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Nonlinear–nonquadratic optimal and inverse optimal control for stochastic dynamical systems

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SUMMARY

In this paper, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for nonlinear stochastic dynamical systems. Specifically, we provide a simplified and tutorial framework for stochastic optimal control and focus on connections between stochastic Lyapunov theory and stochastic Hamilton–Jacobi–Bellman theory. In particular, we show that asymptotic stability in probability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton–Jacobi–Bellman equation and, hence, guaranteeing both stochastic stability and optimality. In addition, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the stochastic stabilization problem. These results are then used to provide extensions of the nonlinear feedback controllers obtained in the literature that minimize general polynomial and multilinear performance criteria. Copyright © 2017 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Under certain conditions, nonlinear controllers offer significant advantages over linear controllers. In particular, if the plant dynamics and/or system measurements are nonlinear [1, 2], the plant/measurement disturbances are either nonadditive or non-Gaussian, the performance measure considered is nonquadratic [3–7], the plant model is uncertain [8–10], or the control signals/state amplitudes are constrained [11, 12], then nonlinear controllers yield better performance than the best linear controllers. In [13], the current status of *deterministic* continuous-time, nonlinear–nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [13] are based on the fact that the steady-state solution of the Hamilton–Jacobi–Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [13, 14].

Building on the results of [13, 14], in this paper, we present a framework for analyzing and designing feedback controllers for nonlinear *stochastic* dynamical systems. Specifically, we consider a feedback stochastic optimal control problem over an infinite horizon involving a nonlinear–nonquadratic performance measure. The performance measure can be evaluated in closed form as long as the nonlinear–nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability in probability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state stochastic

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Hamilton–Jacobi–Bellman equation. The overall framework provides the foundation for extending linear–quadratic control for stochastic dynamical systems to nonlinear–nonquadratic problems with polynomial and multilinear cost functionals.

Our approach focuses on the role of the Lyapunov function guaranteeing stochastic stability of the closed-loop system and its seamless connection to the steady-state solution of the stochastic Hamilton–Jacobi–Bellman equation characterizing the optimal nonlinear feedback controller. In order to avoid the complexity in solving the stochastic steady state, Hamilton–Jacobi–Bellman equation, we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that minimizes a *derived* cost functional that provides the flexibility in specifying the control law. This corresponds to addressing an *inverse optimal stochastic control problem* [15–21].

The inverse optimal control design approach provides a framework for constructing the Lyapunov function for the closed-loop system that serves as an optimal value function and, as shown in [19, 20], achieves desired stability margins. Specifically, nonlinear inverse optimal controllers that minimize a *meaningful* (in the terminology of [19, 20]) nonlinear–nonquadratic performance criterion involving a nonlinear–nonquadratic, nonnegative-definite function of the state and a quadratic positive-definite function of the feedback control are shown to possess sector margin guarantees to component decoupled input nonlinearities in the conic sector $(\frac{1}{2}, \infty)$.

The contents of this paper are as follows. In Section 2, we establish notation and definitions and recall some basic results on stability of nonlinear stochastic dynamical systems. In Section 3, we consider a nonlinear stochastic system with a performance measure evaluated over the infinite horizon. The performance measure is then evaluated in terms of a Lyapunov function that guarantees local and global asymptotic stability in probability. This result is then specialized to general polynomial and multilinear cost functionals. In Section 4, we state a nonlinear–nonquadratic stochastic optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing local and global asymptotic stability in probability of the closed-loop system. In Section 5, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the stochastic stabilization problem. This result is then used to derive extensions of the results in [4, 5] involving nonlinear feedback controllers minimizing polynomial and multilinear performance criteria. In Section 6, we provide two illustrative numerical examples that highlight the stochastic optimal stabilization framework. Finally, in Section 7, we present conclusions and highlight some future research directions.

2. NOTATION, DEFINITIONS, AND MATHEMATICAL PRELIMINARIES

In this section, we establish notation and definitions and review some basic results on stability of nonlinear stochastic dynamical systems [22–26]. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{N}^n$ denotes the set of $n \times n$ nonnegative-definite matrices, and $\mathbb{P}^n$ denotes the set of $n \times n$ positive-definite matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm or an induced matrix norm (depending on context), A^{T} for the transpose of the matrix A, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$, and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [24]. Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure [25, p. 10], with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by the Wiener process $w(t)$ up to time t. We denote a stochastic dynamical system by $\mathcal{G}$ generating a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ adapted to the stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leqslant \tau < t$, such that $\{\omega \in \Omega : x(t, \omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geqslant 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in

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the Borel σ-algebra $\mathfrak{B}^n$. Here, we use the notation $x(t)$ to represent the stochastic process $x(t,\omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega,\mathcal{F},\mathbb{P})$ over the semi-infinite parameter space $[0,\infty)$ by $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, and $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n\times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ comprised of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega,\mathcal{F},\mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely. Furthermore, $\mathbb{E}[\,\cdot\,]$ and $\mathbb{E}^{x_0}[\,\cdot\,]$ denote, respectively, the expectation with respect to the probability measure $\mathbb{P}$ and with respect to the classical Wiener measure $\mathbb{P}^{x_0}$.

Finally, we write tr$(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x)\triangleq\frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x)\triangleq\frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x\in\mathbb{R}^n$, that is, $\mathcal{H}_n\triangleq\{x:\Omega\to\mathbb{R}^n\}$. For an open set $\mathcal{D}\subseteq\mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}}\triangleq\{x\in\mathcal{H}_n : x:\Omega\to\mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0\in\mathbb{R}^n$, $\mathcal{H}_n^{x_0}\triangleq\{x\in\mathcal{H}_n : x\overset{\text{a.s.}}{=}x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V:\mathcal{D}\to\mathbb{R}$ that are two-times continuously differentiable with respect to $x\in\mathcal{D}\subseteq\mathbb{R}^n$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t)=f(x(t))\mathrm{d}t+D(x(t))\mathrm{d}w(t),\quad x(t_0)\overset{\text{a.s.}}{=}x_0,\qquad t\geqslant t_0, \tag{1}$$

where, for every $t\geqslant t_0$, $x(t)\in\mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0)\in\mathcal{H}_n^{x_0}$, $\mathcal{D}\subseteq\mathbb{R}^n$ is an open set with $0\in\mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant t_0},\mathbb{P})$, $x(t_0)$ is independent of $(w(t)-w(t_0)), t\geqslant t_0$, and $f:\mathcal{D}\to\mathbb{R}^n$ and $D:\mathcal{D}\to\mathbb{R}^{n\times d}$ are continuous functions and satisfy $f(0)=0$ and $D(0)=0$. The filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant t_0},\mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x:[t_0,\tau]\times\Omega\to\mathcal{D}$ is said to be a *solution* of (1) on the time interval $[t_0,\tau]$ with initial condition $x(t_0)\overset{\text{a.s.}}{=}x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant t_0}$, $f\in\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, $D\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, and

$$x(t)=x_0+\int_{t_0}^{t}f(x(s))\mathrm{d}s+\int_{t_0}^{t}D(x(s))\mathrm{d}w(s)\ \text{ a.s.},\quad t\in[t_0,\tau], \tag{2}$$

where the integrals in (2) are Itô integrals.

Note that for each fixed $t\geqslant t_0$, the random variable $\omega\mapsto x(t,\omega)$ assigns a vector $x(\omega)$ to every outcome $\omega\in\Omega$ of an experiment, and for each fixed $\omega\in\Omega$, the mapping $t\mapsto x(t,\omega)$ is the *sample path* of the stochastic process $x(t)$, $t\geqslant t_0$. A pathwise solution $t\mapsto x(t)$ of (1) in $(\Omega,\{\mathcal{F}_t\}_{t\geqslant t_0},\mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal pathwise solutions to (1) in $(\Omega,\{\mathcal{F}_t\}_{t\geqslant t_0},\mathbb{P}^{x_0})$ exist on $[t_0,\infty)$, and hence, we assume that (1) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* of (1) are given in [24, Corol. 6.3.5].

Furthermore, we assume that $f:\mathcal{D}\to\mathbb{R}^n$ and $D:\mathcal{D}\to\mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x)-f(y)\|+\|D(x)-D(y)\|_{\mathrm{F}}\leqslant L\|x-y\|,\quad x,y\in\mathcal{D}, \tag{3}$$

and the growth restriction condition

$$\|f(x)\|^2+\|D(x)\|_{\mathrm{F}}^2\leqslant L^2(1+\|x\|^2),\quad x\in\mathcal{D}, \tag{4}$$

for some Lipschitz constant $L>0$, and hence, because $x(t_0)\in\mathcal{H}_n^{\mathcal{D}}$ and $x(t_0)$ are independent of $(w(t)-w(t_0)), t\geqslant t_0$, it follows that there exists a unique solution $x\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ of (1) in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}} \setminus \{0\}$, there exists $\tau_x > 0$ such that if $x_1 : [t_0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [t_0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (1); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, with continuous sample paths almost surely, solve (1), then $\tau_x \leqslant \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_1(t) = x_2(t),\ t_0 \leqslant t \leqslant \tau_x\big) = 1$.

A weaker sufficient condition for the existence of a unique solution to (1) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (3) without the growth condition (4) is given in [27]. Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by ([26, Thm. 3.4], [24, Thm. 9.2.8])

$$\mathbb{P}(x(t) \in B | x(t_0) \overset{\text{a.s.}}{=} x_0) = \mathbb{P}(t - t_0, x_0, 0, B), \quad x_0 \in \mathbb{R}^n, \tag{5}$$

for all $t \geqslant t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(s, x, t, \mathcal{B}), t \geqslant s$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Finally, recall that every continuous process with Feller transition probability function is also a strong Markov process [26, p. 101].

