Finite-Time Stabilization and Optimal Feedback Control for Nonlinear Discrete-Time Systems

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***Abstract*—Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Sufficient conditions for finite-time stability have recently been developed in the literature for discrete-time dynamical systems. In this article, we build on these results to develop a framework for addressing the problem of optimal nonlinear analysis and feedback control for finite-time stability and finite-time stabilization for nonlinear discrete-time controlled dynamical systems. Finite-time stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that satisfies a difference inequality involving fractional powers and a minimum operator. This Lyapunov function can clearly be seen to be the solution to a difference equation that corresponds to a steady-state form of the Bellman equation, and hence, guaranteeing both finite-time stability and optimality. Finally, a numerical example is presented to demonstrate the efficacy of the proposed finite-time discrete stabilization framework.**

***Index Terms*—Bellman theory, discrete-time systems, finite-time stability, finite-time stabilization, optimal control.**

I. Introduction

In [1], the current status of continuous-time, nonlinear–nonquadratic optimal control problems was presented in a simplified and tutorial manner. This framework was extended in [2] and [3] to the discrete-time setting. The basic underlying ideas of the results in [2] and [3] are based on the fact that the steady-state solution of the Bellman equation is a Lyapunov function for the nonlinear system, and thus, guaranteeing both *asymptotic* stability and optimality [4]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear–nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear–nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Bellman equation. The overall framework provides the foundation for extending linear-quadratic control for discrete-time systems to nonlinear–nonquadratic problems.

Using the optimal control framework developed in [1] and [4], the authors in [5] and [6] present a constructive approach for designing optimal and inverse optimal *continuous-time, finite-time* feedback controllers. In this article, we build on these results and extend the framework developed in [2]–[4], and [7] to address the problem of *optimal finite-time discrete stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite-time stability of the closed-loop system. Specifically, using the finite-time stability *analysis* results for discrete-time autonomous *closed* systems presented in [7], we develop optimal control and inverse optimal control problems under nonlinear–nonquadratic costs. In particular, an optimal finite-time control problem is stated and sufficient Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Bellman equation is clearly shown to be a Lyapunov function for the closed-loop, discrete-time system that additionally satisfies a difference inequality involving a fractional power and a minimum operator, and hence, guaranteeing both finite-time stability and optimality. Finally, we explore connections of our approach with inverse optimal control [8]–[12], wherein we parameterize a family of finite-time stabilizing (possibly) sublinear controllers that minimize a derived cost functional involving a combination of quadratic and subquadratic terms. Subquadratic performance criteria have been studied in [13]–[15] for continuous-time systems and have been shown to permit a unified treatment of a broad range of design goals.

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II. Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results on finite-time stability. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ denote the set of positive real numbers, $\overline{\mathbb{R}}_+$ denote the set of nonnegative numbers, $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices, $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of positive integers, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denote transpose. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm in $\mathbb{R}^n$, $\Delta V(x) \triangleq V(f(x)) - V(x)$ for the difference operator of $V : \mathbb{R}^n \to \mathbb{R}$ at x for a given $f(x)$, and $\lceil \alpha \rceil$ for the ceiling function denoting the smallest integer greater than or equal to α.

Consider the discrete-time nonlinear dynamical system

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \tag{1}$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the system state vector, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathcal{D}$, and $f(0) = 0$. We assume that $f(\cdot)$ is continuous on $\mathcal{D}$. Furthermore, we denote the solution to (1) with initial condition $x(0) = x_0$ by $s(\cdot, x_0)$ so that the *map* of the dynamical system given by $s : \overline{\mathbb{Z}}_+ \times \mathcal{D} \to \mathcal{D}$ is continuous on $\mathcal{D}$ and satisfies the *consistency* property $s(0, x_0) = x_0$ and the *semigroup* property $s(\kappa, s(k, x_0)) = s(k+\kappa, x_0)$ for all $x_0 \in \mathcal{D}$ and $k, \kappa \in \overline{\mathbb{Z}}_+$. We use the notation $s(k, x_0)$, $k \in \overline{\mathbb{Z}}_+$, and $x(k)$, $k \in \overline{\mathbb{Z}}_+$, interchangeably as the solution of the nonlinear discrete-time dynamical system (1) with initial condition $x(0) = x_0$. By a *solution* to (1) with initial condition $x(0) = x_0$, we mean a function $x : \overline{\mathbb{Z}}_+ \to \mathcal{D}$ that satisfies (1). Given $k \in \overline{\mathbb{Z}}_+$ and $x \in \mathcal{D}$, we denote the map $s(k, \cdot) : \mathcal{D} \to \mathcal{D}$ by s_k and the map $s(\cdot, x) : \overline{\mathbb{Z}}_+ \to \mathcal{D}$ by s^x.

