A_{in}(x) = -R_{in}^{-1}C_{in}(x), B_{in} = R_{in}^{-1}e, A_{ex}(x) = -R_{ex}^{-1}C_{ex}(x),\) and \(B_{ex} = R_{ex}^{-1}e\).

In this article, we assume that the inspiration process starts from a given initial state \(x_0^{in}\) followed by the expiration process where its initial state will be the final state of the inspiration. An inspiration followed by the expiration is called
a single breathing cycle. We assume that each breathing cycle is followed by another breathing cycle where the initial condition for the latter breathing cycle is the final state of the former breathing cycle. Furthermore, we assume that the duration of inspiration is \(T_{in}\) and that of expiration is \(T_{ex}\), so that the total duration of a breathing cycle is \(T = T_{in} + T_{ex}\). Moreover, the system dynamics switches from inspiration to expiration and back to inspiration. Hence, the dynamics for the breathing process can be characterised by a set of switched dynamical systems as

\[
\dot{x}(t) = f_\sigma(x(t))u(t),
\]

\[
x(0) = x_0, \quad t \geq 0,
\]

\[
y(t) = e^T x(t).
\]

Here, we define the switching signal \(\sigma(t) \in \{1, 2\}\) by

\[
\sigma(t) = \begin{cases} 
1, & \text{if } 0 \leq \text{mod}(t, T) < T_{in}, \\
2, & \text{if } T_{in} \leq \text{mod}(t, T) < T,
\end{cases}
\]

and the switching system functions by

\[
f_1(x) = A_{in}(x)x, \quad G_1(x) = B_{in},
\]

\[
f_2(x) = A_{ex}(x)x, \quad G_2(x) = B_{ex}.
\]

Note that since, by Proposition 4.1 of Chellaboina et al. (2010), \(-R_{in}^{-1}\) and \(-R_{ex}^{-1}\) are essentially non-negative, and \(C_{in}(x)\) and \(C_{ex}(x)\) are diagonal, it follows that \(A_{in}(x)\) and \(A_{ex}(x)\) are essentially non-negative. Moreover, it is also shown in Proposition 4.1 of Chellaboina et al. (2010) that \(B_{in} \geq 0\) and \(B_{ex} \geq 0\). Thus, \(G_\sigma(x) \geq 0\).

Next, the prediction time horizon \(\tau\) in the performance criterion (16) is chosen such that

\[
\text{mod}(T_{in}, \tau) = \text{mod}(T_{ex}, \tau) = 0.
\]

Thus, one period is divided into \(N = \frac{T}{\tau}\) equally spaced intervals with each time interval given by

\[
i \tau \leq \text{mod}(t, T) < (i + 1)\tau, \quad i \in \{0, \ldots, N - 1\}.
\]

In this case, the switching signal \(\sigma(t), t \geq 0\) in (47) can be rewritten as

\[
\sigma(t) = \begin{cases} 
1, & \text{if } i \tau \leq \text{mod}(t, T) < (i + 1)\tau \text{ and } (i + 1)\tau + \tau \leq T_{in}, \\
2, & \text{if } i \tau \leq \text{mod}(t, T) < (i + 1)\tau \text{ and } i \tau \geq T_{in},
\end{cases}
\]

where \(i \in \{0, \ldots, N - 1\}\). Thus, (45) and (46), with the switching signal (50), are in the form of (3) and (4).

As in Section 3, introducing \(\dot{u}(t) = u(t)\) for almost every \(t \geq 0\) and states \(z(t) = [x^T(t), u^T(t)]^T, t \geq 0\), it follows that the augmented nonlinear system dynamics (6) and (7) is satisfied with

\[
\begin{align*}
\dot{f}_p(z) &= \begin{bmatrix} f_p(x) + G_p(x)u(t) \\ 0 \end{bmatrix}, \\
\dot{g} &= \begin{bmatrix} 0_{n_u} \\ 1 \end{bmatrix}, \\
\dot{h}(z) &= [e^T 0]z, \quad p \in \{1, 2\} = \mathcal{P}.
\end{align*}
\]

Now, the time derivatives of the output are given by

\[
\begin{align*}
\dot{y}_p(t) &= e^T f_p(x(t)) + e^T G_p(x(t))u(t), \\
\ddot{y}_p(t) &= a_p(z(t)) + D_p(z(t))v(t), \quad \text{a.e. } t \geq 0,
\end{align*}
\]

where

\[
a_p(z(t)) = \begin{cases} 
\begin{bmatrix} e^T A_{in}(x(t))x(t) + e^T A_{in}^2(x(t))x(t) \\ + e^T A_{in}(x(t))B_{in}u(t), p = 1, \\
e^T A_{ex}(x(t))x(t) + e^T A_{ex}^2(x(t))x(t) \\ + e^T A_{ex}(x(t))B_{ex}u(t), p = 2,
\end{cases}
\]

\[
D_p(z(t)) = \begin{cases} 
\begin{bmatrix} e^T B_{in} \\ e^T B_{ex}, 
\end{bmatrix}, \quad p = 1, \\
\begin{bmatrix} e^T B_{in} \\ e^T B_{ex}, 
\end{bmatrix}, \quad p = 2.
\end{cases}
\]