Definition 2.1 ([25, Def. 7.7]*)*
Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then, the *infinitesimal generator* $\mathcal{L}$ *of* $x(t)$, $t \geqslant t_0$, *with* $x(t_0) \overset{\text{a.s.}}{=} x_0$, *is defined by*

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D}. \tag{6}$$

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geqslant t_0$, satisfies (1), then the limit in (6) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geqslant t_0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (1) and is given by ([25, Thm. 7.9])

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \operatorname{tr} D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D}. \tag{7}$$

The following definition introduces the notions of Lyapunov and asymptotic stability in probability. Recall that an *equilibrium point* $x_{\mathrm{e}} = 0$ of (1) is a point such that $f(0) = 0$ and $D(0) = 0$. In this case, $x_{\mathrm{e}} = 0$ is an equilibrium point of (1) if and only if the zero solution (i.e., the zero stochastic process) $x(\cdot) \overset{\text{a.s.}}{=} 0$ is a solution of (1).

Definition 2.2 ([22]*)* (i) The zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is *Lyapunov stable in probability* if, for every $r > 0$ and $\varepsilon \in (0, 1)$, there exist $\delta = \delta(\varepsilon, r) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geqslant t_0} \|x(t)\| > r\right) \leqslant \varepsilon. \tag{8}$$

(ii) The zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is *locally asymptotically stable in probability* if it is Lyapunov stable in probability and, for every $\varepsilon \in (0, 1)$, there exist $\delta = \delta(\varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t)\| = 0\right) \geqslant 1 - \varepsilon. \tag{9}$$

(iii) The zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t)\| = 0\right) = 1. \tag{10}$$

Remark 2.1
A more general stochastic stability notion can also be introduced here involving stochastic stability and convergence to an invariant (stationary) distribution. In this case, state convergence is not to an equilibrium point but rather to a stationary distribution. This framework can relax the vanishing perturbation assumption $D(0) = 0$ and requires a more involved analysis and synthesis framework showing stability of the underlying Markov semigroup [28].

Finally, we provide sufficient conditions for local and global asymptotic stability in probability for the nonlinear stochastic dynamical system (1).

Theorem 2.1 ([26, Corol. 5.1]*)*
Consider the nonlinear stochastic dynamical system (1) and assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0, \tag{11}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{12}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0. \tag{13}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv}$ to (1) is locally asymptotically stable in probability. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is globally asymptotically stable in probability.

3. STABILITY ANALYSIS AND NONLINEAR–NONQUADRATIC COST EVALUATION OF NONLINEAR STOCHASTIC SYSTEMS

In this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we present sufficient conditions for stability and performance for a given nonlinear stochastic dynamical system with a nonlinear–nonquadratic performance measure. As in deterministic theory [13, 14], the cost functional can be explicitly evaluated as long as it is related to an underlying Lyapunov function. For the following result, let $L : \mathcal{D} \to \mathbb{R}$ with $L(0) = 0$ and let $\mathbb{1}_{[t_0,\tau_m]}(t)$ denote the indicator function defined on the set $[t_0, \tau_m]$, $m \in \mathbb{Z}_+$, that is,

$$\mathbb{1}_{[t_0,\tau_m]}(t) \triangleq \begin{cases} 1, & \text{if } t \in [t_0, \tau_m], \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathfrak{B}^{\mathrm{cost}}_{x_0}$ denote the set of all sample trajectories of (1) for which $\lim_{t\to\infty} \|x(t,\omega)\| = 0$ and $x(\{t \geq t_0\}, \omega) \in \mathfrak{B}^{\mathrm{cost}}_{x_0}$, $\omega \in \Omega$. Finally, define

$$\mathbb{1}_{\mathfrak{B}^{\mathrm{cost}}_{x_0}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq t_0\}, \omega) \in \mathfrak{B}^{\mathrm{cost}}_{x_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.1
Consider the nonlinear stochastic dynamical system given by (1) with nonlinear–nonquadratic performance measure

$$J(x_0, \mathfrak{B}^{\mathrm{cost}}_{x_0}) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}^{\mathrm{cost}}_{x_0}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t)) \mathbb{1}_{\mathfrak{B}^{\mathrm{cost}}_{x_0}}(\omega) \mathrm{d}t\right]. \tag{14}$$

Furthermore, assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0, \tag{15}$$

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$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{16}$$

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{17}$$

$$L(x) + V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) = 0, \qquad x \in \mathcal{D}. \tag{18}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is locally asymptotically stable in probability, and for every $\varepsilon \in (0,1)$, there exist $\delta = \delta(\varepsilon)$ and $\mathfrak{B}_{x_0}^{\text{cost}}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\text{cost}}\right) \geqslant 1-\varepsilon$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subset \mathcal{D}$,

$$J(x_0, \mathfrak{B}_{x_0}^{\text{cost}}) = V(x_0). \tag{19}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (1) is globally asymptotically stable in probability and (19) holds with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\text{cost}}\right) = 1$, $x_0 \in \mathbb{R}^n$.

Proof

Conditions (15)–(17) are a restatement of (11)–(13), and hence, it follows from Theorem 2.1 that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of (1) is locally asymptotically stable in probability. Consequently, for every $\varepsilon \in (0,1)$, there exist $\delta = \delta(\varepsilon)$ and a set of sample trajectories $x(\{t \geqslant t_0\}, \omega) \in \mathfrak{B}_{x_0}^{\text{cost}}$, $\omega \in \Omega$, such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\text{cost}}\right) \geqslant 1 - \varepsilon$.

Next, using (1) and Itô's (chain rule) formula, it follows that the stochastic differential of $V(x(t))$ along the system trajectories $x(t)$, $t \geqslant t_0$, of (1) is given by

$$\mathrm{d}V(x(t)) = \left(V'(x(t))f(x(t)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x(t))V''(x(t))D(x(t))\right)\mathrm{d}t + \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t). \tag{20}$$

Hence, using (18), it follows that

$$\begin{aligned} L(x(t))\mathrm{d}t + \mathrm{d}V(x(t)) &= \left(L(x(t)) + V'(x(t))f(x(t)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x(t))V''(x(t))D(x(t))\right)\mathrm{d}t \\ &\quad + \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t) \\ &= \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t). \end{aligned} \tag{21}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x(t)$, $t \geqslant t_0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, multiplying (21) with $\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)$ and integrating over $[t_0, \min\{t_n, \tau_m\}]$, where $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned} &\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\mathrm{d}t \\ &\quad = -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\mathrm{d}V(x(s)) + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(s))}{\partial x}D(x(s))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\mathrm{d}w(s) \\ &\quad = V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) \\ &\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x}D(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\mathrm{d}w(t) \\ &\quad = V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) \\ &\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x}D(x(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\mathrm{d}w(t). \end{aligned} \tag{22}$$

Taking the expectation on both sides of (22) yields

$$\begin{aligned}
&\mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\,\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\Bigg[V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega) \\
&\quad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\mathrm{d}w(t)\Bigg] \\
&= V(x_0)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\rm cost}\right) - \mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right].
\end{aligned} \tag{23}$$

Next, let $\mathfrak{B}_{x_0}^m$ denote the set of all the sample trajectories $x(t)$, $t \geqslant t_0$, of (1) such that $\tau_m = \infty$ and note that, by regularity of solutions [26, p. 75], $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^m) \to 1$ as $m \to \infty$. Now, noting that $L(x) \geqslant 0$, $x \in \mathcal{D}$, the sequence of random variables $\{f_{m,n}\}_{m,n=0}^{\infty} \subseteq \mathcal{H}_1$, where

$$f_{m,n} \triangleq \int_{t_0}^{\min\{t_n,\tau_m\})} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\mathrm{d}t,$$

is a pointwise nondecreasing sequence in n and m of nonnegative $\mathcal{F}_t$-measurable random variables on Ω. Next, defining the improper integral

$$\int_{t_0}^{\infty} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the Lebesgue monotone convergence theorem [29] that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\,\mathrm{d}t\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\,\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_{t_0}^{\tau_m} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\,\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\mathrm{d}t\right] \\
&= J(x_0,\mathfrak{B}_{x_0}^{\rm cost})\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\rm cost}\right).
\end{aligned} \tag{24}$$

Next, because the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of (1) is asymptotically stable in probability and $V(x(\min\{t_n,\tau_m\}))$ is a positive supermartingale by [26, Lemma 5.4], it follows from [26, Theorem 5.1] that

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right] &= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[V(x(\tau_m))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}V(x(\tau_m))\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[V\left(\lim_{m\to\infty}x(\tau_m)\right)\mathbb{1}_{\mathfrak{B}_{x_0}^{\rm cost}}(\omega)\right] \\
&= 0.
\end{aligned} \tag{25}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (23) and using (24) and (25) yields (19).

Finally, for $\mathcal{D} = \mathbb{R}^n$, global asymptotic stability in probability is direct consequence of the radially unbounded condition on $V(\cdot)$, and hence, (19) holds with $\mathbb{P}^{x_0}\left(\mathfrak{B}^{\rm cost}_{x_0}\right) = 1$ for all $x_0 \in \mathbb{R}^n$. $\square$

Remark 3.1
Note that for global asymptotic stability in probability, $\mathbb{P}^{x_0}\left(\mathfrak{B}^{\rm cost}_{x_0}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\mathbb{1}_{\mathfrak{B}^{\rm cost}_{x_0}}(\omega) \overset{\rm a.s.}{=} 1$. In this case,

$$J(x_0, \mathfrak{B}^{\rm cost}_{x_0}) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}^{\rm cost}_{x_0}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t))\mathbb{1}_{\mathfrak{B}^{\rm cost}_{x_0}}(\omega){\rm d}t\right] = \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t)){\rm d}t\right].$$

Thus, in the remainder of this section, we omit the dependence on $\mathfrak{B}^{\rm cost}_{x_0}$ in the cost functional for all the results concerning global asymptotic stability in probability.

It is important to note that if (18) holds, then (17) is equivalent to $L(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$. Next, we specialize Theorem 3.1 to linear stochastic systems. For this result, let $A \in \mathbb{R}^{n\times n}$, let $\sigma \in \mathbb{R}^d$, and let $R \in \mathbb{R}^{n\times n}$ be a positive-definite matrix.