If $f(\cdot)$ is continuous, then it follows that $f(s(k-1,\cdot))$ is also continuous since it is constructed as a composition of continuous functions. Hence, $s(k,\cdot)$ is continuous on $\mathcal{D}$. If $f(\cdot)$ is such that $f:\mathbb{R}^n\to\mathbb{R}^n$, then we can construct the *solution sequence* or *discrete trajectory* $x(k)=s(k,x_0)$ to (1) iteratively by setting $x(0)=x_0$ and using $f(\cdot)$ to define $x(k)$ recursively by $x(k+1)=f(x(k))$. This iterative process can be continued indefinitely, and hence, a solution to (1) exists for all $k\geq 0$.

Alternatively, if $f(\cdot)$ is such that $f:\mathcal{D}\to\mathbb{R}^n$, then the solution may cease to exist at some point if $f(\cdot)$ maps $x(k)$ into some point $x(k+1)$ outside the domain of $f(\cdot)$. In this case, the solution sequence $x(k)=s(k,x_0)$ will be defined on the maximal interval of existence $x(k)$, $k\in\mathcal{I}^+_{x_0}\subset\overline{\mathbb{Z}}_+$. Furthermore, note that the solution sequence $x(k)$, $k\in\mathcal{I}^+_{x_0}$, is uniquely defined for every initial condition $x_0\in\mathcal{D}$ irrespective of whether or not $f(\cdot)$ is a continuous function. That is, any other solution sequence $y(k)$ starting from x_0 at $k=0$ will take exactly the same values as $x(k)$ and can be continued to the same interval as $x(k)$. It is important to note that if $k\in\overline{\mathbb{Z}}_+$, then uniqueness of solutions backward in time need not necessarily hold. This is due to the fact that $(k,x_0)=f^{-1}(s(k+1,x_0))$, $k\in\overline{\mathbb{Z}}_+$, and there is no guarantee that $f(\cdot)$ is invertible for all $k\in\overline{\mathbb{Z}}_+$. However, if $f:\mathcal{D}\to\mathcal{D}$ is a homeomorphism for all $k\in\overline{\mathbb{Z}}_+$, then the solution sequence is unique for all $k\in\mathbb{Z}$.

The following definition introduces the notion of finite-time stability for discrete systems.

Definition II.1: Consider the nonlinear dynamical system (1). The zero solution $x(k)\equiv 0$ to (1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N}\subseteq\mathcal{D}$ of the origin and a function $K:\mathcal{N}\setminus\{0\}\to\mathbb{Z}_+$, called the *settling time function*, such that the following statements hold.

1) *Finite-time convergence.* For every $x\in\mathcal{N}\setminus\{0\}$, $s^x(k)\in\mathcal{N}\setminus\{0\}$ is defined on $k\in\{0,\ldots,K(x)-1\}$, and $s(k,x)=0, k\geq K(x)$.
2) *Lyapunov stability.* For every $\varepsilon>0$, there exists $\delta>0$ such that $\mathcal{B}_\delta(0)\subset\mathcal{N}$ and for every $x\in\mathcal{B}_\delta(0)\setminus\{0\}$, $s(k,x)\in\mathcal{B}_\varepsilon(0)$ for all $k\in\{0,\ldots,K(x)-1\}$.

The zero solution $x(k)\equiv 0$ to (1) is *globally finite-time stable* if it is finite-time stable with $\mathcal{N}=\mathcal{D}=\mathbb{R}^n$.

Note that if the zero solution $x(k)\equiv 0$ to (1) is finite-time stable, then it is asymptotically stable [4, p. 765]; however, the converse is not true.

