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Mixed-norm $H_2/L_1$ controller synthesis via fixed-order dynamic compensation: a Riccati equation approach

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One of the fundamental problems in feedback control design is the ability of the control system to reject uncertain exogenous disturbances. Since a single performance objective is seldom adequate to capture multiple and often conflicting system disturbances, in this paper we develop a Riccati equation approach for mixed $H_2/L_1$ controller synthesis via fixed-order dynamic compensation. This multiobjective problem is treated by forming a convex combination of both the $H_2$ (quadratic) and $L_1$ bound (worst-case peak amplitude response) performance measures. For flexibility in controller synthesis, we adopt the approach of fixed-structure controller design which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization. Finally, using a quasi-Newton continuation algorithm, we demonstrate the effectiveness of the proposed mixed-norm $H_2/L_1$ Riccati equation approach via several design examples.

1. Introduction

One of the fundamental problems in feedback control design is the ability of the control system to reject uncertain exogenous disturbances. The key difference between controller design approaches for disturbance rejection can be traced to the modelling and treatment of system disturbances. For example, the Wiener–Hopf–Kalman control design theory (Anderson and Moore 1991, Kwakernaak and Sivan 1972) ($H_2$ theory) is based on a stochastic white noise disturbance model possessing a fixed covariance (power spectral density) which allows controller design for wide-band disturbance rejection while $H_\infty$ theory (Francis 1987, Zames 1981) is predicated on a deterministic disturbance model consisting of bounded energy (square-integrable) $L_2$ signals and allows controller design for narrow-band disturbance rejection. Hence, $H_2$ and $H_\infty$ performance quantify disturbance rejection in the frequency-domain involving mean-square power in the presence of white noise disturbances and worst-case peak frequency system gain amplification in the presence of $L_2$ disturbances, respectively. Alternatively, to address time-domain point-wise-in-time performance $L_1$ theory (Vidyasagar 1986) which captures worst-case peak amplitude response due to bounded amplitude persistent $L_\infty$ disturbances is clearly appropriate.

For finite-dimensional time-invariant linear systems the $H_2$ and $H_\infty$ control design problems have been thoroughly investigated (see, for example, Anderson and Moore (1991), Doyle et al. (1989), Kwakernaak and Sivan (1972) and the numerous references therein). In particular, the $H_2$ and $H_\infty$ control design problems were formulated and solved in the state space setting wherein the existence of an $H_2$
optimal and an $H_\infty$ (sub)optimal controller whose order is equal to the plant order is equivalent to the existence of solutions to a set of decoupled algebraic Riccati equations. Alternatively, the $L_1$ optimal control problem was formulated by Vidyasagar (1986) and solved by Dahleh and Pearson (1987b) by using the Youla parametrization of all stabilizing controllers to cast the problem into an infinite-dimensional constrained convex optimization problem. However, unlike $H_2$ and $H_\infty$ controllers, the resulting optimal $L_1$ controllers are irrational and hence infinite dimensional. Even though recent work on suboptimal $L_1$ controllers (Blanchini and Szaiaier 1994, Wang et al. 1995) yield rational (finite dimensional) controllers (by solving a particular discrete-time $\ell_1$ problem) that guarantee $L_1$ performance arbitrarily close to the optimal $L_1$ performance, the order of these controllers may be very high. In order to minimize controller complexity a new and novel approach to point-wise-in-time peak-to-peak gain minimization was formulated by Nagpal et al. (1994). Specifically, by approximating the closure of the set of reachable closed-loop system states due to unit-peak system disturbances by an ellipsoid (Schweppe 1973), Nagpal et al. (1994) obtain an upper bound to the $L_1$ norm ($L_\infty$ equi-induced norm with Euclidean spatial norms as opposed to spectral spatial norms). In this case the $L_1$ synthesis problem reduces to the evaluation of a linear matrix inequality feasibility problem.

Since a single performance objective is seldom adequate to capture multiple and often conflicting design objectives mixed-norm controller synthesis frameworks have been developed in recent years (Bernstein and Haddad 1989, Haddad and Bernstein 1990, Khargonekar and Rotea 1991, Doyle et al. 1995, Szaiaier et al. 1995). Specifically, to simultaneously reject white noise and bounded energy $L_2$ disturbances Bernstein and Haddad (1989) develop mixed-norm $H_2/H_\infty$ controllers characterized by means of a system of modified Riccati equations. Alternatively, the mixed-norm $H_2/\ell_1$ discrete-time problem for single-input/single-output systems with a single disturbance input and a single performance variable was studied by Salapaka et al. (1995) and Voulgaris (1994). However, as in the pure discrete-time $\ell_1$ case (Dahleh and Pearson 1987a, McDonald and Pearson 1991, Staffans 1991, Dahleh and Diaz-Bobillo 1995) the resulting mixed $H_2/\ell_1$ controller order could be arbitrarily high. In recent research Sznaier et al. (1995) posed a constrained (non-optimal) mixed $H_2$ bound/$L_1$ bound control problem using the linear matrix inequality feasibility framework developed by Nagpal et al. (1994). However, since the approach of Sznaier et al. (1995) is based on a matrix inequality feasibility problem, there are no optimality guarantees of the resulting mixed-norm $H_2$ bound/$L_1$ bound controller. Furthermore, in many practical applications it may be desirable to minimize the $H_2$ cost directly. That is, there may exist a ‘gap’ between the actual $H_2$ performance and the $H_2$ performance bound.

In this paper we develop an optimal mixed-norm $H_2/L_1$ bound and $H_2/\ell_1$ bound controller synthesis framework for continuous-time systems and discrete-time systems. Specifically, these multiobjective problems are treated by forming a convex combination of the $L_1$ norm bound proposed by Nagpal et al. (1994) and the $H_2$ closed-loop system norm. The proposed approach is reminiscent of scalarization techniques for Pareto optimization (Haddad and Bernstein 1990). For flexibility in controller synthesis we adopt the approach of fixed-structure controller synthesis (Bernstein et al. 1989) which allows consideration of arbitrary controller structures, including order, internal structure, and decentralization. The mixed-norm $H_2/L_1$ synthesis framework presented in the paper also enables the development of
controllers that are robust with respect to one-block unstructured uncertainty with bounded induced $L_{\infty}$ norms while minimizing nominal $H_2$ performance.

2. Preliminaries

In this section we establish definitions and notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote real and complex numbers, let $\mathcal{N}$ denote the set of nonnegative integers, let $()^T$ and $(\cdot)^*$ denote transpose and complex conjugate transpose, respectively, and let $I_n$ or $I$ denote the $n \times n$ identity matrix. Furthermore, we write $\| \cdot \|$ for the Euclidean vector norm, $\| \cdot \|_F$ for the Frobenius matrix norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, ‘tr’ for the trace operator, and $M \geq 0$ (resp., $M > 0$) to denote the fact that the Hermitian matrix $M$ is non-negative (resp., positive) definite. For a continuous-time (resp., discrete-time) linear time-invariant system with input $u$ and output $y$, $G(s)$ [resp., $G(z)$] and $G(t)$ [resp., $G(k)$] denote real-rational transfer function and impulse response matrix function, respectively.

Let $L_{\infty}$ (resp., $\ell_{\infty}$) denote the space of bounded Lebesgue measurable functions on $[0, \infty)$ (resp., bounded sequences on $\mathcal{N}$). For continuous-time systems, the $H_2$ norm of an asymptotically stable transfer function is defined as

$$
\| G \|_2 = \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \| G(j\omega) \|_F^2 \, d\omega \right]^{1/2} = \left[ \int_{0}^{\infty} \| G(t) \|_F^2 \, dt \right]^{1/2}
$$

while for discrete-time systems the $H_2$ norm is defined as

$$
\| G \|_2 = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \| G(e^{j\omega}) \|_F^2 \, d\omega \right]^{1/2} = \left[ \sum_{k=0}^{\infty} \| G(k) \|_F^2 \right]^{1/2}
$$

For a measurable function $z: [0, \infty) \to \mathbb{R}^r$ (resp., sequence $z: \mathcal{N} \to \mathbb{R}^r$) define the $L_{\infty}$ (resp., $\ell_{\infty}$) function norm with a Euclidean spatial norm by

$$
\| z(\cdot) \|_{\infty,2} = \text{ess sup}_{t \geq 0} \| z(t) \|_2 \quad (\text{resp.,} \quad \| z(\cdot) \|_{\ell_{\infty,2}} = \sup_{k \in \mathcal{N}} \| z(k) \|_2).
$$

For $u(\cdot)$, $y(\cdot) \in L_{\infty}$ on $[0, \infty)$ with Euclidean spatial norms the $L_1$ norm of the convolution operator $G: L_{\infty} \to L_{\infty}$ of a continuous-time linear time-invariant system with input $u$ and output $y$ is the equi-induced signal norm

$$
\| G \|_1 = \sup_{u(\cdot) \in L_{\infty}} \frac{\| y(\cdot) \|_{L_{\infty,2}}}{\| u(\cdot) \|_{L_{\infty,2}}}.
$$