Corollary 3.1
Consider the linear stochastic dynamical system with multiplicative noise given by

$${\rm d}x(t) = Ax(t){\rm d}t + x(t)\sigma^{\rm T}{\rm d}w(t), \quad x(0) = x_0 \ \ \text{a.s.}, \quad t \geqslant 0, \tag{26}$$

and with quadratic performance measure

$$J(x_0) \triangleq \mathbb{E}^{x_0}\left[\int_0^{\infty} x^{\rm T}(t)Rx(t){\rm d}t\right]. \tag{27}$$

Furthermore, assume that there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\rm T} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R. \tag{28}$$

Then, the zero solution $x(t) \overset{\rm a.s.}{\equiv} 0$ to (26) is globally asymptotically stable in probability and

$$J(x_0) = x_0^{\rm T} P x_0, \qquad x_0 \in \mathbb{R}^n. \tag{29}$$

Proof
The result is a direct consequence of Theorem 3.1 with $f(x) = Ax$, $D(x) = x\sigma^{\rm T}$, $L(x) = x^{\rm T}Rx$, $V(x) = x^{\rm T}Px$, and $\mathcal{D} = \mathbb{R}^n$. Specifically, conditions (15) and (16) are trivially satisfied. Now,

$$\begin{aligned} V'(x)f(x) + \frac{1}{2}{\rm tr}\, D^{\rm T}(x)V''(x)D(x) &= x^{\rm T}(A^{\rm T}P + PA)x + \frac{1}{2}{\rm tr}(x\sigma^{\rm T})^{\rm T}2P(x\sigma^{\rm T}) \\ &= x^{\rm T}\left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\rm T} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)\right]x, \end{aligned}$$

and hence, it follows from (28) that $L(x) + V'(x)f(x) + \frac{1}{2}{\rm tr}D^{\rm T}(x)V''(x)D(x) = 0$, $x \in \mathbb{R}^n$, so that all the conditions of Theorem 3.1 are satisfied. Finally, because $V(\cdot)$ is radially unbounded, the zero solution $x(t) \overset{\rm a.s.}{\equiv} 0$ to (26) is globally asymptotically stable in probability. $\square$

Next, we specialize Theorem 3.1 to linear and nonlinear stochastic systems with multilinear cost functionals. First, however, we give several definitions involving multilinear functions and a key lemma establishing the existence and uniqueness of specific multilinear forms. Define $x^{[q]} \triangleq x \otimes x \otimes \cdots \otimes x$ and $\overset{q}{\oplus} A \triangleq A \oplus A \oplus \cdots \oplus A$, where x and A appear q times and q is a positive integer. A scalar function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *q-multilinear* if q is a positive integer and $\psi(x)$ is a linear combination of terms of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, where i_j is a nonnegative integer for $j = 1, \ldots, n$ and $i_1 + i_2 + \cdots + i_n = q$. Furthermore, a q-multilinear function $\psi(\cdot)$ is *nonnegative definite* (resp., *positive definite*) if $\psi(x) \geqslant 0$ for all $x \in \mathbb{R}^n$ (resp., $\psi(x) > 0$ for all nonzero $x \in \mathbb{R}^n$). Note that if q is odd, then $\psi(x)$ cannot be positive definite. If $\psi(\cdot)$ is a q-multilinear function, then $\psi(\cdot)$ can be represented by means of Kronecker products, that is, $\psi(x)$ is given by $\psi(x) = \Psi x^{[q]}$, where $\Psi \in \mathbb{R}^{1 \times n^q}$. Note that every polynomial function can be written as a multilinear function; the converse, however, is not true.

The following lemma is needed for several of the main results of this paper.

Lemma 3.1
Let $A \in \mathbb{R}^{n \times n}$ and $\sigma \in \mathbb{R}^d$ be such that $A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n$ is Hurwitz, and let $h : \mathbb{R}^n \to \mathbb{R}$ be a q-multilinear function. Then, there exists a unique q-multilinear function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax + h(x), \qquad x \in \mathbb{R}^n. \tag{30}$$

Furthermore, if $h(x)$ is nonnegative (resp., positive) definite, then $g(x)$ is nonnegative (resp., positive) definite.

Proof
Let $h(x) = \Psi x^{[q]}$ and define $g(x) \triangleq \Gamma x^{[q]}$, where $\Gamma \triangleq -\Psi(\overset{q}{\oplus}(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n))^{-1}$, and note that $\overset{q}{\oplus}(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)$ is invertible because $A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n$ is Hurwitz by assumption. Now, note that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
& g'(x)Ax + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) \\
&= (\Gamma x^{[q]})' Ax + \frac{1}{2} x^{\mathrm{T}} (\Gamma x^{[q]})'' x \|\sigma\|^2 \\
&= \Gamma\left(\sum_{i_q=1}^{q} x \otimes \cdots \otimes \overbrace{I_n}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes x \right) Ax + \frac{1}{2}\|\sigma\|^2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{i_q=1}^{q} \sum_{j_q=1, j_q \neq i_q}^{q} x_i \Gamma(x \otimes \cdots \right. \\
&\quad \left. \cdots \otimes \overbrace{e_i}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes \overbrace{e_j}^{j_q^{\text{th}}\text{entry}} \otimes \cdots \otimes x) x_j \right) \\
&= \Gamma\left(\sum_{i_q=1}^{q} x \otimes \cdots \otimes \overbrace{Ax}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes x \right) + \frac{1}{2}\|\sigma\|^2 \left(\sum_{i_q=1}^{q} \sum_{j_q=1, j_q \neq i_q}^{q} \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma(x \otimes \cdots \right. \\
&\quad \left. \cdots \otimes \overbrace{x_i e_i}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes \overbrace{x_j e_j}^{j_q^{\text{th}}\text{entry}} \otimes \cdots \otimes x) \right) \\
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes I_n \right) x^{[q]} + \frac{1}{2}\|\sigma\|^2 \left(\sum_{i_q=1}^{q} \sum_{j_q=1, j_q \neq i_q}^{q} \Gamma(x \otimes \cdots \right. \\
&\quad \left. \cdots \otimes \overbrace{\left(\sum_{i=1}^{n} x_i e_i \right)}^{i_q^{\text{th}}\text{entry}} \otimes \cdots \otimes \overbrace{\left(\sum_{j=1}^{n} x_j e_j \right)}^{j_q^{\text{th}}\text{entry}} \otimes \cdots \otimes x) \right)
\end{aligned}$$

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$$
\begin{aligned}
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\rm th}\text{entry}} \otimes \cdots \otimes I_n\right) x^{[q]} + \frac{1}{2}\|\sigma\|^2 \Gamma\left(\sum_{i_q=1}^{q} \sum_{j_q=1, j_q\neq i_q}^{q} x \otimes \cdots \right.\\
&\qquad \left. \cdots \otimes \overbrace{x}^{i_q^{\rm th}\text{entry}} \otimes \cdots \otimes \overbrace{x}^{j_q^{\rm th}\text{entry}} \otimes \cdots \otimes x\right)\\
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\rm th}\text{entry}} \otimes \cdots \otimes I_n\right) x^{[q]}\\
&\qquad + \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{\frac{1}{2}(q-1)\|\sigma\|^2 I_n}^{i_q^{\rm th}\text{entry}} \otimes \cdots \otimes I_n\right) x^{[q]}\\
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)}^{i_q^{\rm th}\text{entry}} \otimes \cdots \otimes I_n\right) x^{[q]}\\
&= \Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right) x^{[q]}\\
&= -\Psi x^{[q]}\\
&= -h(x).
\end{aligned}
$$

To prove uniqueness, suppose, *ad absurdum*, that $\hat{g}(x) = \hat{\Gamma} x^{[q]}$ satisfies (30). Then, it follows that

$$
\Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right) x^{[q]} = \hat{\Gamma}\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right) x^{[q]}.
$$

Because $\overset{q}{\oplus}(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)$ is Hurwitz and $e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} = (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t})^{[q]}$, it follows that, for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\Gamma x^{[q]} &= \Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)^{-1} x^{[q]}\\
&= -\Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)\int_0^\infty e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} x^{[q]} \mathrm{d}t\\
&= -\Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)\int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t})^{[q]} x^{[q]} \mathrm{d}t\\
&= -\Gamma\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)\int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t\\
&= -\hat{\Gamma}\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)\int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t\\
&= \hat{\Gamma} x^{[q]},
\end{aligned}
$$

which shows that $g(x) = \hat{g}(x)$, $x \in \mathbb{R}^n$, leading to a contradiction.

Finally, if $h(x)$ is nonnegative definite, then it follows that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} g(x) &= -\Psi\left(\overset{q}{\oplus}\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)\right)^{-1} x^{[q]} \\ &= \Psi \int_0^\infty e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} x^{[q]} \mathrm{d}t \\ &= \Psi \int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t \\ &\geqslant 0. \end{aligned}$$

If, in addition, $x \neq 0$, then $e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x \neq 0$, $t \geqslant 0$. Hence, if $h(x)$ is positive definite, then $g(x)$, $x \in \mathbb{R}^n$, is positive definite. □

Next, assume that $\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)$ is Hurwitz, where $q \geqslant 2$ is a given integer, let P be given by (28), and consider the case in which $D(\cdot)$ $L(\cdot)$, $f(\cdot)$, and $V(\cdot)$ are given by $D(x) = x\sigma^{\mathrm{T}}$,

$$L(x) = x^{\mathrm{T}} R x + h(x), \quad f(x) = Ax + N(x), \quad V(x) = x^{\mathrm{T}} P x + g(x), \tag{31}$$

where $h : \mathcal{D} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$ are nonlinear and nonquadratic, and $N : \mathcal{D} \to \mathbb{R}^n$ is nonlinear. In this case, (18) holds if and only if

$$\begin{aligned} 0 = {} & x^{\mathrm{T}} R x + h(x) + x^{\mathrm{T}}(A^{\mathrm{T}} P + PA)x + 2x^{\mathrm{T}} P N(x) + g'(x)(Ax + N(x)) \\ & + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}\left[2P + g''(x)\right](x\sigma^{\mathrm{T}}),\ x \in \mathcal{D}, \end{aligned} \tag{32}$$

or, equivalently,

$$\begin{aligned} 0 = {} & x^{\mathrm{T}}\left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R\right]x \\ & + \frac{1}{2}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)(Ax + N(x)) + h(x) + 2x^{\mathrm{T}} P N(x),\ x \in \mathcal{D}. \end{aligned} \tag{33}$$

Because $\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)$ is Hurwitz, we can choose P to satisfy (28). Now, suppose $N(x) \equiv 0$ and let P satisfy (28). Then, (33) specializes to

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax + h(x), \quad x \in \mathcal{D}. \tag{34}$$

Next, given $h(\cdot)$, we determine the existence of a function $g(\cdot)$ satisfying (34). Here, we focus our attention on multilinear functionals for which (34) holds with $\mathcal{D} = \mathbb{R}^n$. Specifically, let $h(x)$ be a nonnegative-definite q-multilinear function, where q is necessarily even. Furthermore, let $g(x)$ be the nonnegative-definite q-multilinear function given by Lemma 3.1. Then, because $\frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax \leqslant 0$, $x \in \mathbb{R}^n$, it follows that $x^{\mathrm{T}} P x + g(x)$ is a Lyapunov function for (26). Hence, Lemma 3.1 can be used to generate Lyapunov functions of specific multilinear structures.