For continuous-time dynamical systems with a finite-time stable equilibrium, the vector field $f(\cdot)$ is necessarily non-Lipschitzian at the system equilibrium because of backward nonuniqueness at the system equilibrium. This leads to standard existence and uniqueness results not applying to solutions reaching the system equilibrium. Consequently, finite-time stability is defined over the time interval that the solution takes to reach the system equilibrium with solutions after the settling time function being given as a separate result (see [16, Prop. 2.3]). In contrast, for finite-time discrete autonomous systems, forward uniqueness is always guaranteed, and hence, such a result is not necessary. In other words, we can extend $K(\cdot)$ to all of $\mathcal{N}$ by defining $K(0)\triangleq 0$ for the equilibrium point $x_{\rm e}=0$. It is easy to see from Definition II.1 that

$$K(x)=\min\{k\in\mathbb{Z}_+ : s(k,x)=0\},\quad x\in\mathcal{N}. \tag{2}$$

The next proposition shows that if the settling-time function of a finite-time stable system is lower semicontinuous at the origin, then it is lower semicontinuous on $\mathcal{N}$.

Proposition II.1 (see[7]): Consider the nonlinear dynamical system (1). Assume that the zero solution $x(k)\equiv 0$ to (1) is finite-time stable, let $\mathcal{N}\subseteq\mathcal{D}$ be as in Definition II.1, and let $K:\mathcal{N}\to\mathbb{Z}_+$ be the settling-time function. Then, $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$.

Remark II.1: In the case of continuous-time systems, it is known that the settling-time function $T(\cdot)$ of a finite-time stable equilibrium is continuous in the domain of convergence if and only if it is continuous at the equilibrium (see [16, Prop. 2.4]). In the case of discrete-time systems, the integer-valued function $K(\cdot)$ is continuous at a point only if it is locally constant. Thus, if $K(\cdot)$ is continuous at an equilibrium point $x_{\rm e}$, then $x_{\rm e}$ necessarily has to satisfy $K(x)=K(x_{\rm e})$, $x\in\mathcal{B}_\varepsilon(x_{\rm e})$. On the other hand, since $f(\cdot)$ is continuous, the set of equilibrium points is closed. Hence, $K(\cdot)$ can be continuous at any equilibrium point only in the uninteresting case where the set of equilibria is either empty or the whole space $\mathcal{D}$.

Next, we provide sufficient conditions for finite-time stability of the nonlinear dynamical system given by (1). For the results in this section, define $\Delta V(x)\triangleq V(f(x))-V(x)$ for a given continuous function $V:\mathcal{D}\to\mathbb{R}$.

Theorem II.1 (see[7]): Consider the nonlinear dynamical system (1). Assume that there exist a continuous function $V:\mathcal{D}\to\mathbb{R}$, real numbers $\alpha\in(0,1)$ and $c>0$, and a neighborhood $\mathcal{M}\subseteq\mathcal{D}$ of the origin such that

$$V(0)=0 \tag{3}$$

$$V(x)>0,\quad x\in\mathcal{M}\setminus\{0\} \tag{4}$$

$$\Delta V(x)\leq -c\min\left\{\frac{V(x)}{c},V(x)^\alpha\right\},\quad x\in\mathcal{M}\setminus\{0\}. \tag{5}$$

Then, the zero solution $x(k)\equiv 0$ to (1) is finite-time stable. Moreover, there exist an open neighborhood $\mathcal{N}$ of the origin and a settling-time function $K:\mathcal{N}\to\overline{\mathbb{Z}}_+$ such that either

$$K(x_0)\leq\left\lceil \log_{[1-cV(x_0)^{\alpha-1}]}\frac{c^{\frac{1}{1-\alpha}}}{V(x_0)}\right\rceil+1$$
$$x_0\in\mathcal{N},\quad V(x_0)>c^{\frac{1}{1-\alpha}} \tag{6}$$

or

$$K(x_0)=1,\quad x_0\in\mathcal{N}\setminus\{0\},\quad V(x_0)\leq c^{\frac{1}{1-\alpha}} \tag{7}$$

where $K(\cdot)$ is lower semicontinuous on $\mathcal{N}$. If, in addition, $\mathcal{N}=\mathcal{D}=\mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (5) holds on $\mathbb{R}^n$, then the zero solution $x(k)\equiv 0$ to (1) is globally finite-time stable.