Similarly, for $u(\cdot)$, $y(\cdot) \in \ell_{\infty}$ on $\mathcal{N}$ with Euclidean spatial norms the $\ell_1$ norm of the convolution operator $G: \ell_{\infty} \to \ell_{\infty}$ of a discrete-time linear time-invariant system with input $u$ and output $y$ is the equi-induced signal norm

$$
\| G \|_1 = \sup_{u(\cdot) \in \ell_{\infty}} \frac{\| y(\cdot) \|_{\ell_{\infty,2}}}{\| u(\cdot) \|_{\ell_{\infty,2}}}.
$$

From an input–output point of view the $L_1$ (resp., $\ell_1$) norm captures the worst-case peak amplification from input disturbance signals to output signals, where the signal size is taken to be the supremum over time of the signal’s point-wise-in-time Euclidean norm. Note that the input–output signal norms for inducing the $L_1$ (resp., $\ell_1$) norm considered in this paper are different from the input–output signal norms considered by Dahleh and Pearson (1987a, b) where $u(\cdot)$, $y(\cdot) \in L_{\infty}$ on $[0, \infty)$ (resp., $\ell_{\infty}$ on $\mathcal{N}$) with spectral spatial norms are used to capture the maximum peak-to-peak system gain.
3. Combined $H_2/L_1$ fixed-order dynamic compensation: continuous-time case

In this section we introduce the mixed $H_2/L_1$ fixed-order dynamic compensation problem. Without the $L_1$ performance criterion the problem considered here corresponds to the standard fixed-order $H_2$ control problem.

**Combined $H_2/L_1$ fixed-order dynamic output feedback control problem**

Consider the $n^{th}$-order stabilizable and detectable system

\[
\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t) + D_{1\infty}w_\infty(t), \quad t \in [0, \infty) \tag{5}
\]

\[
y(t) = Cx(t) + D_2w(t) + D_{2\infty}w_\infty(t)
\]

with $H_2$ and $L_1$ performance variables, respectively,

\[
z_2(t) = E_1x(t) + E_2u(t)
\]

\[
z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t)
\]

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^d$, $w_\infty \in \mathbb{R}^{d_\infty}$, $y \in \mathbb{R}^p$, $z_2 \in \mathbb{R}^p$, $z_\infty \in \mathbb{R}^{p_\infty}$, and $w(\cdot)$ denotes a unit-intensity white noise signal and $w_\infty(\cdot)$ denotes an $L_\infty$ signal such that $\|w_\infty(\cdot)\|_{\infty,2} \leq 1$. We seek an $n_c^{th}$-order ($1 \leq n_c \leq n$) dynamic output feedback controller

\[
\dot{x}_c(t) = A_c x_c(t) + B_c y(t)
\]

\[
u(t) = C_c x_c(t)
\]

such that the following design criteria are satisfied:

(i) the closed-loop system (5), (6), (9), and (10) is asymptotically stable; and

(ii) for $\mu \in [0, 1]$ the cost functional

\[
J(A_c, B_c, C_c) = \mu \| \tilde{G} \|_2^2 + (1 - \mu) \| \tilde{H} \|_1^2
\]

is minimized, where $\tilde{G}$ corresponds to the closed-loop impulse response matrix function from disturbances $w(\cdot)$ to $H_2$ performance variables $z_2(\cdot)$ and $\| \tilde{H} \|_1$ is the $L_1$ convolution operator norm from $L_\infty$ disturbances $w_\infty(\cdot)$ to $L_\infty$ performance variables $z_\infty(\cdot)$ of the closed-loop system defined by

\[
\| \tilde{H} \|_1 = \sup_{w_\infty(\cdot) \in L_\infty} \frac{\| z_\infty(\cdot) \|_{\infty,2}}{\| w_\infty(\cdot) \|_{\infty,2}}
\]

Note that if criterion (i) is satisfied then $\| \tilde{H} \|_1$ is bounded. As shown in figure 1, the problem statement involves both $H_2$ and $L_1$ performance variables $z_2$ and $z_\infty$, respectively, with disturbance inputs $w$ and $w_\infty$. For generality $z_2$ is not necessarily equal to $z_\infty$.

**Remark 1:** The cost functional (11) involves a convex combination of two scalar costs. By varying $\mu \in [0, 1]$ (11) can be viewed as a scalar representation of a multiobjective cost (see, for example, Haddad and Bernstein (1990) and the references therein). By setting $\mu = 0$ we obtain an $L_1$ optimal fixed-order dynamic compensation problem. Alternatively, setting $\mu = 1$ recovers the standard $H_2$ optimal fixed-order dynamic compensation problem. The practical value of this formulation is
the case $\mu \in (0,1)$ in which the optimization problem yields a trade-off between $H_2$ and $L_1$ performance.

4. Combined $H_2/L_1$ fixed-order dynamic compensation: decentralized static output feedback formulation

In this section we use the fixed-structure control framework of Bernstein et al. (1989) and Erwin et al. (1996) to transform the combined mixed-norm $H_2/L_1$ fixed-order strictly proper, centralized dynamic compensation problem to a decentralized static output feedback setting. Specifically, note that for every dynamic controller (9), (10) the closed-loop system (5), (6), (9), and (10) can be written as

$$
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_c(t)
\end{bmatrix} =
\begin{bmatrix}
A & B C_c \\
B_c C & A_c
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_c(t)
\end{bmatrix} +
\begin{bmatrix}
D_1 \\
B_c D_2
\end{bmatrix} w(t) +
\begin{bmatrix}
D_{1\infty} \\
B_c D_{2\infty}
\end{bmatrix} w_\infty(t)
$$

(13)

Furthermore, by treating $A_c$, $B_c$, and $C_c$ as decentralized static output feedback gains we obtain

$$
\hat{x}(t) = \mathcal{A}\tilde{x}(t) + \sum_{i=1}^{3} B_i \hat{u}_i(t) + \mathcal{D}_1 w(t) + \mathcal{D}_{1\infty} w_\infty(t), \quad t \in [0, \infty)
$$

(14)

$$
\hat{y}_i(t) = c_i \tilde{x}(t) + \mathcal{D}_2 w(t) + \mathcal{D}_{2\infty} w_\infty(t), \quad i = 1, 2, 3
$$

(15)

$$
\hat{z}_2(t) = \mathcal{E}_1 \tilde{x}(t) + \sum_{i=1}^{3} \mathcal{E}_2 \hat{u}_i(t)
$$

(16)

$$
\hat{z}_{2\infty}(t) = \mathcal{E}_{1\infty} \tilde{x}(t) + \sum_{i=1}^{3} \mathcal{E}_{2\infty} \hat{u}_i(t)
$$

(17)

$$
\hat{u}_1(t) = A_c \hat{y}_1(t), \quad \hat{u}_2(t) = B_c \hat{y}_2(t), \quad \hat{u}_3(t) = C_c \hat{y}_3(t)
$$

(18)

where

![Figure 1. Mixed-norm $H_2/L_1$ control problem.](image)
\[ \hat{x}(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad A = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} B \\ 0 \end{bmatrix} \]

\[ c_1 = \begin{bmatrix} 0 & I \end{bmatrix}, \quad c_2 = \begin{bmatrix} C & 0 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 & I \end{bmatrix} \]

\[ D_1 = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \quad D_{2_1} = 0, \quad D_{2_2} = D_2, \quad D_{2_3} = 0 \]

\[ D_{1_\infty} = \begin{bmatrix} D_{1_\infty} \\ 0 \end{bmatrix}, \quad D_{2_0} = 0, \quad D_{2_0_1} = D_{2_\infty}, \quad D_{2_0_2} = 0 \]

\[ \varepsilon_1 = \begin{bmatrix} E_1 & 0 \end{bmatrix}, \quad \varepsilon_{2_1} = 0, \quad \varepsilon_{2_2} = 0, \quad \varepsilon_{2_3} = E_2 \]

\[ \varepsilon_{1_\infty} = \begin{bmatrix} E_{1_\infty} & 0 \end{bmatrix}, \quad \varepsilon_{2_0} = 0, \quad \varepsilon_{2_0_1} = 0, \quad \varepsilon_{2_0_2} = 0, \quad \varepsilon_{2_0_3} = E_{2_\infty} \]

Next, defining

\[ \hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \\ \hat{u}_3(t) \end{bmatrix}, \quad \hat{y}(t) = \begin{bmatrix} \hat{y}_1(t) \\ \hat{y}_2(t) \\ \hat{y}_3(t) \end{bmatrix} \]