To demonstrate the aforementioned discussion, suppose $h(x)$ in (31) is of the more general form given by

$$h(x) = \sum_{\nu=2}^{r} h_{2\nu}(x), \tag{35}$$

where, for $\nu = 2, 3, \ldots, r$, $h_{2\nu} : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative-definite 2ν-multilinear function. Now, using Lemma 3.1, it follows that there exists a nonnegative-definite 2ν-multilinear function $g_{2\nu} : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''_{2\nu}(x)(x\sigma^{\mathrm{T}}) + g'_{2\nu}(x)Ax + h_{2\nu}(x), \quad x \in \mathbb{R}^n, \quad \nu = 2, 3, \ldots, r. \tag{36}$$

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Defining $g(x) \triangleq \sum_{\nu=2}^{r} g_{2\nu}(x)$ and summing (36) over ν yields (34). Because (18) is satisfied with $L(x)$ and $V(x)$ given by (31), respectively, (19) implies that

$$J(x_0) = x_0^{\mathrm{T}} P x_0 + g(x_0). \tag{37}$$

To illustrate condition (34) with quartic Lyapunov functions, let

$$V(x) = x^{\mathrm{T}} P x + (x^{\mathrm{T}} M x)^2, \tag{38}$$

where P satisfies (28) and assume M is an $n \times n$ symmetric matrix. In this case, $g(x) = (x^{\mathrm{T}} M x)^2$ is a nonnegative-definite four-multilinear function and (34) yields

$$\begin{aligned} h(x) &= -4(x^{\mathrm{T}} M x) x^{\mathrm{T}} M A x - \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} \left[8 M x x^{\mathrm{T}} M + 4(x^{\mathrm{T}} M x) M\right](x\sigma^{\mathrm{T}}) \\ &= -2(x^{\mathrm{T}} M x) x^{\mathrm{T}} \left[\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} M + M\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)\right] x. \end{aligned} \tag{39}$$

Now, letting M satisfy

$$0 = \left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} M + M\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right) + \hat{R}, \tag{40}$$

where $\hat{R}$ is an $n \times n$ symmetric matrix, it follows from (39) that $h(x)$ satisfying (34) is of the form

$$h(x) = 2(x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x). \tag{41}$$

If $\hat{R}$ is nonnegative definite, then M is nonnegative definite, and hence, $h(x)$ is a nonnegative-definite four-multilinear function. Thus, if $V(x)$ is a quartic Lyapunov function of the form given by (38) and $L(x)$ is given by

$$L(x) = x^{\mathrm{T}} R x + 2(x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x), \tag{42}$$

where M satisfies (40), then condition (34), and hence, (18) is satisfied.

The following proposition generalizes the aforementioned results to general polynomial cost functionals.

Proposition 3.1

Let $A \in \mathbb{R}^{n\times n}$ and $\sigma \in \mathbb{R}^d$ be such that $A + \frac{1}{2}(2r-1)\|\sigma\|^2 I_n$ is Hurwitz, and let $R \in \mathbb{R}^{n\times n}$, $R > 0$, and $\hat{R}_q \in \mathbb{R}^{n\times n}$, $\hat{R}_q \geqslant 0$, $q = 2, \ldots, r$. Consider the linear stochastic dynamical system (26) with performance measure

$$J(x_0) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty \left\{x^{\mathrm{T}}(t) R x(t) + \sum_{q=2}^{r}\left[(x^{\mathrm{T}}(t)\hat{R}_q x(t))(x^{\mathrm{T}}(t) M_q x(t))^{q-1}\right]\right\} \mathrm{d}t\right], \tag{43}$$

where $M_q \in \mathbb{R}^{n\times n}$, and $M_q \geqslant 0$, $q = 2, \ldots, r$, satisfy

$$0 = \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)^{\mathrm{T}} M_q + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right) + \hat{R}_q. \tag{44}$$

Then, there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R \tag{45}$$

and the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (26) is globally asymptotically stable in probability and

$$J(x_0) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \frac{1}{q}(x_0^{\mathrm{T}} M_q x_0)^q, \qquad x_0 \in \mathbb{R}^n. \tag{46}$$

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Proof
The existence of a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ for some $R > 0$ follows from converse Lyapunov theory using the fact that $A+\frac{1}{2}\|\sigma\|^2 I_n$ is Hurwitz. The result now is a direct consequence of Theorem 3.1 with $f(x) = Ax$, $D(x) = x\sigma^{\mathrm{T}}$, $L(x) = x^{\mathrm{T}}Rx + \sum_{q=2}^{r}[(x^{\mathrm{T}}\hat{R}_q x)(x^{\mathrm{T}}M_q x)^{q-1}]$, $V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{r}\frac{1}{q}(x^{\mathrm{T}}\ M_q x)^q$, and $\mathcal{D} = \mathbb{R}^n$. Specifically, conditions (15) and (16) are trivially satisfied. Now,

$$\begin{aligned}
&V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x)\\
&\quad = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}(A^{\mathrm{T}}M_q + M_q A)x\\
&\qquad + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}\big[2P + 4(q-1)(x^{\mathrm{T}}M_q x)^{q-2}M_q x x^{\mathrm{T}}M_q + 2(x^{\mathrm{T}}M_q x)M_q\big](x\sigma^{\mathrm{T}})\\
&\quad = x^{\mathrm{T}}\left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)\right]x\\
&\qquad + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}\left[\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)^{\mathrm{T}} M_q\right.\\
&\qquad \left. + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)\right]x,
\end{aligned}$$

and hence, it follows from (44) and (45) that $L(x) + V'(x)f(x) + \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x) = 0$, $x \in \mathbb{R}^n$, so that all the conditions of Theorem 3.1 are satisfied. Finally, because $V(\cdot)$ is radially unbounded, (26) is globally asymptotically stable in probability. □

Remark 3.2
Proposition 3.1 requires the solutions of $r-1$ Lyapunov equations in (44) to obtain a closed-form expression for the nonlinear–nonquadratic cost functional (43).

4. STOCHASTIC OPTIMAL NONLINEAR CONTROL

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear–nonquadratic cost functional. We use the framework developed in Theorem 3.1 to obtain a characterization of optimal feedback controllers that guarantee closed-loop local and global stabilization in probability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the stochastic Hamilton–Jacobi–Bellman equation. To address the problem of characterizing stochastic optimal stabilizing feedback controllers, consider the controlled nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = F(x(t),u(t))\mathrm{d}t + D(x(t),u(t))\mathrm{d}w(t), \quad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geqslant t_0, \tag{47}$$

where, for every $t \geqslant t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $x(0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^{U}$, $U \subseteq \mathbb{R}^m$ is open set with $0 \in U$, $w(t)$ is a d-dimensional independent standard Wiener process, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is jointly continuous in x and u with $F(0,0) = 0$, and $D : \mathcal{D} \times U \to \mathbb{R}^{n\times d}$ is jointly continuous in x and u with $D(0,0) = 0$.

Here, we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (47) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (47) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant t_0}$ such that $u(t) \in \mathcal{H}_m$, $t \geqslant t_0$, and, for all $t \geqslant s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leqslant s$, and $x(t_0)$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume that $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (3) and (4) hold for the controlled drift and diffusion terms $F(x,u)$ and $D(x,u)$ uniformly in u. In

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this case, it follows from Theorem 2.2.4 of [30] that there exists a pathwise unique solution to (47) in $(\Omega, \{\mathcal{F}_{t\geqslant t_0}\}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geqslant t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geqslant t_0$, satisfies (47), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control because $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geqslant t_0$, the *closed-loop system* (47) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)))\mathrm{d}t + D(x(t), \phi(x(t)))\mathrm{d}w(t) \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geqslant t_0. \tag{48}$$

Next, we present a main theorem for stochastic stabilization characterizing feedback controllers that guarantee local and global closed-loop stability in probability and minimize a nonlinear–nonquadratic performance measure. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and, for every $\varepsilon \in (0, 1)$, define the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0, \varepsilon) \triangleq \Big\{ u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (47) is such that } \mathbb{P}^{x_0}\left(\mathfrak{B}^{u(\cdot)}_{x_0}\right) \geqslant 1 - \varepsilon,$$
$$\text{where } \mathfrak{B}^{u(\cdot)}_{x_0} \triangleq \left\{ x(\{t \geqslant t_0\}, \omega) : \lim_{t\to\infty} \|x(t, \omega)\| = 0,\ \omega \in \Omega \right\} \Big\}.$$

Theorem 4.1

Consider the nonlinear stochastic controlled dynamical system (47) with performance measure

$$J\left(x_0, u(\cdot), \mathfrak{B}^{u(\cdot)}_{x_0}\right) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}^{u(\cdot)}_{x_0}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}^{u(\cdot)}_{x_0}}(\omega)\mathrm{d}t\right], \tag{49}$$

where $u(\cdot)$ is an admissible control and $\mathbb{1}_{\mathfrak{B}^{u(\cdot)}_{x_0}}(\omega)$ denotes the indicator function of the set $\mathfrak{B}^{u(\cdot)}_{x_0}$. Assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$V(0) = 0, \tag{50}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{51}$$

$$\phi(0) = 0, \tag{52}$$

$$V'(x)F(x, \phi(x)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x, \phi(x))V''(x)D(x, \phi(x)) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{53}$$

$$H(x, \phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{54}$$

$$H(x, u) \geqslant 0, \qquad x \in \mathcal{D}, \qquad u \in U, \tag{55}$$

where

$$H(x, u) \triangleq L(x, u) + V'(x)F(x, u) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x, u)V''(x)D(x, u). \tag{56}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (48) is locally asymptotically stable in probability and, for every $\varepsilon \in (0, 1)$, there exist $\delta = \delta(\varepsilon)$ and $\mathfrak{B}^{\phi(x(\cdot))}_{x_0}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}^{\phi(x(\cdot))}_{x_0}\right) \geqslant 1 - \varepsilon$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$,

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}^{\phi(x(\cdot))}_{x_0}\right) = V(x_0). \tag{57}$$

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In addition, if $x_0 \in \mathcal{B}_\delta(0)$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes (49) in the sense that

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = \min_{u(\cdot)\in\mathcal{S}(x_0,\varepsilon)} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right). \tag{58}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (48) is globally asymptotically stable in probability and (58) holds with $\varepsilon = 0$ and $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = 1$, $x_0 \in \mathbb{R}^n$.