Remark II.2: Note that Theorem II.1 also holds for the case where (5) is replaced by

$$\Delta V(x)\leq -\min\{V(x),c\},\quad x\in\mathcal{M}\setminus\{0\} \tag{8}$$

and with the settling-time function $K:\mathcal{N}\to\overline{\mathbb{Z}}_+$ given by

$$K(x_0)\leq\left\lceil\frac{V(x_0)}{c}\right\rceil,\quad x_0\in\mathcal{N}. \tag{9}$$

For details of this fact, see [7, Th. 4.2]. A similar remark holds for Theorems III.1 and IV.1 below.

III. Optimal Finite-Time Stabilization

In this section, we obtain a characterization of optimal feedback controllers that guarantee closed-loop, finite-time stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Bellman equation. To address the problem of characterizing finite-time stabilizing feedback controllers, consider the controlled discrete-time nonlinear dynamical system

$$x(k+1) = F(x(k), u(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+ \tag{10}$$

where $x(k) \in \mathcal{D} \subseteq \mathbb{R}^n$, $k \in \overline{\mathbb{Z}}_+$, is the state vector, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(k) \in U \subseteq \mathbb{R}^m$, $k \in \overline{\mathbb{Z}}_+$, is the control input with $0 \in U$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is continuous in x and u, and $F(0, 0) = 0$. The control $u(\cdot)$ in (10) is restricted to the class of *admissible* controls consisting of functions $u(\cdot)$ such that $u(k) \in U$ for all $k \in \overline{\mathbb{Z}}_+$, where the control constraint set U is given.

A continuous function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(k) = \phi(x(k))$, $k \in \overline{\mathbb{Z}}_+$, where $\phi(\cdot)$ is a control law and $x(k)$ satisfies (10), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(k) = \phi(x(k))$, the closed-loop system has the form

$$x(k+1) = F(x(k), \phi(x(k))), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+. \tag{11}$$

We now consider the problem of finite-time stabilization.

Definition III.1: Consider the controlled dynamical system given by (10). The feedback control law $u = \phi(x)$ is *finite-time stabilizing* if the closed-loop system (11) is finite-time stable. Furthermore, the feedback control law $u = \phi(x)$ is *globally finite-time stabilizing* if the closed-loop system (11) is globally finite-time stable.

Next, we present a main theorem for characterizing sufficient conditions for feedback controllers that guarantee finite-time stability for a nonlinear discrete-time system and minimize a nonlinear–nonquadratic performance functional. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ and define the set of regulation control input signals for the nonlinear system (10) by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot)\text{is admissible and } x(\cdot)\text{given by (10) satisfies } \lim_{k\to\infty} x(k) = 0\}.$$

Note that since finite-time convergence is a stronger condition than asymptotic convergence, $\mathcal{S}(x_0)$ includes the set of all finite-time convergent controllers.

Theorem III.1: Consider the nonlinear controlled dynamical system (10) with performance functional

$$J(x_0, u(\cdot)) \triangleq \sum_{k=0}^{\infty} L(x(k), u(k)) \tag{12}$$

where $u(\cdot)$ is an admissible control. Assume there exist a continuous function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0, 1)$, a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin, and a control law $\phi : \mathcal{D} \to U$ such that

$$\phi(0) = 0 \tag{13}$$

$$V(0) = 0 \tag{14}$$

$$V(x) > 0, \quad x \in \mathcal{M} \setminus \{0\} \tag{15}$$

$$V(F(x, \phi(x))) - V(x) \le -c \min\left\{\frac{V(x)}{c}, V(x)^\alpha\right\}, \quad x \in \mathcal{M} \setminus \{0\} \tag{16}$$

$$L(x, \phi(x)) + V(F(x, \phi(x))) - V(x) = 0, \quad x \in \mathcal{D} \tag{17}$$

$$L(x, u) + V(F(x, u)) - V(x) \ge 0, (x, u) \in \mathcal{D} \times U. \tag{18}$$

Then, with the feedback control $u = \phi(x)$, the zero solution $x(k) \equiv 0$, $k \in \overline{\mathbb{Z}}_+$, to (10) is finite-time stable. Moreover, there exist an open neighborhood $\mathcal{D}_0 \subset \mathcal{M}$ of the origin and a settling-time function $K : \mathcal{D}_0 \to \overline{\mathbb{Z}}_+$ such that either