(14)–(17) can be rewritten as

\[ \hat{x}(t) = A \hat{x}(t) + B \hat{u}(t) + D_1 w(t) + D_{1_\infty} w_{1_\infty}(t), \quad t \in [0, \infty) \]  

(19)

\[ \hat{y}(t) = c \hat{x}(t) + D_2 w(t) + D_{2_0} w_{2_0}(t) \]  

(20)

\[ z_2(t) = \varepsilon_1 \hat{x}(t) + \varepsilon_2 \hat{u}(t) \]  

(21)

\[ z_{2_\infty}(t) = \varepsilon_{1_\infty} \hat{x}(t) + \varepsilon_{2_0} \hat{u}(t) \]  

(22)

where

\[ \mathcal{B} = \begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}, \quad \mathcal{E}_2 = \begin{bmatrix} \varepsilon_{2_1} & \varepsilon_{2_2} & \varepsilon_{2_3} \end{bmatrix}, \quad \mathcal{E}_{2_\infty} = \begin{bmatrix} \varepsilon_{2_0} & \varepsilon_{2_0_1} & \varepsilon_{2_0_2} & \varepsilon_{2_0_3} \end{bmatrix} \]

\[ C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} D_{2_1} \\ D_{2_2} \\ D_{2_3} \end{bmatrix}, \quad D_{2_0} = \begin{bmatrix} D_{2_0_1} \\ D_{2_0_2} \\ D_{2_0_3} \end{bmatrix} \]

Furthermore, by rewriting the decentralized controls (18) in the compact form

\[ \hat{u}(t) = \mathcal{K} \hat{y}(t) \]  

(23)

where

\[ \mathcal{K} = \begin{bmatrix} A_c & 0 & 0 \\ 0 & B_c & 0 \\ 0 & 0 & C_c \end{bmatrix} \]

the closed-loop system is given by
\[ \hat{x}(t) = \tilde{A} \hat{x}(t) + \tilde{D} w(t) + \tilde{D}_\infty w_\infty(t), \quad t \in [0, \infty) \]  
(24)

\[ z_2(t) = \tilde{E} \hat{x}(t) \]  
(25)

\[ z_\infty(t) = \tilde{E}_\infty \hat{x}(t) \]  
(26)

where

\[ \tilde{A} = \mathcal{A} + B \mathcal{K} C, \quad \tilde{D} = D_1 + B \mathcal{K} D_2, \quad \tilde{D}_\infty = D_1 \infty + B \mathcal{K} D_2 \infty \]
\[ \tilde{E} = \mathcal{E}_1 + \mathcal{E}_2 \mathcal{K} C, \quad \tilde{E}_\infty = \mathcal{E}_1 \infty + \mathcal{E}_2 \mathcal{K} C \]

Note that the closed-loop transfer function from disturbances \( w \) to \( H_2 \) performance variables is characterized by the triple \((\tilde{A}, \tilde{D}, \tilde{E})\) while the closed-loop transfer function from disturbances \( w_\infty \) to \( L_1 \) performance variables is characterized by the triple \((\tilde{A}, \tilde{D}_\infty, \tilde{E}_\infty)\).

It is useful to note that if \( \tilde{A} \) is asymptotically stable for a given feedback gain \( \mathcal{K} \in \mathbb{R}^{(2n + m) \times (2n + l)} \), then \( \| \tilde{G} \|_2^2 \) is given by

\[ \| \tilde{G} \|_2^2 = \int_0^\infty \| \tilde{E} e^{\tilde{A} t} \tilde{D} \|_F^2 \, dt = \text{tr} \tilde{Q} \tilde{R} \]  
(27)

where \( \tilde{R} = \tilde{E}^T \tilde{E} \) and \( \tilde{Q} \) is the unique, \( \tilde{n} \times \tilde{n} \) nonnegative-definite solution to the algebraic Lyapunov equation

\[ 0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V}, \]  
(28)

where \( \tilde{n} = n + n_c \) and \( \tilde{V} = \tilde{D} \tilde{D}^T \).

Since minimizing the \( L_1 \) norm directly results in irrational (infinite dimensional) controllers (Blanchini and Sznajer 1994, Dahleh and Diaz-Bobillo 1995, Dahleh and Pearson 1987a) we minimize an upper bound on the \( L_1 \) norm to avoid this complexity. Next, we present a key lemma that provides an upper bound on the \( L_1 \) performance in terms of a solution to a modified Lyapunov equation. It is important to stress that the \( L_1 \) performance bound provided is not new to this paper. Specifically, using set theoretic arguments involving closed convex sets and support functions, a similar bound was given by Schwepppe (1973) for systems with unknown but bounded disturbances. Alternatively, here we provide for the first time a constructive algebraic proof of the \( L_1 \) norm bound using Riccati and Lyapunov equations. For the statement of this result the following proposition in needed.

**Proposition 1:** Let \( \alpha > 0 \) and \( \mathcal{K} \in \mathbb{R}^{(2n + m) \times (2n + l)} \) be given and assume there exists a positive-definite matrix \( \mathcal{P} \in \mathbb{R}^{\tilde{n} \times \tilde{n}} \) satisfying

\[ 0 \geq \tilde{A}^T \mathcal{P} + \mathcal{P} \tilde{A} + \alpha \mathcal{P} + \alpha^{-1} \mathcal{P} \tilde{D}_\infty \tilde{D}_\infty^T \mathcal{P} \]  
(29)

Then the \( L_1 \) norm of the convolution operator \( \tilde{H} \) of the closed-loop system from disturbances \( w_\infty \) to performance variables \( z_\infty \) satisfies the bound

\[ \| \tilde{H} \|_1^2 \leq \sigma_{\max}(\tilde{E}_\infty \mathcal{P}^{-1} \tilde{E}_\infty^T) \]  
(30)

**Proof:** First, note that the closed-loop system (24), (26) (with \( \tilde{D} = 0 \)) from unit-peak disturbances \( w_\infty \) to \( L_1 \) performance variables \( z_\infty \) with \( \hat{x}(0) = 0 \) has the state-space representation
\[
\dot{x}(t) = \tilde{A}x(t) + \tilde{D}\omega(t), \quad \tilde{x}(0) = 0, \quad t \in [0, \infty) \tag{31}
\]

\[
z_\infty(t) = \tilde{E}_\infty \tilde{x}(t) \tag{32}
\]

where \(w_\infty(t) \in \mathcal{W}_\infty \triangleq \{w_\infty(t) : \|w_\infty(t)\|^2_2 \leq 1, t \geq 0\}\). Next, let \(T \geq 0\) and consider the shifted system

\[
\dot{\hat{x}}(t) = \tilde{A}_\alpha \hat{x}(t) + \tilde{D}_\omega \omega_\infty(t), \quad \hat{x}(0) = 0, \quad t \in [0, \infty) \tag{33}
\]

\[
z_\infty(t) = e^{-\alpha(t-T)/2} \tilde{E}_\infty \hat{x}(t) \tag{34}
\]

where \(\tilde{A}_\alpha \triangleq \tilde{A} + \alpha I/2\), \(\hat{x}(t) \triangleq e^{\alpha(t-T)/2} \tilde{x}(t)\), and \(\omega_\infty(t) \triangleq e^{\alpha(t-T)/2} \omega(t)\). Note that (31), (32) is equivalent to (33), (34). Furthermore, note that \(w_\infty(t) \in \mathcal{W}_\infty\) implies

\[
v_\infty(t) \in \mathcal{V}_\infty \triangleq \left\{v_\infty(t) : \int_0^T v_\infty(t) \omega_\infty(t) \, dt \leq \alpha^{-1}\right\}
\]

Next, consider the positive definite function \(V(\tilde{x}) = \tilde{x}^T P \tilde{x}\) with total derivative \(\dot{V}(\tilde{x})\) along the state trajectories \(\tilde{x}(t), t \geq 0\), of (33) given by

\[
\dot{V}(\tilde{x}) = \tilde{x}^T (\tilde{A}^T P + P \tilde{A}) \tilde{x} + \tilde{x}^T P \tilde{D}_\omega \omega_\infty + v_\infty \tilde{D}_\omega^T P \tilde{x}
\]

\[
\leq - \alpha^{-1} \tilde{x}^T P \tilde{D}_\omega \tilde{D}_\omega^T P \tilde{x} + \tilde{x}^T P \tilde{D}_\omega \omega_\infty + v_\infty \tilde{D}_\omega^T P \tilde{x}
\]

\[
= - (\alpha^{-1/2} \tilde{D}_\omega^T P \hat{x} - \alpha^{1/2} \omega_\infty) (\alpha^{-1/2} \tilde{D}_\omega^T P \hat{x} - \alpha^{1/2} \omega_\infty) + \alpha \omega_\infty^T \omega_\infty
\]

\[
\leq \alpha \omega_\infty^T \omega_\infty \tag{35}
\]