Proof

Local and global asymptotic stability in probability are a direct consequence of (50)–(53) by applying Theorem 3.1 to the closed-loop system (48). Furthermore, using (54), condition (57) is a restatement of (19) as applied to the closed-loop system. Consequently, for every $\varepsilon \in (0, 1)$, there exist $\delta = \delta(\varepsilon)$ and a set of sample trajectories $x(\{t \geqslant t_0\}, \omega) \in \mathfrak{B}_{x_0}^{\phi(x(\cdot))}$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) \geqslant 1 - \varepsilon$.

Next, let $x_0 \in \mathcal{B}_\delta(0)$, let $u(\cdot) \in \mathcal{S}(x_0, \varepsilon)$, and let $x(t)$, $t \geqslant t_0$, be the solution of (47). Then using Itô's (chain rule) formula, it follows that

$$\begin{aligned} L(x(t), u(t))\mathrm{d}t + \mathrm{d}V(x(t)) = &\bigg(L(x(t), u(t)) + V'(x(t))F(x, u(t)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x(t), u(t)) \\ &\cdot V''(x(t))D(x(t), u(t))\bigg)\mathrm{d}t + \frac{\partial V(x)}{\partial x}D(x, u)\mathrm{d}w(t), \end{aligned}$$

and hence,

$$L(x(t), u(t))\mathrm{d}t = -\mathrm{d}V(x(t)) + H(x(t), u(t))\mathrm{d}t + \frac{\partial V(x(t))}{\partial x}D(x(t), u(t))\mathrm{d}w(t). \tag{59}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x(t)$, $t \geqslant t_0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, multiplying (59) with $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)$ and integrating over $[0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned} &\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t), u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\, \mathrm{d}t \\ &\quad = -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}V(x(t)) + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t), u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t \\ &\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x(t), u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}w(t) \\ &\quad = V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) - V(x(\min\{t_n, \tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \\ &\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t), u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t \\ &\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t), u(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}w(t). \end{aligned} \tag{60}$$

Next, taking the expectation on both sides of (60) and using (55) yields

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$$\begin{aligned}
&\mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\,\mathrm{d}t\right] \\
&\quad = \mathbb{E}^{x_0}\Bigg[V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \\
&\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t \\
&\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t),u(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}w(t)\Bigg]
\end{aligned} \tag{61}$$

$$\geqslant V(x_0)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) - \mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right]. \tag{62}$$

Next, let $\mathfrak{B}_{x_0}^m$ denote the set of all the sample trajectories of $x(t)$, $t \geqslant t_0$, such that $\tau_m = \infty$ and note that, by regularity of solutions [26, p. 75], $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^m) \to 1$ as $m \to \infty$. Now, noting that for all $u(\cdot) \in \mathcal{S}(x_0, \varepsilon)$,

$$\int_0^\infty \left|L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right| \mathrm{d}t \overset{\text{a.s.}}{<} \infty,$$

let the random variable

$$g \triangleq \sup_{t\geqslant 0, m>0} \int_0^{\min\{t,\tau_m\}} \left|L(x(s),u(s))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right| \mathrm{d}s.$$

In this case, the sequence in n and m of $\mathcal{F}_t$-measurable random variables $\{f_{m,n}\}_{m,n=0}^\infty \subseteq \mathcal{H}_1$ on Ω for all $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, where

$$f_{m,n} \triangleq \int_0^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t$$

satisfies $|f_{m,n}| \overset{\text{a.s.}}{<} g$, $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Now, defining the improper integral

$$\int_0^\infty L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the dominated convergence theorem [29] that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&\quad = \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&\quad = \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_{t_0}^{\tau_m} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&\quad = \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&\quad = J\left(x_0,u(\cdot),\mathfrak{B}_{x_0}^{u(\cdot)}\right)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right).
\end{aligned} \tag{63}$$

Next, using the fact that $u(\cdot) \in \mathcal{S}(x_0, \varepsilon)$ and $V(\cdot)$ is continuous, it follows that for every $m > 0$, $V(x((\min\{t_n,\tau_m\}))$ is bounded for all $n \in \mathbb{Z}_+$. Thus, using the dominated convergence theorem

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[29] and the fact that $\|x(t,\omega)\| \to 0$ as $t \to \infty$ for all $x(\{t \geqslant t_0\},\omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}$, we obtain

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right] &= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right]\\ &= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}V(x(\tau_m))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right]\\ &= \mathbb{E}^{x_0}\left[V\left(\lim_{m\to\infty}x(\tau_m)\right)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right]\\ &= 0.\end{aligned} \tag{64}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (62) and using the fact that $u(\cdot) \in \mathcal{S}(x_0,\varepsilon)$, (63), (64), and $J\left(x_0,\phi(x(\cdot)),\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = V(x_0)$ yields (58).

Finally, for $\mathcal{D} = \mathbb{R}^n$, global asymptotic stability in probability of closed-loop system is direct consequence of the radially unbounded condition on $V(\cdot)$, and hence, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = 1$ for all $x_0 \in \mathbb{R}^n$. In this case, the proof of (58) follows using identical arguments as in the proof of the local result. □

Note that (54) is the steady-state stochastic Hamilton–Jacobi–Bellman equation. To see this, recall that the stochastic Hamilton–Jacobi–Bellman equation is given by ([24])

$$\frac{\partial}{\partial t}V(t,x(t)) + \min_{u\in U} H\left(t,x(t),u,\frac{\partial}{\partial x}V(t,x(t)),\frac{\partial^2}{\partial x^2}V(t,x(t))\right) = 0, \quad t \geqslant t_0, \tag{65}$$

which characterizes the optimal control for stochastic time-varying systems on a finite or infinite interval. For infinite horizon time-invariant systems, $V(t,x) = V(x)$, and hence (65), reduces to (54) and (55). Conditions (54) and (55) guarantee optimality with respect to the set of admissible stabilizing controllers $\mathcal{S}(x_0,\varepsilon)$. However, it is important to note that an explicit characterization of the set $\mathcal{S}(x_0,\varepsilon)$ is not required. In addition, the optimal stabilizing *feedback* control law $u = \phi(x)$ is independent of the initial condition x_0. Finally, in order to ensure asymptotic stability in probability of the closed-loop system (47), Theorem 4.1 requires that $V(\cdot)$ satisfy (50), (51), and (53), which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (47). However, for optimality, $V(\cdot)$ need not satisfy (51) and (53). Specifically, if $V(\cdot)$ is a two-times continuously differentiable function such that (50) is satisfied and $\phi(\cdot) \in \mathcal{S}(x_0,\varepsilon)$, then (54) and (55) imply (57) and (58).

The optimal feedback control $\phi(\cdot)$ that guarantees global asymptotic stability in probability gives $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right) = 1$, and hence, $\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. Moreover, all the admissible controls $u(\cdot)$ that guarantee global attraction in probability satisfy $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\varepsilon = 0$ and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$\begin{aligned}J\left(x_0,u(\cdot),\mathfrak{B}_{x_0}^{u(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right]\\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),u(t))\mathrm{d}t\right]\end{aligned} \tag{66}$$

and

$$\begin{aligned}J\left(x_0,\phi(\cdot),\mathfrak{B}_{x_0}^{\phi(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),\phi(x(t)))\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega)\mathrm{d}t\right]\\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),\phi(x(t)))\mathrm{d}t\right].\end{aligned} \tag{67}$$

Thus, in the remainder of the paper, we omit the dependence on $\mathfrak{B}_{x_0}^{\phi(\cdot)}$ and $\mathfrak{B}_{x_0}^{u(\cdot)}$ in the cost functional, and we write $\mathcal{S}(x_0)$ for $\mathcal{S}(x_0,\varepsilon)$ for all the results concerning globally stabilizing controllers in probability.

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Next, we specialize Theorem 4.1 to linear stochastic dynamical systems and provide connections to the stochastic optimal linear–quadratic regulator problem with multiplicative noise. For the following result, let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\sigma \in \mathbb{R}^d$, $R_1 \in \mathbb{P}^n$, and $R_2 \in \mathbb{P}^m$ be given.

Corollary 4.1
Consider the linear controlled stochastic dynamical system with multiplicative noise given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \qquad x(0) \overset{\text{a.s.}}{=} x_0, \qquad t \geqslant 0, \tag{68}$$

and with quadratic performance measure

$$J(x_0, u(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty [x^{\mathrm{T}}(t)R_1x(t) + u^{\mathrm{T}}(t)R_2u(t)]\mathrm{d}t\right], \tag{69}$$

where $u(\cdot)$ is an admissible control. Furthermore, assume that there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PBR_2^{-1}B^{\mathrm{T}}P. \tag{70}$$

Then, with the feedback control $u = \phi(x) \triangleq -R_2^{-1}B^{\mathrm{T}}Px$, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (68) is globally asymptotically stable in probability and

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}}Px_0, \qquad x_0 \in \mathbb{R}^n. \tag{71}$$

Furthermore,

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \tag{72}$$

where $\mathcal{S}(x_0)$ is the set of regulation controllers for (68) and $x_0 \in \mathbb{R}^n$.