$$K(x_0) \le \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathcal{D}_0, \quad V(x_0) > c^{\frac{1}{1-\alpha}} \tag{19}$$

or

$$K(x_0) = 1, \quad x_0 \in \mathcal{D}_0 \setminus \{0\}, \quad V(x_0) \le c^{\frac{1}{1-\alpha}} \tag{20}$$

and

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathcal{D}_0. \tag{21}$$

In addition, if $x_0 \in \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(\cdot)) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{22}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $V(\cdot)$ is radially unbounded, and (16) holds on $\mathbb{R}^n \setminus \{0\}$, then the closed-loop system (11) is globally finite-time stable.

Proof: Local and global finite-time stability for all initial conditions $x_0 \in \mathcal{D}_0$ for some neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ and $x_0 \in \mathbb{R}^n$, along with the existence of a settling-time function $K : \mathcal{D}_0 \to [0, \infty)$ such that either (19) or (20) holds are a direct consequence of (13)–(16) by applying Theorem II.1 to the closed-loop system given by (11).

Next, let $u(\cdot) = \phi(x(\cdot))$ and $x(k)$, $k \in \overline{\mathbb{Z}}_+$, satisfy (11). Then, since

$$0 = -\Delta V(x(k)) + V(F(x(k), \phi(x(k)))) - V(x(k)), k \in \overline{\mathbb{Z}}_+ \tag{23}$$

it follows from (17) that

$$\begin{aligned} L(x(k), \phi(x(k))) = & -\Delta V(x(k)) + L(x(k), \phi(x(k))) \\ & + V(F(x(k), \phi(x(k)))) - V(x(k)) \\ = & -\Delta V(x(k)), \quad k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{24}$$

Now, summing (24) over $[0, \kappa]$ yields

$$\sum_{k=0}^{\kappa} L(x(k), u(k)) = -V(x(\kappa+1)) + V(x_0), \quad k \in \overline{\mathbb{Z}}_+. \tag{25}$$

Using (14) and letting $\kappa \to \infty$, it follows from (25) and the continuity of $V(\cdot)$ that

$$\sum_{k=0}^{\infty} L(x(k), u(k)) = -V\left(\lim_{\kappa\to\infty} x(\kappa)\right) + V(x_0), \quad k \in \overline{\mathbb{Z}}_+ \tag{26}$$

and hence, (21) is a direct consequence of (26) using the fact that $\lim_{\kappa\to\infty} x(\kappa) = x(K(x_0)) = 0$.

Next, let $x_0 \in \mathcal{D}_0$, let $u(\cdot) \in \mathcal{S}(x_0)$, and let $x(k)$, $k \in \overline{\mathbb{Z}}_+$, be the solution of (10). Then, it follows that

$$0 = -\Delta V(x(k), u(k)) + V(F(x(k), u(k))) - V(x(k)), k \in \overline{\mathbb{Z}}_+ \tag{27}$$

where $\Delta V(x, u) \triangleq V(F(x, u)) - V(x)$. Hence

$$\begin{aligned} L(x(k), u(k)) = & -\Delta V(x(k), u(k)) + L(x(k), u(k)) \\ & + V(F(x(k), u(k))) - V(x(k)), k \in \overline{\mathbb{Z}}_+. \end{aligned} \tag{28}$$

Now, it follows from (14), (17), (18), (21), (28) and the fact that $u(\cdot) \in \mathcal{S}(x_0)$ that

$$\begin{aligned} J(x_0, u(\cdot)) &= \sum_{k=0}^{\infty} [-\Delta V(x(k), u(k)) + L(x(k), u(k)) \\ &\quad + V(F(x(k), u(k))) - V(x(k))] \\ &= -\lim_{k\to\infty} V(x(k)) + V(x_0) \\ &\quad + \sum_{k=0}^{\infty} [L(x(k), u(k)) \\ &\quad + V(F(x(k), u(k))) - V(x(k))] \\ &\geq -V\left(\lim_{k\to\infty} x(k)\right) + V(x_0) \\ &= -V\left(\lim_{k\to K} x(k)\right) + V(x_0) \\ &= V(x_0) \\ &= J(x_0, \phi(x(\cdot))) \end{aligned} \tag{29}$$