Now, it follows from (35) that

\[
V[\hat{x}(T)] - V[\hat{x}(0)] \leq \alpha \int_0^T \omega_\infty(t) \omega_\infty(t) \, dt \leq 1 \tag{36}
\]

and hence, since \(\hat{x}(T) = \tilde{x}(T)\) and \(V[\hat{x}(0)] = 0, \tilde{x}^T(T) P \tilde{x}(T) \leq 1, t \geq 0\). Thus, for all \(T \geq 0\),

\[
\|z_\infty(T)\|_2^2 = \|\tilde{E}_\infty \tilde{x}(T)\|^2_2
\]

\[
\leq \sup_{z^T P z \leq 1} \|\tilde{E}_\infty z\|^2_2
\]

\[
= \sup_{z^T \tilde{Z}^T \tilde{Z} \leq 1} \|\tilde{E}_\infty P^{-1/2} \tilde{Z}\|^2_2
\]

\[
= \sigma_{\max}(P^{-1/2} \tilde{E}_\infty^T P^{-1/2})
\]

\[
= \sigma_{\max}(\tilde{E}_\infty^T \tilde{E}_\infty^T)
\]

\[
\leq \sigma_{\max}(P^{-1/2} \tilde{E}_\infty^T P^{-1/2})
\]

\[
\leq \sup_{T \geq 0} \|z_\infty(T)\|_2^2, \tag{37}
\]

The result is now immediate by noting that \(\|H\|_2^2 \leq \sup_{T \geq 0} \|z_\infty(T)\|_2^2\).

**Remark 2:** Note that we can replace the Riccati inequality (29) by the Riccati equation

\[
0 = \tilde{A}^T P + P \tilde{A} + \alpha P + \alpha^{-1} P \tilde{D}_\omega \tilde{D}_\omega^T P \tag{38}
\]
in Proposition 1. In this case if \((\tilde{A}, \tilde{D} )\) is controllable and \(\tilde{A}_\alpha\) is asymptotically stable then there exists a positive-definite solution to (38). In particular,

\[
P = \left[ \int_0^\infty e^{\tilde{A}_\alpha t} \tilde{D} \tilde{D}^T e^{\tilde{A}_\alpha^T t} \, dt \right]^{-1}
\]  

(39)

is positive definite and satisfies (38).

**Lemma 1:** Let \(\alpha > 0\) and \(\kappa \in \mathbb{R}^{(2n_1+m)\times(2n_1+l)}\) be given and assume there exists a positive-definite matrix \(\tilde{Q} \in \mathbb{R}^{n \times n}\) satisfying

\[
0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \alpha \tilde{Q} + \frac{1}{\alpha} \tilde{V}
\]  

(40)

where \(\tilde{V} \triangleq \tilde{D} \tilde{D}^T \). Then \(\tilde{A}\) is Hurwitz. Furthermore, the \(L_1\) norm of the convolution operator \(\tilde{H}\) of the closed-loop system from disturbances \(w_\infty\) to performance variables \(z_\infty\) satisfies the bound

\[
\| \tilde{H} \|_1^2 \leq \sigma_{\max}(\tilde{E}_\infty \hat{\tilde{Q}} \hat{\tilde{E}}_\infty^T)
\]  

(41)

**Proof:** Asymptotic stability of \(\tilde{A}\) follows from standard Lyapunov theory. The \(L_1\) norm bound (41) is a direct consequence of Proposition 1 by letting \(\hat{\tilde{Q}} = P^{-1}\) so that \(P^{-1}(29)\) yields

\[
0 \geq \tilde{A} \hat{\tilde{Q}} + \hat{\tilde{Q}} \tilde{A}^T + \alpha \hat{\tilde{Q}} + \frac{1}{\alpha} \tilde{V}
\]  

(42)

Now, the minimal solution to (42) is given by (40) providing the tightest \(L_1\) norm bound.

**Remark 3:** Note that (40) has a positive-definite solution if and only if \(\tilde{A} + \alpha \tilde{R} / 2\) is Hurwitz which implies \(\alpha < -2\alpha_R(\tilde{A})\), where \(\alpha_R(\tilde{A})\) denotes the spectral abscissa of \(\tilde{A}\).

**Remark 4:** Note that in order to provide the tightest upper bound for the \(L_1\) norm of the closed-loop system we can replace (41) by

\[
\| \tilde{H} \|_1^2 \leq \inf_{0 < \alpha < \frac{-2\alpha_R(\tilde{A})}{\alpha_R(\tilde{A})}} \sigma_{\max}(\tilde{E}_\infty \hat{\tilde{Q}} \hat{\tilde{E}}_\infty^T)
\]  

(43)

where \(\hat{\tilde{Q}}\) satisfies (40).

**Remark 5:** As noted earlier even though the constructive algebraic proof of Lemma 1 is new to this paper, the bound (41) appears in Schewpepe (1973). Furthermore, referring to Schewpepe’s work, within the context of \(L_\infty\) induced signals, the \(L_1\) norm bound (41) also appears without proof in Nagpal et al. (1994) and Sznaier et al. (1995).

Lemma 1 shows that the \(L_1\) norm constraint is enforced when a positive-definite solution to (40) is known to exist and \(\tilde{A}\) is Hurwitz. Furthermore, \(H_2\) performance can be captured by tr \(\tilde{O} \tilde{R}\) where \(\tilde{O}\) is the non-negative definite solution to (28). Since \(\sigma_{\max}(\cdot)\) is not differentiable, in order to consider a well posed mixed-norm \(H_2 / L_1\) optimization problem we have the following result.
Lemma 2: Let $\hat{Q} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ be a non-negative-definite matrix. Then $\lim_{q \to \infty} \left[ \text{tr} \left( \hat{E}_\infty \hat{Q} \hat{E}_\infty^T \right)^q \right]^{1/q} = \sigma_{\max}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$.

Proof: Let $\sigma_i(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$, $i = 1, \ldots, r$, denote the singular values of $\hat{E}_\infty \hat{Q} \hat{E}_\infty^T$, where $r = \text{rank}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$. Since $\hat{E}_\infty \hat{Q} \hat{E}_\infty^T \geq 0$ it follows that $\text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)^q = \sum_{i=1}^r \sigma_i^q(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$. The result is now immediate by noting that

$$\lim_{q \to \infty} \left[ \sum_{i=1}^r \sigma_i^q(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T) \right]^{1/q} = \max_{i=1, \ldots, r} \sigma_i(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T) = \sigma_{\max}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T).$$

Now, it follows from Lemma 2 that we can replace (41) by

$$\|H\|_1^2 \leq \sigma_{\max}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T) \leq \left[ \text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)^q \right]^{1/q}$$

where $q \geq 1$ is some ‘large’ positive number. (Note that for the special case of multi-input/single-output closed-loop systems rank $\hat{E}_\infty^T \hat{E}_\infty = 1$ and hence $\sigma_{\max}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T) = \text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$.) Hence, the combined $H_2/L_1$ fixed-order dynamic output feedback control problem can be recast as the following auxiliary optimization problem.

Auxiliary optimization problem: Let $q \geq 1$ be given. For $\mu \in [0, 1]$ determine $\kappa \in \mathbb{R}^{(2n_m + m) \times (2n_m + l)}$ that minimizes

$$J(\kappa) = \mu \text{tr} \hat{Q} \hat{R} + (1 - \mu) \left[ \text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)^q \right]^{1/q}$$

where $\hat{Q} \geq 0$ and $\hat{Q} > 0$ satisfy (28) and (40), respectively.

Remark 6: In the case where $\tilde{D} = \tilde{D}_\infty$, the solution to (28) satisfies the bound

$$\tilde{Q} \leq \hat{Q}$$

Furthermore, if $\varepsilon_1 = \varepsilon_{1\infty}$ and $\varepsilon_2 = \varepsilon_{2\infty}$ then

$$\|\tilde{G}\|_2^2 \leq \text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)$$

5. Optimality conditions for mixed-norm $H_2/L_1$ fixed-order dynamic compensation: continuous-time case

In this section we state optimality conditions for characterizing dynamic output feedback controllers guaranteeing closed-loop stability and mixed $H_2/L_1$ performance.