Proof
The result is a direct consequence of Theorem 4.1 with $F(x,u) = Ax + Bu$, $D(x,u) = x\sigma^{\mathrm{T}}$, $L(x,u) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u$, $V(x) = x^{\mathrm{T}}Px$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Specifically, conditions (50) and (51) are trivially satisfied. Next, it follows from (70) that $H(x,\phi(x)) = 0$, and hence, $V'(x)F(x,\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$. Thus, $H(x,u) = H(x,u) - H(x,\phi(x)) = [u-\phi(x)]^{\mathrm{T}}R_2[u-\phi(x)] \geqslant 0$ so that all the conditions of Theorem 4.1 are satisfied. Finally, because $V(\cdot)$ is radially unbounded, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (68), with $u(t) = \phi(x(t)) = -R_2^{-1}B^{\mathrm{T}}Px(t)$, is globally asymptotically stable in probability. □

The optimal feedback control law $\phi(x)$ in Corollary 4.1 is derived using the properties of $H(x,u)$ as defined in Theorem 4.1. Specifically, because $H(x,u) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u + x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + 2x^{\mathrm{T}}PBu + \|\sigma\|^2x^{\mathrm{T}}Px$, it follows that $\frac{\partial^2 H}{\partial u^2} = R_2 > 0$. Now, $\frac{\partial H}{\partial u} = 2R_2u + 2B^{\mathrm{T}}Px = 0$ gives the unique global minimum of $H(x,u)$. Hence, because $\phi(x)$ minimizes $H(x,u)$, it follows that $\phi(x)$ satisfies $\frac{\partial H}{\partial u} = 0$ or, equivalently, $\phi(x) = -R_2^{-1}B^{\mathrm{T}}Px$.

5. INVERSE OPTIMAL STOCHASTIC CONTROL FOR NONLINEAR AFFINE SYSTEMS

In this section, we specialize Theorem 4.1 to affine in the control systems. Specifically, we construct nonlinear feedback controllers using a stochastic optimal control framework that minimizes a nonlinear–nonquadratic performance criterion. This is accomplished by choosing the controller such that the mapping of the infinitesimal generator of the Lyapunov function is negative definite along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing (in probability) solutions to the stochastic Hamilton–Jacobi–Bellman equation. Thus, these results provide a family of globally stabilizing controllers parameterized by the cost functional that is minimized.

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The controllers obtained in this section are predicated on an *inverse optimal stochastic control problem* [15–21]. In particular, to avoid the complexity in solving the stochastic steady-state Hamilton–Jacobi–Bellman equation, we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function for the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the stochastic Hamilton–Jacobi–Bellman equation. Hence, by varying parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing in probability controllers that can meet closed-loop system response constraints.

Consider the nonlinear stochastic affine in the control dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{73}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $D : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfies $D(0) = 0$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Furthermore, we consider performance integrands $L(x,u)$ of the form

$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u, \tag{74}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{P}^m$ so that (49) becomes

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right]. \tag{75}$$

Theorem 5.1
Consider the nonlinear controlled affine stochastic dynamical system (73) with performance measure (75). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that

$$V(0) = 0, \tag{76}$$

$$L_2(0) = 0, \tag{77}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{78}$$

$$\begin{aligned} V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] \\ + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \end{aligned} \tag{79}$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{80}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\frac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}}, \tag{81}$$

and the performance measure (75), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x), \tag{82}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{83}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{84}$$

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Proof
The result is a direct consequence of Theorem 4.1 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(x,u) = f(x) + G(x)u$, $D(x,u) = D(x)$, and $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u$. Specifically, with (74), the Hamiltonian has the form

$$H(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u + V'(x)(f(x) + G(x)u) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x).$$

Now, the feedback control law (81) is obtained by setting $\frac{\partial H}{\partial u} = 0$. With (81), it follows that (76), (78), and (79) imply (50), (51), and (53), respectively. Next, because $V(\cdot)$ is two-times continuously differentiable and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, because by assumption $L_2(0) = 0$, it follows that $\phi(0) = 0$, which implies (52). Next, with $L_1(x)$ given by (82) and $\phi(x)$ given by (81), (54) holds. Finally, because $H(x,u) = H(x,u) - H(x,\phi(x)) = [u-\phi(x)]^{\mathrm{T}}R_2(x)[u-\phi(x)]$ and $R_2(x)$ is positive definite for all $x \in \mathbb{R}^n$, condition (55) holds. The result now follows as a direct consequence of Theorem 4.1. □

Note that (79) is equivalent to

$$\mathcal{L}V(x) \triangleq V'(x)[f(x) + G(x)\phi(x)] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{85}$$

with $\phi(x)$ given by (81). Furthermore, conditions (76), (78), and (85) ensure that $V(\cdot)$ is a Lyapunov function for the closed-loop system (80). As discussed in [14], it is important to recognize that the function $L_2(x)$, which appears in the integrand of the performance measure (74), is an arbitrary function of $x \in \mathbb{R}^n$ subject to conditions (77) and (79). Thus, $L_2(x)$ provides flexibility in choosing the control law.

With $L_1(x)$ given by (82) and $\phi(x)$ given by (81), $L(x,u)$ can be expressed as

$$\begin{aligned} L(x,u) &= u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) + L_2(x)(u-\phi(x)) \\ &\quad - V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \left[u + \frac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x)\left[u + \frac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] - V'(x)[f(x) \\ &\quad + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) - \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x). \end{aligned} \tag{86}$$

Because $R_2(x) > 0$, $x \in \mathbb{R}^n$, the first term on the right-hand side of (86) is nonnegative, while (85) implies that the second, third, and fourth terms collectively are nonnegative. Thus, it follows that

$$L(x,u) \geqslant -\frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{87}$$

which shows that $L(x,u)$ may be negative. As a result, there may exist a control input u for which the performance measure $J(x_0,u)$ is negative. However, if the control u is a regulation controller, that is, $u \in \mathcal{S}(x_0)$, then it follows from (83) and (84) that

$$J(x_0, u(\cdot)) \geqslant V(x_0) \geqslant 0, \qquad x_0 \in \mathbb{R}^n, \qquad u(\cdot) \in \mathcal{S}(x_0). \tag{88}$$

Furthermore, in this case, substituting $u = \phi(x)$ into (86) yields

$$L(x,\phi(x)) = -V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x), \tag{89}$$

which, by (85), is positive.

Next, we specialize Theorem 5.1 to linear stochastic systems controlled by nonlinear controllers that minimize a polynomial cost functional. For the following result, let $\sigma \in \mathbb{R}^d$, $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^m$, and $\hat{R}_q \in \mathbb{N}^n$, $q = 2,\ldots,r$, be given, where r is a positive integer, and define $S \triangleq BR_2^{-1}B^{\mathrm{T}}$.

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Corollary 5.1
Consider the linear controlled stochastic dynamical system (68). Assume that there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PSP, \tag{90}$$

$$0 = \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)^{\mathrm{T}} M_q + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right) + \hat{R}_q, \quad q = 2, \ldots, r. \tag{91}$$

Then, the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = (Ax(t) + B\phi(x(t)))\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{92}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right)x, \tag{93}$$

and the performance measure (75) with $R_2(x) = R_2$, $L_2(x) = 0$, and

$$\begin{aligned} L_1(x) = x^{\mathrm{T}}\Bigg(& R_1 + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} S \\ & \cdot \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]\Bigg)x, \end{aligned} \tag{94}$$

is minimized in the sense of (83). Finally,

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}}Px_0 + \sum_{q=2}^{r}\frac{1}{q}(x_0^{\mathrm{T}}M_q x_0)^q, \qquad x_0 \in \mathbb{R}^n. \tag{95}$$

Proof
The result is a direct consequence of Theorem 5.1 with $f(x) = Ax$, $G(x) = B$, $D(x) = x\sigma^{\mathrm{T}}$, $L_2(x) = 0$, $R_2(x) = R_2$, and

$$V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{r}\frac{1}{q}(x^{\mathrm{T}}M_q x)^q.$$

Specifically, (76)–(78) are trivially satisfied. Next, it follows from (90), (91), and (93) that

$$\begin{aligned} & V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\ & = -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}\hat{R}_q x - \phi^{\mathrm{T}}(x)R_2\phi(x) \\ & \quad - x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} S\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]x, \end{aligned}$$

which implies (79), so that all the conditions of Theorem 5.1 are satisfied. □

Corollary 5.1 requires the solutions of $r-1$ modified Riccati equations in (91) to obtain the optimal controller (93). It is important to note that the derived performance measure weighs the state variables by arbitrary even powers. Furthermore, $J(x_0, u(\cdot))$ has the form

$$J(x_0,u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left\{x^{\mathrm{T}}(t)\left(R_1 + \sum_{q=2}^{r}(x^{\mathrm{T}}(t)M_q x(t))^{q-1}\hat{R}_q\right)x(t) + u^{\mathrm{T}}(t)R_2u(t)\right.\right.$$
$$\left.\left. + \phi_{\mathrm{NL}}^{\mathrm{T}}(x(t))R_2\phi_{\mathrm{NL}}(x(t))\right\}\mathrm{d}t\right],$$

where $\phi_{\mathrm{NL}}(x)$ is the nonlinear part of the optimal feedback control

$$\phi(x) = \phi_{\mathrm{L}}(x) + \phi_{\mathrm{NL}}(x),$$

where $\phi_{\mathrm{L}}(x) \triangleq -R_2^{-1}B^{\mathrm{T}}Px$ and $\phi_{\mathrm{NL}}(x) \triangleq -R_2^{-1}B^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_qx$.

Remark 5.1
Corollary 5.1 generalizes the stochastic nonlinear–nonquadratic optimal control problem considered in [5] to polynomial performance criteria. Specifically, unlike the results of [5], Corollary 5.1 is not limited to sixth-order cost functionals and cubic nonlinear controllers but rather addresses a polynomial performance criterion of an arbitrary even order.

Remark 5.2
General nonquadratic cost functions can result in nonlinear controllers that yield a faster than exponential closed-loop system response. Alternatively, when the nonlinear–nonquadratic performance measure involves terms of order x^p, where $p < 2$, then we have a *subquadratic* cost criterion, which pays close attention to the system state near the origin. In this case, the optimal controller is *sublinear* and, hence, exhibits finite settling time behavior [31].