which yields (22). ■

Note that (17) is the steady-state Bellman equation for the controlled nonlinear dynamical system (10) with performance criterion (12). Furthermore, conditions (16)–(18) guarantee optimality as well as finite-time stability with respect to the set of admissible stabilizing controllers $\mathcal{S}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_0)$ is not required. In order to ensure finite-time stability of the closed-loop system (11), Theorem III.1 requires that $V(\cdot)$ satisfy (14)–(16), which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (11). However, for optimality, $V(\cdot)$ need not satisfy (15) and (16). Specifically, if $V(\cdot)$ is a continuous function such that (14) is satisfied and $\phi(\cdot) \in \mathcal{S}(x_0)$, then (17) and (18) imply (21) and (22).

Finally, setting $\mathcal{M} = \mathcal{D}$ in Theorem III.1 and replacing (16) with

$$V(F(x, \phi(x))) - V(x) < 0, \quad x \in \mathcal{D} \tag{30}$$

Theorem III.1 reduces to [4, Th. 14.4] characterizing the classical asymptotically stabilizing optimal control problem for time-invariant systems on an infinite interval.

IV. Finite-Time Stabilization for Affine Dynamical Systems and Connections to Inverse Optimal Control

In this section, we specialize the results of Section III to discrete-time nonlinear affine dynamical systems of the form

$$x(k+1) = f(x(k)) + G(x(k))u(k), \quad x(0) = x_0, k \in \overline{\mathbb{Z}}_+ \tag{31}$$

where, for every $k \in \overline{\mathbb{Z}}_+$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are such that $f(\cdot)$ and $G(\cdot)$ are continuous in x and $f(0) = 0$. Furthermore, we consider performance summands $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u \tag{32}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2(x) > 0$, $x \in \mathbb{R}^n$, so that (12) becomes

$$\begin{aligned} J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} [&L_1(x(k)) + L_2(x(k))u(k) \\ &+ u^{\mathrm{T}}(k) R_2(x(k))u(k)] \,. \end{aligned} \tag{33}$$

Theorem IV.1: Consider the controlled discrete-time nonlinear affine system (31) with performance measure (33). Assume that there exist functions $V : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{12} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $P_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and real numbers $c > 0$ and $\alpha \in (0, 1)$ such that $V(\cdot)$ is continuous, $P_2(\cdot)$ is nonnegative definite

$$L_2(0) = 0 \tag{34}$$

$$P_{12}(0) = 0 \tag{35}$$

$$V(0) = 0 \tag{36}$$

$$V(x) > 0, \quad x \in \mathbb{R}^n, \quad x \neq 0 \tag{37}$$

$$\begin{aligned} &V\left(f(x) - \frac{1}{2}G(x)[R_2(x) + P_2(x)]^{-1}[L_2(x) + P_{12}(x)]^{\mathrm{T}}\right) \\ &- V(x) \leq -c\min\left\{\frac{V(x)}{c}, V(x)^{\alpha}\right\}, \quad x \in \mathbb{R}^n, \quad x \neq 0 \end{aligned} \tag{38}$$

$$\begin{aligned} V(f(x) + G(x)u) &= V(f(x)) + P_{12}(x)u + u^{\mathrm{T}} P_2(x)u \\ &\qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \end{aligned} \tag{39}$$

$$\begin{aligned} 0 = L_1(x) &- \frac{1}{4}[L_2(x) + P_{12}(x)][R_2(x) + P_2(x)]^{-1} \\ &\cdot [L_2(x) + P_{12}(x)]^{\mathrm{T}} + V(f(x)) - V(x) \end{aligned} \tag{40}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{41}$$

Then, the zero solution $x(k) \equiv 0$ to

$$x(k+1) = f(x(k)) + G(x(k))\phi(x(k)), \quad x(0) = x_0, k \in \overline{\mathbb{Z}}_+ \tag{42}$$

is globally finite-time stable with the feedback control

$$u = \phi(x) = -\frac{1}{2}[R_2(x) + P_2(x)]^{-1}[L_2(x) + P_{12}(x)]^{\mathrm{T}}. \tag{43}$$