Theorem 1: Let $\alpha > 0$, $q \geq 1$, and let $\kappa \in \mathbb{R}^{(2n_m + m) \times (2n_m + l)}$ be such that $\tilde{A}$ is asymptotically stable and $J(\kappa)$ is minimized. Then there exist $\tilde{n} \times \tilde{n}$ nonnegative-definite matrices $\tilde{Q}$ and $\tilde{P}$ and $\tilde{n} \times \tilde{n}$ positive-definite matrices $\hat{Q}$ and $\hat{P}$ satisfying.

$$0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V}$$

$$0 = \tilde{A}^T \hat{P} + \hat{P} \tilde{A} + \mu \hat{R}$$

$$0 = \tilde{A} \hat{Q} + \hat{Q} \tilde{A}^T + \alpha \hat{P} + \frac{1}{\alpha} \tilde{V}_{\infty}$$

$$0 = \tilde{A}^T \hat{P} + \hat{P} \tilde{A} + \alpha \hat{P} + (1 - \mu) \left[ \text{tr}(\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)^q \right]^{1-q} \hat{E}_\infty^T (\hat{E}_\infty \hat{Q} \hat{E}_\infty^T)^{q-1} \hat{E}_\infty$$

such that $(A_c, B_c, C_c)$ satisfy
\[ 0 = b_1^T (\tilde{P}\tilde{Q} + \hat{P}) c_1^T \]  
\[ 0 = b_2^T (\tilde{P}\tilde{Q} + \hat{P}) c_2^T + \left( b_2^T \tilde{P}\tilde{D}d_2 + \frac{1}{\alpha} b_2^T \tilde{P}\tilde{D}_{\infty}d_{2\infty} \right), \]  
\[ 0 = b_3^T (\tilde{P}\tilde{Q} + \hat{P}) c_3^T + \mu \epsilon_{23}^T \tilde{E}\tilde{Q} c_3^T + (1 - \mu) \left[ \text{tr}(\tilde{E}_{\infty}^{\infty}\hat{E}_{\infty}^{\infty})^q \right]^{1-q}/q \epsilon_{2\infty}^T (\tilde{E}_{\infty}^{\infty}\hat{E}_{\infty}^{\infty})^{q-1} \tilde{E}_{\infty}^{\infty}Q c_3^T. \]

Furthermore,
\[ \| \tilde{G} \|^2 = \text{tr} \tilde{Q} \tilde{R}, \]  
\[ \| \tilde{H} \|^2 \leq \sigma_{\text{max}}(\tilde{E}_{\infty}^{\infty}\hat{E}_{\infty}^{\infty}) \]

Proof: The result follows from standard Lagrange multiplier arguments. Specifically, since \( \kappa \) minimizes \( J (\cdot) \) given by (45) subject to (28) and (40), \( \partial L / \partial \tilde{Q} = 0, \partial L / \partial A_c = 0, \partial L / \partial B_c = 0, \) and \( \partial L / \partial C_c = 0 \) where \( L \) is the Lagrangian given by
\[ L (\kappa, \tilde{P}, \hat{P}) \triangleq J (\kappa) + \text{tr} \tilde{P} \left[ \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} \right] + \text{tr} \hat{P} \left[ \hat{A}\hat{Q} + \hat{Q}\hat{A}^T + \alpha \hat{Q} + \frac{1}{\alpha} \tilde{V}_{\infty} \right], \]
and \( \tilde{P}, \hat{P} \in \mathbb{R}^{n \times \bar{n}} \) are Lagrange parameters. Now, \( \partial L / \partial \tilde{Q} = 0, \partial L / \partial \hat{Q} = 0, \partial L / \partial A_c = 0, \partial L / \partial B_c = 0, \) and \( \partial L / \partial C_c = 0 \) yield (49), (51), (52), (53), and (54), respectively. Equations (48) and (50) are a restatement of (28) and (40), respectively.

Equations (48)–(54) provide optimality conditions that yield dynamic controllers for fixed-order (i.e., full- and reduced-order) mixed-norm \( H_2/L_1 \) output feedback compensation. In the design equations (50)–(54) one can view \( \alpha \) as a free parameter and optimize the combined \( H_2/L_1 \) performance criterion (45) with respect to \( \alpha \). In particular, setting \( \partial J / \partial \alpha = 0 \) yields
\[ \alpha = \left[ \frac{\text{tr} \hat{P}\tilde{P}_{\infty}}{\text{tr} \hat{P}\hat{P}} \right]^{1/2}. \]

It is important to note that since (44), or, equivalently, (56) provides a sufficient condition for capturing the \( L_1 \) norm, the conservatism of the bound (56) is difficult to predict and will depend upon the actual value of \( \hat{Q} \) determined by solving (50). Since, as shown in Proposition 1, the \( L_1 \) norm bound (44) approximates the closure of the set of reachable closed-loop system states due to unit-peak system disturbances by an ellipsoidal region, there might exist cases wherein the approximated reachable set is a poor approximation of the actual reachable set (Venkatesh and Dahleh 1995). Nevertheless, however, unlike the optimal \( L_1 \) approach presented in Dahleh and Pearson (1987 a, b) resulting in irrational controllers, the present framework allows for the design of reduced-order dynamic compensators with mixed \( H_2/L_1 \) performance specifications.

As mentioned in the introduction, the mixed-norm \( H_2/L_1 \) synthesis framework presented in this section can also be used to analyse controllers with respect to one-block unstructured uncertainty with bounded induced \( L_\infty \) norms while minimizing nominal \( H_2 \) performance. Specifically, let \( \kappa \in \mathbb{R}^{(2n+ m) \times (2n+ i)} \) be given such that \( \tilde{A} \) is
asymptotically stable and consider the uncertain closed-loop system shown in Figure 2 where

\[
\tilde{H}(s) = \begin{bmatrix}
\tilde{A} \\
\tilde{E}_\infty \\
0
\end{bmatrix}
\]

represents the nominal closed-loop system and \( w_\infty = \Delta z_\infty \), where \( \Delta \in \Delta \triangleq \{ \Delta : L_\infty \to L_\infty : \Delta(\cdot) \) is a linear time-invariant stable convolution operator such that \( \| \Delta \|_1 \leq \gamma^{-1} \}_1 \), where \( \gamma > 0 \). Now it follows from the small gain theorem (Desoer and Vidyasagar 1975) that if \( K \) is such that \( \| \tilde{H} \|_1 \leq 1/\gamma \max(\tilde{E}_\infty \tilde{Q} \tilde{E}_\infty^T) < \gamma \) then the feedback interconnection in Figure 2 is asymptotically stable for all \( \Delta \in \Delta \) with nominal \( H_2 \) cost given by \( \text{tr} \tilde{Q} \tilde{R} \).

6. Combined \( H_2/\ell_1 \) fixed-order dynamic compensation: discrete-time case

In this section we introduce the mixed \( H_2/\ell_1 \) discrete-time fixed-order dynamic compensation problem.

6.1. Combined \( H_2/\ell_1 \) fixed-order dynamic output feedback control problem

Consider the \( n \)-th-order stabilizable and detectable system

\[
\begin{align*}
\dot{x}(k+1) &= Ax(k) + Bu(k) + D_1w(k) + D_1w_\infty(k), \quad k \in \mathbb{N} \\
y(k) &= Cx(k) + D_2w(k) + D_2w_\infty(k)
\end{align*}
\tag{58}
\tag{59}
\]

with \( H_2 \) and \( \ell_1 \) performance variables, respectively,

\[
\begin{align*}
z_2(k) &= E_1x(k) + E_2u(k), \\
z_\infty(k) &= E_1w(k) + E_2w_\infty(k)
\end{align*}
\tag{60}
\tag{61}
\]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( w \in \mathbb{R}^d \), \( w_\infty \in \mathbb{R}^{d_\infty} \), \( y \in \mathbb{R}^l \), \( z_2 \in \mathbb{R}^p \), \( z_\infty \in \mathbb{R}^{p_\infty} \), and \( w(\cdot) \) denotes a unit-covariance discrete-time white noise signal and \( w_\infty(\cdot) \) denotes an \( \ell_\infty \) signal such that \( \|w_\infty(\cdot)\|_{\ell_\infty,2} \leq 1 \). We seek an \( n_c \)-th-order \((1 \leq n_c \leq n) \) dynamic output feedback controller

\[
\begin{align*}
x_c(k+1) &= A_cx_c(k) + B_cy(k) \\
u(k) &= C_cx_c(k)
\end{align*}
\tag{62}
\tag{63}
\]

such that the following design criteria are satisfied:

(i) the closed-loop system (58), (59), (62), and (63) is asymptotically stable; and
(ii) for \( \mu \in [0,1] \) the cost functional

\[
J(A_c,B_c,C_c) = \mu\|\tilde{G}\|^2 + (1 - \mu)\|\tilde{H}\|^2
\tag{64}
\]
is minimized, where \( \widetilde{G} \) corresponds to the closed-loop discrete-time impulse response matrix function from disturbances \( w(\cdot) \) to \( H_2 \) performance variables \( z_2(\cdot) \) and \( \|\widetilde{H}\|_1 \) is the \( \ell_1 \) convolution operator norm from \( \ell_\infty \) disturbances \( w_\infty(\cdot) \) to \( \ell_\infty \) performance variables \( z_\infty(\cdot) \) of the closed-loop system defined by

\[
\|\widetilde{H}\|_1 \triangleq \sup_{w_\infty(\cdot) \in \ell_\infty} \frac{\|z_\infty\|_{\ell_\infty}}{\|w_\infty\|_{\ell_\infty}}
\]  