Next, we specialize Theorem 5.1 to linear stochastic systems controlled by nonlinear controllers that minimize a multilinear cost functional. For the following result, recall the definition of S and let $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^m$, and $\hat{R}_{2q} \in \mathcal{N}^{(2q,n)}$, $q = 2,\ldots,r$, be given, where r is a given integer and $\mathcal{N}^{(k,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geqslant 0, x \in \mathbb{R}^n\}$.

Corollary 5.2
Consider the linear controlled stochastic dynamical system (68). Assume that there exist $P \in \mathbb{P}^n$ and $\hat{P}_q \in \mathcal{N}^{(2q,n)}$, $q = 2,\ldots,r$, such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PSP, \tag{96}$$

$$0 = \hat{P}_q\left[\overset{2q}{\oplus}\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)\right] + \hat{R}_{2q}, \qquad q = 2,\ldots,r. \tag{97}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (92) is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(Px + \frac{1}{2}g'^{\mathrm{T}}(x)\right), \tag{98}$$

where $g(x) \triangleq \sum_{q=2}^{r}\hat{P}_q x^{[2q]}$, and the performance measure (75) with $R_2(x) = R_2$, $L_2(x) = 0$, and

$$L_1(x) = x^{\mathrm{T}}R_1x + \sum_{q=2}^{r}\hat{R}_{2q}x^{[2q]} + \frac{1}{4}g'(x)Sg'^{\mathrm{T}}(x) \tag{99}$$

is minimized in the sense of (83). Finally,

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}, \qquad x_0 \in \mathbb{R}^n. \tag{100}$$

Proof
The result is a direct consequence of Theorem 5.1 with $f(x) = Ax$, $G(x) = B$, $D(x) = x\sigma^{\mathrm{T}}$, $L_2(x) = 0$, $R_2(x) = R_2$, and $V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}$. Specifically, (76)–(78) are trivially satisfied. Next, it follows from (96)–(98) that

$$\begin{aligned} V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) \\ = -x^{\mathrm{T}} R_1 x - \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} - \phi^{\mathrm{T}}(x) R_2 \phi(x) - \frac{1}{4} g'(x) S g'^{\mathrm{T}}(x), \end{aligned}$$

which implies (79) so that all the conditions of Theorem 5.1 are satisfied. □

Note that because

$$g'(x)(A - SP)x + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) = \sum_{q=2}^{r} \hat{P}_q \left[\overset{2q}{\oplus}\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)\right] x^{[2q]},$$

it follows that (97) can be equivalently written as

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)(A - SP)x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]}, \quad x \in \mathbb{R}^n,$$

and hence, it follows from Lemma 3.1 that there exists a unique $\hat{P}_q \in \mathcal{N}^{(2q,n)}$ such that (97) is satisfied.

Remark 5.3
Corollary 5.2 generalizes the deterministic nonlinear feedback controller results obtained by Bass and Webber in [4] to stochastic nonlinear feedback control.

6. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the efficacy of the proposed approach.

Example 6.1
Consider the two-state controlled nonlinear stochastic dynamical system given by

$$\mathrm{d}x_1(t) = -x_1(t)\mathrm{d}t + u_1(t)\mathrm{d}t + x_2^2(t)\mathrm{d}w(t), \qquad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \qquad t \geqslant 0, \tag{101}$$

$$\mathrm{d}x_2(t) = -x_2^3(t)\mathrm{d}t + u_2(t)\mathrm{d}t + x_1(t)\mathrm{d}w(t), \qquad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{102}$$

and note that (101) and (102) can be cast in the form of (73) with $f(x) = [-x_1,\ -x_2^3]^{\mathrm{T}}$, $G(x) = I_2$, and $D(x) = [x_2^2,\ x_1]^{\mathrm{T}}$, where $x \triangleq [x_1\ x_2]^{\mathrm{T}}$. To construct an inverse optimal globally stabilizing control law for (101) and (102), let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ and let $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2 u$, where $R_2 > 0$. Now, $L_2(x) = x^{\mathrm{T}}$ satisfies (77) so that the inverse optimal control law (81) is given by $\phi(x) = -R_2^{-1}x$. In this case, the performance measure (75), with $L_1(x) = x^{\mathrm{T}} R_2^{-1} x + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$, is minimized in the sense of (83). Furthermore, because $V(x)$ is radially unbounded and

$$\mathcal{L}V(x) = -x^{\mathrm{T}} R_2^{-1} x - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^4 < 0, \qquad x \in \mathbb{R}^2, \qquad x \neq 0, \tag{103}$$

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the feedback control law $\phi(x) = -R_2^{-1}x$ is globally stabilizing in probability.

Let $x(0) = [1,\ -1]^{\mathrm{T}}$ a.s. and $R_2 = 4I_2$. Figure 1 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figure 2 shows the sample average along with the standard deviation of the corresponding control signal versus time for 20 sample paths.

Example 6.2

Consider the pitch axis longitudinal dynamics model of the F-16 fighter aircraft system for nominal flight conditions at 3000 ft and Mach number of 0.6 with stochastic disturbances given by ([14])

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{104}$$

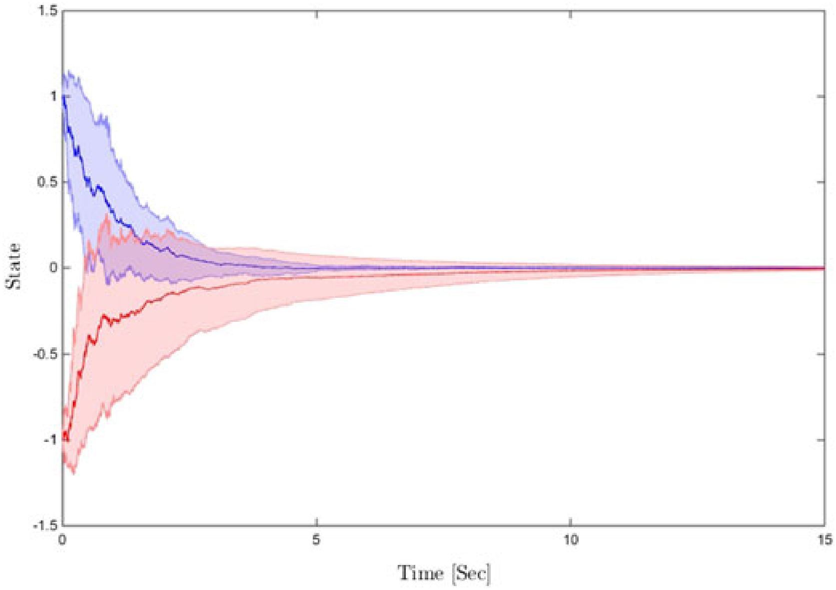

Figure 1. Sample average along with the sample standard deviation of the closed-loop system trajectory versus time: $x_1(t)$ in blue and $x_2(t)$ in red. [Colour figure can be viewed at wileyonlinelibrary.com]

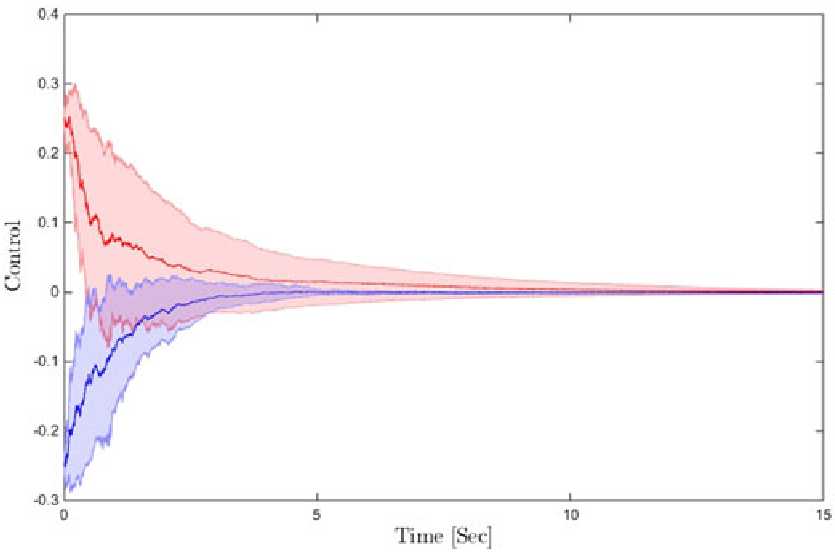

Figure 2. Sample average along with the sample standard deviation of the control signal versus time: $u_1(t)$ in blue and $u_2(t)$ in red. [Colour figure can be viewed at wileyonlinelibrary.com]

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where $x \triangleq [x_1\ x_2\ x_3]^{\mathrm{T}}$, $u \triangleq [u_1\ u_2]^{\mathrm{T}}$, x_1 is the pitch angle, x_2 is the pitch rate, x_3 is the angle of attack, u_1 is the elevator deflection, u_2 is the flaperon deflection, and

$$A = \begin{bmatrix} 0 & 1.00 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}, \qquad \sigma = 0.5.$$

In order to design an inverse optimal control law for the controlled stochastic dynamical system (104), consider the Lyapunov function candidate given by

$$V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{3} \frac{1}{q} (x^{\mathrm{T}} M_q x)^q, \tag{105}$$

where $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, 3$. Now, letting $L(x, u) = L_1(x) + u^{\mathrm{T}} R_2 u$, where $R_2 > 0$, it follows from Corollary 5.1 that

$$P = \begin{bmatrix} 0.3773 & 0.0039 & -0.0307 \\ 0.0039 & 0.0032 & 0.0010 \\ -0.0307 & 0.0010 & 0.0906 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.0740 & -0.0778 & -0.0266 \\ -0.0778 & 0.0836 & 0.0236 \\ -0.0266 & 0.0236 & 0.0354 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.0005 & -0.0003 & -0.0013 \\ -0.0003 & 0.0008 & -0.0011 \\ -0.0013 & -0.0011 & 0.0140 \end{bmatrix},$$

satisfy (90) and (91) for $R_1 = 0.3 I_3$, $R_2 = 0.01 I_2$, $\hat{R}_2 = 0.1 I_3$, and

$$\hat{R}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}.$$

In this case, the feedback control law

$$\phi(x) = -R_2^{-1} B^{\mathrm{T}} \left(P + \sum_{q=2}^{3} (x^{\mathrm{T}} M_q x)^{q-1} M_q \right) x$$