Moreover, there exists a settling-time function $K : \mathbb{R}^n \to \overline{\mathbb{Z}}_+$ such that either

$$\begin{aligned} K(x_0) \leq \left\lceil \log_{[1-cV(x_0)^{\alpha-1}]} \frac{c^{\frac{1}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathbb{R}^n \\ V(x_0) > c^{\frac{1}{1-\alpha}} \end{aligned} \tag{44}$$

or

$$K(x_0) = 1, \quad x_0 \in \mathbb{R}^n \setminus \{0\}, \quad V(x_0) \leq c^{\frac{1}{1-\alpha}}. \tag{45}$$

and the performance measure (33) is minimized in the sense of (22). Finally

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad x_0 \in \mathbb{R}^n. \tag{46}$$

Proof: The result is a direct consequence of Theorem III.1 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(x, u) = f(x) + G(x)u$, and $L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R_2(x)u$. Specifically, the feedback control law (43) follows from (17) and (18) by setting

$$\begin{aligned} \frac{\partial}{\partial u} [&L_1(x) + V(f(x)) + [L_2(x) + P_{12}(x)]u \\ &+ u^{\mathrm{T}}(R_2(x) + P_2(x))u - V(x)] = 0. \end{aligned} \tag{47}$$

Now, with $u = \phi(x)$ given by (43), conditions (36)–(38) and (40) imply (14)–(17), respectively.

Next, since

$$\begin{aligned} &L(x,u)+V(f(x,u)+G(x)u)-V(x)\\ &\quad=L(x,u)+V(f(x,u)+G(x)u)-V(x)-L(x,\phi(x))\\ &\qquad-V(f(x,\phi(x))+G(x)\phi(x))+V(x)\\ &\quad=(u-\phi(x))^{\mathrm{T}}(R_2(x)+P_2(x))(u-\phi(x)) \end{aligned} \tag{48}$$

and $R_2(x)+P_2(x)>0, x\in\mathbb{R}^n$, condition (18) holds. The result now follows as a direct consequence of Theorem III.1. ■

Remark IV.1: Condition (39) requires that $V(f(x)+G(x)u)$ is quadratic in u. In the local case, this condition is without loss of generality if the Lyapunov function V has a nondegenerate minimum at the origin. In the global case, a sufficient (but not necessary) condition for (39) holding is the case when V is quadratic. For details, see [17].

Next, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem*. In particular, to avoid the complexity in solving the steady-state Bellman equation (40), we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of finite-time stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance summand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the finite-time stabilizing feedback control law, wherein the coupling is introduced via the Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance summand, the proposed framework can be used to characterize a class of globally finite-time stabilizing controllers that can meet closed-loop response constraints.

Theorem IV.2: Consider the controlled discrete-time nonlinear affine dynamical system (31) with performance measure (33). Assume that there exist a continuous radially unbounded function $V:\mathbb{R}^n\to\mathbb{R}$, $L_2:\mathbb{R}^n\to\mathbb{R}^{1\times m}$, $P_{12}:\mathbb{R}^n\to\mathbb{R}^{1\times m}$, a nonnegative-definite function $P_2:\mathbb{R}^n\to\mathbb{R}^{m\times m}$, and real numbers $c>0$ and $\alpha\in(0,1)$ such that (34)–(39) hold. Then, with the feedback control

$$u=\phi(x)=-\frac{1}{2}[R_2(x)+P_2(x)]^{-1}[L_2(x)+P_{12}(x)]^{\mathrm{T}} \tag{49}$$

the zero solution $x(k)\equiv 0$ to (42) is globally finite-time stable. Moreover, there exists a settling-time function $K:\mathbb{R}^n\to\overline{\mathbb{Z}}_+$ such that either (44) or (45) hold and the performance measure (33), with

$$L_1(x)=\phi^{\mathrm{T}}(x)(R_2(x)+P_2(x))\phi(x)-V(f(x))+V(x) \tag{50}$$

is minimized in the sense of (22). Finally, (46) holds.