(65)

### 7. Combined \( H_2/\ell_1 \) fixed-order dynamic compensation: decentralized static output feedback formulation

As in the continuous-time case, we transform the combined \( H_2/\ell_1 \) fixed-order strictly proper, centralized dynamic compensation problem to a decentralized static output feedback setting. Specifically, the closed-loop system (58), (59), (62), and (63) can be written as

\[
\begin{bmatrix} x(k + 1) \\ x_c(k + 1) \end{bmatrix} = \begin{bmatrix} A & BC_c \\ BcC & A_c \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} D_1 \\ BcD_2 \end{bmatrix} w(k) + \begin{bmatrix} D_1\infty \\ BcD_2\infty \end{bmatrix} w_\infty(k)
\]  

(66)

Furthermore, by treating \( A_c, B_c, \) and \( C_c \) as decentralized static output feedback gains we obtain

\[
\bar{x}(k + 1) = A\bar{x}(k) + B\hat{u}(k) + D_1w(k) + D_2w_\infty(k), \quad k \in \mathcal{N}
\]  

(67)

\[
\hat{y}(k) = C\bar{x}(k) + D_2w(k) + D_2w_\infty(k)
\]  

(68)

\[
z_2(k) = \varepsilon_1\bar{x}(k) + \varepsilon_2\hat{u}(k)
\]  

(69)

\[
z_\infty(k) = \varepsilon_1\infty\bar{x}(k) + \varepsilon_2\infty\hat{u}(k)
\]  

(70)

where

\[
\hat{u}(k) = \kappa\hat{y}(k).
\]  

(71)

Now, the closed-loop system is given by

\[
\bar{x}(k + 1) = \bar{A}\bar{x}(k) + \bar{D}w(k) + \bar{D}_\infty w_\infty(k), \quad k \in \mathcal{N}
\]  

(72)

\[
z_2(k) = \bar{E}\bar{x}(k)
\]  

(73)

\[
z_\infty(k) = \bar{E}_\infty\bar{x}(k)
\]  

(74)

If \( \bar{A} \) is asymptotically stable for a given feedback gain \( \kappa \in \mathbb{R}^{(2n_r+m)\times(2n_r+i)} \), then \( \|\bar{G}\|^2_2 \) is given by

\[
\|\bar{G}\|^2_2 = \sum_{k=0}^{\infty} \|\bar{E}\bar{A}^k\bar{D}\|^2_1 = \text{tr} \bar{Q} \bar{R}
\]  

(75)

where \( \bar{Q} \) is the unique, \( \bar{n} \times \bar{n} \) nonnegative-definite solution to the algebraic Lyapunov equation

\[
\bar{Q} = \bar{A}\bar{Q}\bar{A}^T + \bar{V}
\]  

(76)

Next, we present a key proposition and lemma that provide an upper bound on the \( \ell_1 \) performance in terms of a solution to a modified Riccati and Lyapunov equation, respectively.
Proposition 2: Let $\alpha > 1$ and $\kappa \in \mathbb{R}^{(2n+m) \times (2n+1)}$ be given and assume there exists a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying
\[ P \geq \alpha A^T P A + \alpha^2 A^T P \tilde{D}_\infty \left[ (\alpha - 1)I_{d_0} - \alpha \tilde{D}_\infty^T P \tilde{D}_\infty \right]^{-1} \tilde{D}_\infty^T P \tilde{A} \]
(77)

Then the $\ell_1$ norm of the convolution operator $\tilde{H}$ of the closed-loop system from disturbances $w_\infty$ to performance variables $z_\infty$ satisfies the bound
\[ \|\tilde{H}\|_1 \leq \sigma_{\text{max}} \left( E_\infty P^{-1} E_\infty^T \right) \]
(78)

Proof: First, note that the closed-loop system (72), (74) (with $\tilde{D} = 0$) from unit-peak disturbances $w_\infty$ to $\ell_1$ performance variables $z_\infty$ with $\tilde{x}(0) = 0$ has the state-space representation
\[ \tilde{x}(k+1) = \tilde{A} \tilde{x}(k) + \tilde{D}_\infty w_\infty(k), \quad \tilde{x}(0) = 0, \quad k \in \mathcal{N} \]
(79)
\[ z_\infty(k) = \tilde{E}_\infty \tilde{x}(k) \]
(80)
where $w_\infty(\cdot) \in \mathcal{W}_\infty \triangleq \{w_\infty(\cdot) : \|w_\infty(k)\|_2^2 \leq 1, \ k \in \mathcal{N}\}$. Next, let $N \geq 0$ and consider the dilated system
\[ \hat{x}(k+1) = \sqrt{\alpha} \hat{A} \hat{x}(k) + \sqrt{\alpha} \tilde{D}_\infty v_\infty(k), \quad \hat{x}(0) = 0, \quad k \in \mathcal{N} \]
(81)
\[ z_\infty(k) = \sqrt{\alpha} \hat{E}_\infty \hat{x}(k) \]
(82)
where $\hat{x}(k) \triangleq \alpha^{(k-1)/2} \tilde{x}(k)$ and $v_\infty(k) \triangleq \alpha^{(k-1)/2} w_\infty(k)$. Note that (79), (80) is equivalent to (81), (82). Furthermore, note that $w_\infty(\cdot) \in \mathcal{W}_\infty$ implies $v_\infty(\cdot) \in \mathcal{V}_\infty \triangleq \{v_\infty(\cdot) : \sum_{k=0}^{N-1} v_\infty(k) v_\infty(k) \leq (\alpha-1)^{-1}\}$. Next, consider the positive definite function
\[ V(\hat{x}) = \hat{x}^T \hat{P} \hat{x} \]
along the state trajectories $\hat{x}(k), k \in \mathcal{N}$, of (81) given by
\[ V[\hat{x}(k+1)] - V[\hat{x}(k)] = \alpha \left[ \hat{A} \hat{x}(k) + \tilde{D}_\infty v_\infty(k) \right] \hat{P} \left[ \hat{A} \hat{x}(k) + \tilde{D}_\infty v_\infty(k) \right] - \hat{x}^T(k) \hat{P} \hat{x}(k) \leq 2 \alpha \hat{x}^T(k) \hat{A}^T \hat{P} \tilde{D}_\infty v_\infty(k) + \alpha \hat{x}^T(k) \tilde{D}_\infty^T \hat{P} \tilde{D}_\infty v_\infty(k) \]
\[ - \alpha \hat{x}^T(k) \hat{A}^T \hat{P} \tilde{D}_\infty M^1 \tilde{D}_\infty^T \hat{P} \tilde{A} \hat{x}(k) \]
\[ = - \left[ \alpha \tilde{D}_\infty \hat{P} \hat{A} \hat{x}(k) - M v_\infty(k) \right] M^1 \left[ \alpha \tilde{D}_\infty \hat{P} \hat{A} \hat{x}(k) - M v_\infty(k) \right] \]
\[ + (\alpha-1) v_\infty(k) v_\infty(k) \]
\[ \leq (\alpha-1) v_\infty(k) v_\infty(k), \]
(83)
where $M \triangleq (\alpha - 1)I_{d_0} - \alpha \tilde{D}_\infty^T \hat{P} \tilde{D}_\infty$. Now, it follows from (83) that
\[ V[\hat{R}(N)] - V[\hat{R}(0)] \leq (\alpha - 1) \sum_{k=0}^{N-1} v_\infty(k) v_\infty(k) \leq 1 \]
(84)
and hence, since $\hat{x}(N) = \tilde{x}(N)$ and $V[\hat{x}(0)] = 0$, $\tilde{x}(N) \hat{P} \tilde{x}(N) \leq 1$, $N \geq 0$. Thus, for all $N \geq 0$,
\[ \| z_\infty(N) \|_2^2 = \| \tilde{E}_\infty \mathcal{X}(N) \|_2^2 \]

\[ \leq \sup_{z^* \in \mathcal{P}, z \leq 1} \| \tilde{E}_\infty z \|_2^2 \]

\[ = \sup_{z^* \in \mathcal{P}, z \leq 1} \| \tilde{E}_\infty P^{-1/2} z \|_2^2 \]

\[ = \sigma_{\text{max}}(P^{-1/2} \tilde{E}_\infty E_\infty P^{-1/2}) \]

\[ = \sigma_{\text{max}}(\tilde{E}_\infty P^{-1} E_\infty^T) \] (85)

The result is now immediate by noting that \( \| H \|_2^2 \leq \sup_{N \geq 0} \| w_\infty(z) \| \| z_\infty(N) \|_2^2. \)