Since \(D_1(z(t)) = e^T R_{in}^{-1}e > 0\) and \(D_2(z(t)) = e^T R_{ex}^{-1}e > 0, i)\) of Assumption 3.1 is satisfied. To characterise the system zero dynamics, define

\[
Z_p^{\ast} = \left\{ z = \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n_x} \times U : e^T x = 0 \right\}
\]

\[
= \left\{ z = \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n_x} \times U : x = 0, u = 0 \right\}. \tag{54}
\]

Thus, \(ii)\) in Assumption 3.1 is automatically satisfied. Now, it follows from (53) that the system has relative degree two, that is, \(r = 2\), and hence, by (11),

\[
\phi_p(t) = \dot{e}_p(t) + 2e(t) + \int_0^t e(\tau) \text{d}\tau, \quad p \in \mathcal{P}, \quad \text{a.e. } t \geq 0.
\]

\[
\text{Hence, } \psi_p(t) = 2\phi_p(t) + e(t), \quad p \in \mathcal{P}, \quad \text{a.e. } t \geq 0.
\]

For our simulation, we consider a two-compartment lung model and use the values for lung resistance and compliance found in Crooke et al. (2002). In particular, we set \(a_{in} = 0.018 \ell/cm \text{H}_2\text{O}, b_{in} = 0.0233, a_{ex} = 0.025 \ell/cm \text{H}_2\text{O}, a_{in} = 0.2532 \ell/cm \text{H}_2\text{O}, b_{in} = -0.01, x_{in} = 0.3 \ell, x_{ex} = 0.48 \ell, a_{ex}^x = 0.02 \ell/cm \text{H}_2\text{O}, b_{ex}^x = 0.078, a_{ex}^x = 0.038 \ell/cm \text{H}_2\text{O}, a_{ex}^x = 0.1025 \ell/cm \text{H}_2\text{O}, b_{ex} = -0.15, x_{ex} = 0.23 \ell, \text{and } x_{ex} = 0.43 \ell, i = 1, 2, \) Here, we assume...
that the bronchial tree has a dichotomy structure (see Section 4). The airway resistance varies with the branch generation and typical values can be found in Hofman and Meyer (1999). Furthermore, the expiratory resistance will be higher than the inspiratory resistance by a factor 2 to 3. Here, we assume that the factor is 2.5. The initial conditions are set as $x_0 = [0.01, 0.05]^T$ and $u_0 = 0$. The prediction time steps $\tau$ and $\delta$ are both set to be 0.1. We choose the control input constraints to be $u = 0$ cm H$_2$O and $\overline{u} = 25$ cm H$_2$O, and the control rate constraints to be $v = -100$ cm H$_2$O/s and $\overline{v} = 50$ cm H$_2$O/s. Finally, we set $Q = 100$ and $R = 0$.

Note that since $R = 0$, the optimal performance criterion (16) becomes

$$J^\ast_p(v(t)) = \frac{1}{2} \phi^T_p(t + \tau) Q \phi_p(t + \tau), \ p \in \mathcal{P}$$

for almost every $t \geq 0$. Since $J^\ast_p(v(t))$ is strictly convex and $\mathcal{V}_t$ is convex, it follows that there exist an optimal control $v^\ast(t), \ t \geq 0$, which satisfies the amplitude constraints and is bounded over certain time intervals as shown in Figure 5 and Figure 6, respectively. Next, we change the control input constraints to $u = 0$ cm H$_2$O and $\overline{u} = 35$ cm H$_2$O, and the control rate constraints to $v = -100$ cm H$_2$O/s and $\overline{v} = 100$ cm H$_2$O/s. Figures 8 and 9 show that the optimal control input $u^\ast(t), \ t \geq 0$, and the control rate $v^\ast(t), \ t \geq 0$, satisfy the amplitude and rate constraints and are not saturated. Figure 10 shows that controller drives the total lung compartment volume to the reference trajectory asymptotically, which agrees with the analysis in Section 3.

6. Conclusion
In this article, we designed a predictive tracking controller for a nonlinear multi-compartment respiratory system to
track a given reference lung volume pattern that accounts for amplitude and rate control constraints. The predictive control law is derived by minimising a quadratic performance criterion involving a prediction of the system response over a prescribed time step. This proposed approach gives an explicit form of the control law, and thus, avoids online optimisation.

Although model predictive control is well suited for addressing mechanical ventilation control problems involving control constraints, it is in general not easy to use if the system model involves nonlinear uncertainties and time-varying dynamics. One of the key reasons for this is that it can be difficult to guarantee closed-loop system stability, which typically involves adding weighting parameters in the control architecture. Furthermore, online computation can be expensive for time-varying systems with fast changing dynamics.

The model predictive control framework presented in this article is based on a nominal lung mechanics model. Physiological variables, however, vary from patient to patient, as well as within the same patient under different conditions, making it very challenging to develop models and effective control law architectures for active mechanical ventilation. Using a predictive tracking algorithm, Singh et al. (1995) showed that the tracking errors $e(t)$, $t \geq 0$, remain bounded in the face of small perturbations to the parameters in the system model. However, in the presence of amplitude or rate input constraints, the predictive tracking algorithm does not provide robustness. In future research, we plan to extend the single-step-ahead scheme developed in Theorem 3.1 to address robust stability and system tracking error convergence in the face of parametric system uncertainty.

Funding
This research was supported in part by the QNRF under NPRP [grant number 4-187-2-060].

Note
1. In this article, it is important to distinguish between a square non-negative (respectively, positive) matrix and a non-negative-definite (respectively, positive-definite matrix).

References