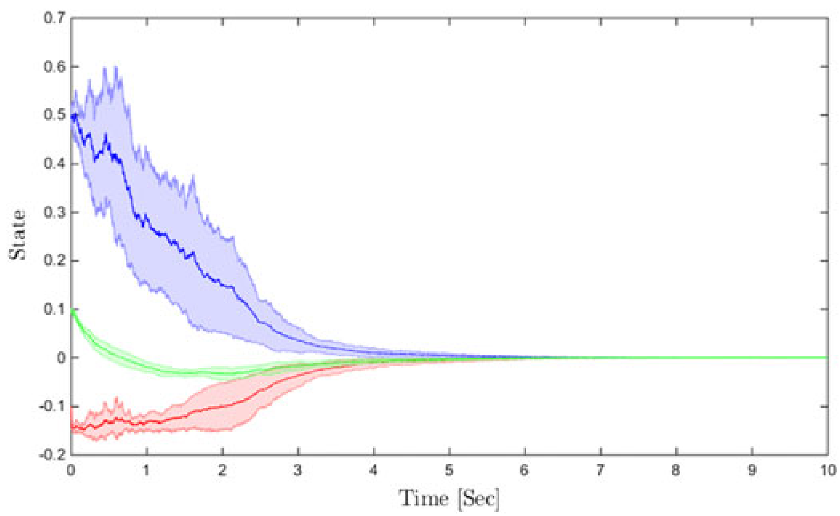

Figure 3. Sample average along with the sample standard deviation of the closed-loop system trajectory versus time: $x_1(t)$ in blue, $x_2(t)$ in red, and $x_3(t)$ in green. [Colour figure can be viewed at wileyonlinelibrary.com]

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globally stabilizes in probability the controlled dynamical system (104). Furthermore, the performance measure (75), with

$$L_1(x) = x^{\mathrm{T}}\left(R_1 + \sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q + \left[\sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} S \cdot \left[\sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]\right)x,$$

is minimized in the sense of (83).

Figure 3 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figure 4 shows the sample average along with the standard deviation of

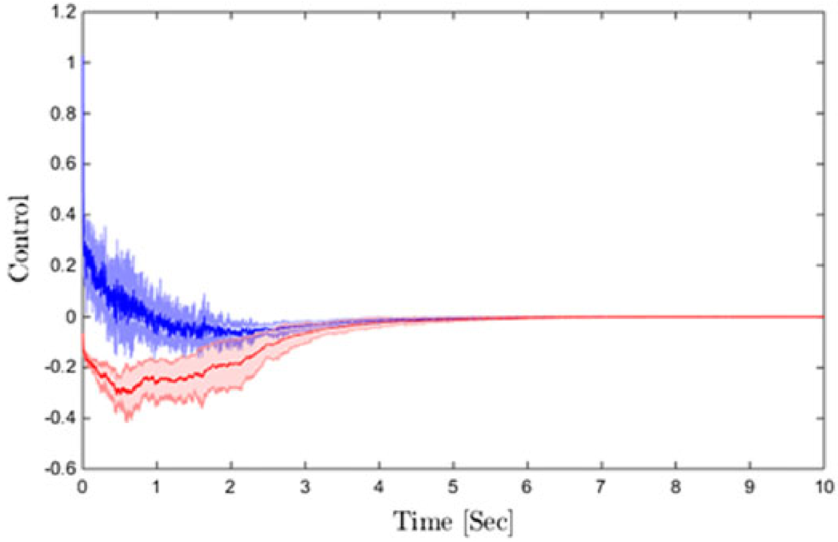

Figure 4. Sample average along with the sample standard deviation of the control signal versus time: $u_1(t)$ in blue and $u_2(t)$ in red. [Colour figure can be viewed at wileyonlinelibrary.com]

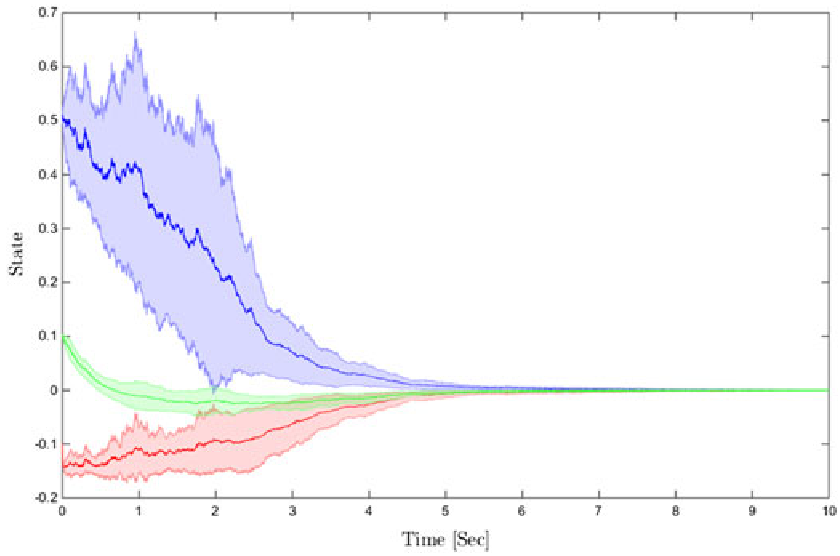

Figure 5. Sample average along with the sample standard deviation of the closed-loop system trajectory versus time: $x_1(t)$ in blue, $x_2(t)$ in red, and $x_3(t)$ in green. [Colour figure can be viewed at wileyonlinelibrary.com]

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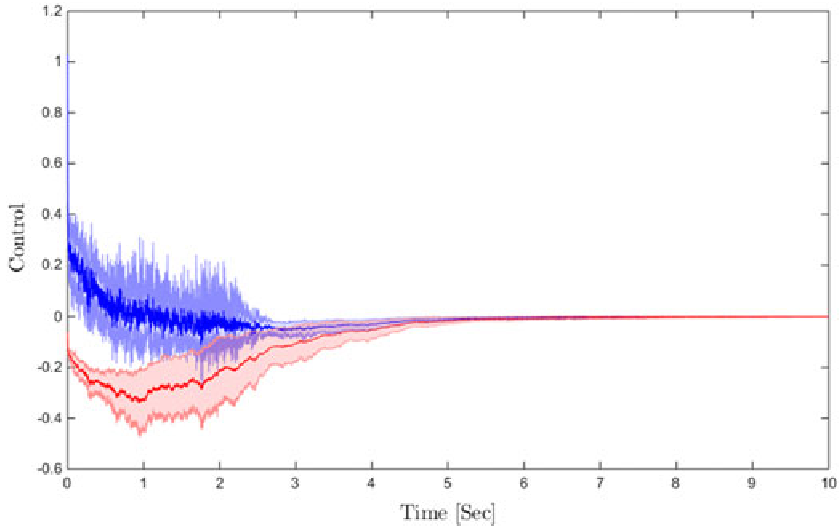

Figure 6. Sample average along with the sample standard deviation of the control signal versus time: $u_1(t)$ in blue and $u_2(t)$ in red. [Colour figure can be viewed at wileyonlinelibrary.com]

the corresponding control signal versus time for $x(0) \overset{\text{a.s.}}{=} [0.5,\ -0.1,\ 0.1]^{\mathrm{T}}$ for 20 sample paths. This controller is compared with the Speyer controller [5] involving a sixth-order cost functional and a cubic-order controller ($q = 2$ in (105)) in Figures 5 and 6. $\triangle$

7. CONCLUSION

In this paper, an optimal control problem for stochastic stabilization is stated, and sufficient conditions are derived to characterize a stochastic optimal nonlinear feedback controller that guarantees asymptotic stability in probability of the closed-loop system. Specifically, we utilized a steady-state stochastic Hamilton–Jacobi–Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function guaranteeing stability in probability of the closed-loop system. This result was then used to develop inverse optimal feedback controllers for affine nonlinear stochastic systems and linear stochastic systems with polynomial and multilinear performance criteria.

In spite of the appealing nature of the classical stochastic Hamilton–Jacobi–Bellman theory, its current state of development entails limitations in addressing the design of static and dynamic *output-feedback* compensators. In contrast, the simplified and tutorial exposition of the stochastic optimal control framework presented in this paper can potentially be used to develop a *fixed-structure* stochastic Hamilton–Jacobi–Bellman theory in which one can prespecify the structure of the feedback law with respect to, for example, the order of nonlinearities appearing in the dynamic compensator. The actual gain maps can then be determined by solving algebraic relations in much the same way full-state feedback controllers can be obtained. In this case, the structure of the nonlinear–nonquadratic Lyapunov function, nonlinear–nonquadratic cost functional, and nonlinear feedback controller can be fixed, while the performance can be optimized with respect to the controller gains.

To demonstrate how fixed-structure stochastic Hamilton–Jacobi–Bellman synthesis can be performed assume that A (which can denote a closed-loop system) is Hurwitz, let P be given by (28), and consider the case where $D(x) = x\sigma^{\mathrm{T}}$ and $L(x)$, $f(x)$, and $V(x)$ are given by (31). To satisfy (18), we require that (33) holds. Equation (33) is the basic constraint that must be satisfied by the closed-loop system in order for $J(x_0)$ to be given by (19).

Now, for the simplicity of exposition, consider the linear controlled dynamical system with multiplicative noise given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{106}$$

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$$y(t) = Cx(t), \tag{107}$$

and constrain the *output feedback* control law to be given by $u = \phi(y)$, where $\phi(\cdot)$ is a finitely parameterized control law (e.g., linear plus cubic plus quintic). Then the closed-loop system takes the form

$$\mathrm{d}x(t) = (Ax(t) + B\phi(Cx(t)))\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geqslant 0, \tag{108}$$

which has the form of (1) with $f(x)$ given by (31). Minimizing $J(x_0)$ given by (19) subject to (33) now reduces to a system of algebraic relations in the coefficients of the different powers of x. Hence, the proposed framework allows for the synthesis of fixed-structure static and dynamic output-feedback controllers. This line of research is currently under development.

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