**Remark 7:** Note that the Riccati inequality (77) is equivalent to the linear matrix inequality

\[ \begin{bmatrix} P & 0 \\ 0 & (\alpha - 1)I_{d_\alpha} \end{bmatrix} \geq \alpha \begin{bmatrix} \tilde{A}^T & \tilde{D}_\infty \\ \tilde{D}_\infty^T & P \end{bmatrix} \] (86)

or, equivalently,

\[ \begin{bmatrix} P & 0 \\ 0 & (\alpha - 1)I_{d_\alpha} \end{bmatrix} \geq \alpha \begin{bmatrix} \tilde{A} & \tilde{D}_\infty \\ \tilde{D}_\infty^T & P \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - 1)I_{d_\alpha} \end{bmatrix} \] (87)

**Lemma 3:** Let \( \alpha > 1 \) and \( \kappa \in \mathbb{R}^{(2n_r+m) \times (2n_r+1)} \) be given and assume there exists a positive-definite matrix \( \hat{Q} \in \mathbb{R}^{\tilde{n} \times \tilde{n}} \) satisfying

\[ \hat{Q} = \alpha \tilde{A} \hat{Q} \tilde{A}^T + \frac{\alpha}{\alpha - 1} \tilde{V}_\infty \] (88)

Then \( \tilde{A} \) is discrete-time asymptotically stable. Furthermore, the \( \ell_1 \) norm of the convolution operator \( \tilde{H} \) of the closed-loop system from disturbances \( w_\infty \) to performance variables \( z_\infty \) satisfies the bound

\[ \| \tilde{H} \|_1^2 \leq \sigma_{\text{max}}(\tilde{E}_\infty \hat{Q} \tilde{E}_\infty^T) \] (89)

**Proof:** Asymptotic stability of \( \tilde{A} \) follows from standard discrete-time Lyapunov theory. The \( \ell_1 \) norm bound (89) is a direct consequence of Proposition 2 and Remark 7 by noting that, using Schur compliments, (87) is equivalent to

\[ \begin{bmatrix} P^{-1} & 0 \\ 0 & (\alpha - 1)I_{d_\alpha} \end{bmatrix} \geq \alpha \begin{bmatrix} \tilde{A} & \tilde{D}_\infty \\ \tilde{D}_\infty & P^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - 1)I_{d_\alpha} \end{bmatrix} \] (90)

and letting \( \hat{Q} = P^{-1} \) so that (90) yields

\[ \hat{Q} \geq \alpha \tilde{A} \hat{Q} \tilde{A}^T + \frac{\alpha}{\alpha - 1} \tilde{V}_\infty \] (91)

Now, the minimal solution to (91) is given by (88) providing the tightest \( \ell_1 \) norm bound.

**Remark 8:** Note that (88) has a positive-definite solution if and only if \( \sqrt{\alpha} \tilde{A} \) is discrete-time asymptotically stable which implies \( \alpha < 1/\rho^2(\tilde{A}) \), where \( \rho(\tilde{A}) \) denotes the spectral radius of \( \tilde{A} \).
Now, as in the continuous-time case, we introduce a combined discrete-time $H_2/\ell_1$ fixed-order dynamic output feedback control problem by considering (45) with $\tilde{Q} \geq 0$ and $\hat{Q} > 0$ satisfying (76) and (88), respectively, and proceed by determining the controller gains that minimize $J(\infty)$.

8. Optimality conditions for mixed-norm $H_2/\ell_1$ fixed-order dynamic compensation: discrete-time case

In this section we state optimality conditions for characterizing discrete-time dynamic output feedback controllers guaranteeing closed-loop stability and mixed $H_2/\ell_1$ performance.

Theorem 2: Let $\alpha > 1$, $q \geq 1$, and let $\infty \in \mathbb{R}^{(2n_c+m) \times (2n_c+l)}$ be such that $\tilde{A}$ is asymptotically stable and $J(\infty)$ is minimized. Then there exists $\tilde{n} \times \tilde{n}$ non-negative-definite matrices $\tilde{Q}$ and $\tilde{P}$ and $\tilde{n} \times \tilde{n}$ positive-definite matrices $\hat{Q}$ and $\hat{P}$ satisfying

$$\tilde{Q} = \tilde{A} \tilde{Q} \tilde{A}^T + \tilde{V}$$
$$\tilde{P} = \tilde{A}^T \tilde{P} \tilde{A} + \mu \tilde{R}$$
$$\hat{Q} = \alpha \hat{A} \hat{Q} \hat{A}^T + \frac{\alpha}{\alpha - 1} \tilde{V}_\infty$$
$$\hat{P} = \alpha \hat{A}^T \hat{P} \hat{A} + (1 - \mu) \left[ \text{tr} \left( \hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T \right) \right] \frac{1}{q} \frac{1}{\text{tr} \hat{P} \hat{V}_\infty} \left( \hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T \right)^{q-1} \hat{E}_\infty$$

such that $(A_c, B_c, C_c)$ satisfy

$$0 = b_1^T (\tilde{P} \tilde{A} \tilde{Q} + \alpha \hat{A} \hat{Q} \hat{A}^T) c_1^T$$
$$0 = b_2^T (\tilde{P} \tilde{A} \tilde{Q} + \alpha \hat{A} \hat{Q} \hat{A}^T) c_2^T + b_2^T \tilde{P} \tilde{D} \tilde{D}_2 \tilde{A}^T + \frac{\alpha}{\alpha - 1} b_2^T \hat{P} \hat{D} \hat{D}_2 \hat{A}^T$$
$$0 = b_3^T (\tilde{P} \tilde{A} \tilde{Q} + \alpha \hat{A} \hat{Q} \hat{A}^T) c_3^T + \mu \epsilon_2 \hat{E} \hat{Q} c_3^T$$
$$+ (1 - \mu) \left[ \text{tr} \left( \hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T \right) \right] \frac{1}{q} \frac{1}{\text{tr} \hat{P} \hat{V}_\infty} \left( \hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T \right)^{q-1} \hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T$$

Furthermore,

$$\| \tilde{Q} \|_2^2 = \text{tr} \tilde{Q} \tilde{R}$$
$$\| \hat{P} \|_1^2 \leq \sigma_{\text{max}} (\hat{E}_\infty \hat{A} \hat{Q} \hat{A}^T \hat{E}_\infty^T)$$

Proof: The proof is analogous to the proof of Theorem 1. □

Equations (92)–(98) provide optimality conditions that yield dynamic controllers for discrete-time fixed-order (i.e. full- and reduced-order) mixed-norm $H_2/\ell_1$ output feedback compensation. As in the continuous-time case, one can view $\alpha$ in the design equations (94)–(98) as a free parameter and optimize the combined $H_2/\ell_1$ performance criterion $J(\infty)$ with respect to $\alpha$. In particular, setting $\partial J / \partial \alpha = 0$ yields

$$\alpha = 1 + \left[ \frac{\text{tr} \hat{P} \hat{V}_\infty}{\text{tr} \hat{P} \tilde{A} \tilde{Q} \tilde{A}^T} \right]^{1/2}$$
9. Quasi-Newton/continuation algorithm

To solve the combined $H_2/L_1$ and $H_2/L_\infty$ fixed-order dynamic output feedback control problem, a general-purpose BFGS quasi-Newton algorithm (Dennis and Schnabel 1983) is developed. In particular, the algorithm is a continuation algorithm with correction steps performed using quasi-Newton corrections with the BFGS inverse Hessian update. The line-search portions of the algorithm were modified to include a constraint-checking subroutine which decreases the length of the search direction vector until it lies entirely within the set of parameters that yield a stable closed-loop system. This modification ensures that the cost function $J$ remains defined at every point in the line search process. Numerical experience indicates that this subroutine is usually invoked only during the first few iterations of a synthesis problem.

One requirement of gradient-based optimization is an initial stabilizing design. For full-order controller design the algorithm was initialized with an LQG controller while for reduced-order control the algorithm was initialized with a balanced truncated LQG controller. For feasible values of $\mu$ the quasi-Newton optimization algorithm was used to find $A_c$, $B_c$, $C_c$, and $\alpha$ satisfying the necessary conditions. After each iteration $\mu$ was decreased and the current values of the controller gains ($A_c$, $B_c$, $C_c$) were then used as the starting point for the next iteration. For details of a similar algorithm see Erwin et al. (1996).

10. Illustrative numerical examples

In this section we provide several numerical examples to demonstrate the proposed mixed-norm $H_2/L_1$ fixed-order dynamic compensation framework.

**Example 1:** Consider the dynamic system (5), (6) with performance variables (7), (8) where (Sznaier et al. 1995)

\[
A = \frac{1}{3} \begin{bmatrix} 2 & -8 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[
D_1 = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \quad D_{1\infty} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{2\infty} = \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

\[
E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad E_{1\infty} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_{2\infty} = 0
\]

Several full-order ($n_c = 2$) controllers were designed with $q = 1$ to examine the tradeoff between $H_1$ and $L_1$ performance objectives. Figure 3 shows $H_2$ and $L_1$ norm variations with respect to $\mu$ which clearly shows an inverse proportionality trend between the two norms. Table 1 shows the values of $H_2$ norm, $L_1$ norm bound, and the actual $L_1$ norm. Figure 4 compares the performance variable $z_{\infty}(t)$ of the mixed $H_2/L_1$ full-order dynamic output feedback controller ($\mu = 0.001$) to an $H_2$ optimal linear-quadratic Gaussian controller with an $L_\infty$ disturbance signal $w_{\infty}(t) = \sin 2.7t$. Note that the mixed-norm $H_2/L_1$ controller reduces the maximum peak of the response by 42.1% over the $H_2$ optimal controller.

**Example 2:** Consider the spring-mass-damper system
Figure 3. Trade-off between $H_2$ and $L_1$ Performance: Example 1.

Figure 4. Comparison of $H_2$ and mixed-norm $H_2/L_1$ controllers: Example 1.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$H_2$ norm</th>
<th>$L_1$ norm bound</th>
<th>Actual $L_1$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>190.5185</td>
<td>30.7007</td>
<td>28.5335</td>
</tr>
<tr>
<td>1</td>
<td>8.9518</td>
<td>48.6016</td>
<td>40.7496</td>
</tr>
</tbody>
</table>

Table 1. Summary of design study: Example 1.
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\xi\omega_n
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u(t) + \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
v(t) + w_\infty(t)
\end{bmatrix}
\]

\[
y(t) = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + \begin{bmatrix}
1 & 0
\end{bmatrix} v(t)
\]

where \(x_1\) and \(x_2\) are the position and velocity of the mass, respectively, and \(\omega_n = 1 \text{ rad/s}\) and \(\xi = 0.4\). Furthermore, let the \(H_2\) and \(L_1\) performance variables be given by

\[
\begin{bmatrix}
z_2(t) = z_\infty(t)
\end{bmatrix}
\]

Several full-order \((n_c = 2)\) controllers were designed to examine the trade-off between \(H_2\) and \(L_1\) performance objectives. Figure 5 shows \(H_2\) and \(L_1\) norm variations with respect to \(\mu\) which clearly shows an inverse proportionality trend between the two norms. For \(\mu = 0.001\) figure 6 shows the reduction in the \(L_1\) norm bound with an increase in the index \(q\). Figure 7 compares the performance variable \(z_\infty(t)\) of the mixed \(H_2/L_1\) full-order dynamic output feedback controller \((\mu = 0.001, q = 5)\) to an \(H_2\) optimal linear-quadratic Gaussian controller with an \(L_\infty\) disturbance signal \(w_\infty(t) = \sin 0.1t\). Note that the mixed-norm \(H_2/L_1\) controller reduces the maximum peak of the response by 76.7% over the \(H_2\) optimal controller.

**Example 3:** Consider the pitch axis longitudinal dynamics model of the F-16 fighter aircraft system given in Semitendorf (1991) with state equations

![Figure 5. Trade-off between \(H_2\) and \(L_1\) performance (\(q = 1\)): Example 2.](image_url)
Figure 6. Variation of $L_1$ norm bound with respect to index $q$ ($\mu = 0.001$): Example 2.

Figure 7. Comparison of $H_2$ and mixed-norm $H_2/L_1$ controllers: Example 2.

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1.00 & 0 \\
0 & -0.87 & 43.22 \\
0 & 0.999 & -1.34
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix} -
\begin{bmatrix}
0 \\
17.25 \\
0.17
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix}
\]

\[
y(t) = x_1(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
w(t) + w_\infty(t)
\end{bmatrix}
\]
where $x_1$ is the pitch angle, $x_2$ is the pitch rate, $x_3$ is the angle of attack, $u_1$ is the elevator deflection, and $u_2$ is the flap deflection. Here, we choose the performance variables given by

$$ z_2(t) = z_{\infty}(t) = \begin{bmatrix} x_1(t) & x_2(t) & u_1(t) & u_2(t) \end{bmatrix} $$

corresponding to the pitch angle, the pitch rate, and the control signals. Our goal is to constrain the maximum response of the pitch angle due to an $L_{\infty}$ sinusoidal disturbance. Figure 8 compares the pitch angle response of the mixed-norm $H_2/L_1$ full-order ($n_c = 3$) dynamic output feedback controller ($\mu = 0.001$) to an $H_2$ optimal linear-quadratic Gaussian controller with an $L_{\infty}$ disturbance signal $w_{\infty}(t) = \sin 0.1t$. Note that the mixed-norm $H_2/L_1$ controller reduces the maximum excursion of the pitch angle by 84.16%. Table 2 shows the values of the $H_2$ norm and $L_1$ norm bounds.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$q$</th>
<th>$H_2$ norm</th>
<th>$L_1$ norm bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>7</td>
<td>48.5427</td>
<td>31.5417</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>37.4885</td>
<td>44.0381</td>
</tr>
</tbody>
</table>

Table 2. Summary of design study: Example 3.

Next, we use the mixed-norm $H_2/L_1$ controller for tracking a step command. Specifically, the pitch angle is commanded to track a unit step by feeding back the error between the pitch angle and the input command to the compensator. Figure 9 compares the pitch angle response to a unit step command input of the the mixed-norm $H_2/L_1$ full-order ($n_c = 3$) dynamic output feedback controller ($\mu = 0.001$) to an $H_2$ optimal linear-quadratic Gaussian controller. As expected, the time domain
performance characteristics (i.e. peak overshoot and settling time) of the mixed-norm $H_2/L_1$ controller are superior to that of the $H_2$ controller. In particular, the mixed-norm $H_2/L_1$ controller provides a 52.78% decrease in overshoot over the $H_2$ optimal controller.

Example 4: Finally, we consider the system formulated in Cannon and Rosenthal (1984) involving four coupled rotating discs with a torque input located at the third disk and angular displacement measurement of the first disc. The plant is of eighth order, has a rigid body mode, and possesses a complex pair of non-minimum phase zeros. This plant is reminiscent of a single-axis spacecraft involving unstable dynamics with flexible appendages and a non-colocated sensor-actuator pair. The problem data are as follows:

$$
A = \begin{bmatrix}
-0.1610 & -6.0040 & -0.5822 & -9.9835 & -0.4073 & -3.9820 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0.0064 \\
0.0024 \\
0.0713 \\
1.0002 \\
0.1045 \\
0.9955
\end{bmatrix},
$$

$$
C = \begin{bmatrix} 1 & 0_{1 \times 7} \end{bmatrix}, \quad D_1 = \begin{bmatrix} B & 0_{8 \times 1} \end{bmatrix},
$$

$$
D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},
$$

$$
E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 5.5 & 110 & 13.2 & 180 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 10^{-2} \end{bmatrix}
$$
Our goal is to constrain the maximum response of the rigid body position due to a $L_\infty$ sinusoidal disturbance. Several reduced-order ($n_c = 4$) controllers were designed with $q = 1$ to examine the trade-off between $H_2$ and $L_1$ performance objectives. Figure 10 shows $H_2$ and $L_1$ norm variations with respect to $\mu$. Figure 11 compares the output variable $y(t) = x_1(t)$ of the mixed $H_2/L_1$ reduced-order dynamic output feedback controller ($\mu = 0.1$) to a reduced-order $H_2$ optimal controller ($\mu = 1$) with an $L_\infty$ disturbance signal $w_\infty(t) = \sin(0.95t)$. Note that the mixed-norm $H_2/L_1$...
controller reduces the maximum peak of the output response by 24.1% over the $H_2$ optimal controller. It is interesting to note that for this example, the mixed-norm $H_2/ L_1$ reduced-order controller designed with $q = 1$ satisfies $\sigma_{\max}(\tilde{E}_\infty Q \tilde{E}_\infty^T) = \text{tr} \tilde{E}_\infty Q \tilde{E}_\infty^T$. Thus, increasing $q$ does not yield different controllers.

11. Conclusion

A Riccati equation approach for mixed-norm $H_2/L_1$ and $H_2/\ell_1$ dynamic compensation reminiscent to the mixed-norm $H_2/H_\infty$ framework of Bernstein and Haddad (1989) was developed. This multiobjective problem is captured by forming a convex combination of both the $H_2$ and $L_1$ performance measures. Using a fixed-structure controller synthesis approach fixed-order mixed-norm $H_2/L_1$ controllers were designed. Several design examples were presented to demonstrate the effectiveness of the proposed mixed-norm $H_2/L_1$ Riccati equation approach.

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References