Proof: The proof is similar to the proof of Theorem IV.1, and hence, is omitted. ■

V. Illustrative Numerical Example

In this section, we provide an illustrative numerical example to demonstrate the proposed optimal finite-time discrete stabilization framework. Consider the discrete-time nonlinear affine system given by

$$x_1(k+1)=f_1(x_1(k),x_2(k))+u_1(k),\quad x_1(0)=x_{10}\qquad k\in\overline{\mathbb{Z}}_+ \tag{51}$$

$$x_2(k+1)=f_2(x_1(k),x_2(k))+u_2(k),\quad x_2(0)=x_{20} \tag{52}$$

where, for $k\in\overline{\mathbb{Z}}_+$, $x_1(k)$, $x_2(k)$, $u_1(k)$, and $u_2(k)\in\mathbb{R}$, and $f_1(0,0)=0$ and $f_2(0,0)=0$. For this example, we apply Theorem IV.2 to find an *inverse optimal* globally finite-time stabilizing control law $u=[u_1,u_2]^{\mathrm{T}}=\phi(x)$, where $x=[x_1,x_2]^{\mathrm{T}}$. Note that (51) and (52) can be cast in the form of (31) with $n=2$, $m=2$, $f(x)=[f_1(x_1,x_2),f_2(x_1,x_2)]^{\mathrm{T}}$, and $G(x)=I_2$.

To construct an inverse optimal finite-time controller for (51) and (52), let $V(x)=x^{\mathrm{T}}x$, $L(x,u)$ be given by (32), $R_2(x)=I_2$, $P_2(x)=I_2$

$$P_{12}(x)=[2f_1(x_1,x_2),\quad 2f_2(x_1,x_2)] \tag{53}$$

and let

$$\begin{aligned} L_2(x)=\Bigg[&2f_1(x)-4x_1+4c\,\mathrm{sign}(x_1)\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\},\\ &2f_2(x)-4x_2+4c\,\mathrm{sign}(x_2)\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}\Bigg] \end{aligned} \tag{54}$$

where $\mathrm{sign}(z)\triangleq z/|z|$, $z\neq 0$, and $\mathrm{sign}(0)\triangleq 0$. Now, the inverse optimal control law (49) is given by

$$\begin{aligned} u=\phi(x)&=-\frac{1}{2}[R_2(x)+P_2(x)]^{-1}[L_2(x)+P_{12}(x)]^{\mathrm{T}}\\ &=-\frac{1}{4}[L_2(x)+P_{12}(x)]^{\mathrm{T}}\\ &=\begin{bmatrix} x_1-c\,\mathrm{sign}(x_1)\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}-f_1(x_1,x_2)\\ x_2-c\,\mathrm{sign}(x_2)\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}-f_2(x_1,x_2)\end{bmatrix} \end{aligned} \tag{55}$$

and the performance functional (33), with

$$L_1(x)=\phi^{\mathrm{T}}(x)\phi(x)-V(f(x))+V(x) \tag{56}$$

is minimized in the sense of (22).

Furthermore, note that (36) and (37) hold and, since

$$\begin{aligned} &V\left(f(x)-\frac{1}{2}G(x)[R_2(x)+P_2(x)]^{-1}[L_2(x)+P_{12}]^{\mathrm{T}}\right)\\ &\quad-V(x)\\ &=V\left[f(x)-\frac{1}{4}[L_2(x)+P_{12}(x)]^{\mathrm{T}}\right]-V(x)\\ &=\left(x_1-c\,\mathrm{sign}(x_1)\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}\right)^2-x_1^2\\ &\quad+\left(x_2-c\,\mathrm{sign}(x_2)\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}\right)^2-x_2^2\\ &=-2c\,|x_1|\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}\\ &\quad+\left(c\,\mathrm{sign}(x_1)\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}\right)^2\\ &\quad-2c\,|x_2|\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}\\ &\quad+\left(c\,\mathrm{sign}(x_2)\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}\right)^2\\ &\le -c\,\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}|x_1|\\ &\quad-c\,\min\left\{\frac{|x_2|}{c},|x_2|^{\alpha}\right\}|x_2|\\ &\le -c^2\left(\min\left\{\frac{|x_1|}{c},|x_1|^{\alpha}\right\}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&- c^2 \left(\min\left\{\frac{|x_1|}{c}, |x_1|^\alpha\right\}\right)^2 \\
&= -c^2 \min\left\{\frac{|x_1|^2}{c^2}, |x_1|^{2\alpha}\right\} - c^2 \min\left\{\frac{|x_2|^2}{c^2}, |x_2|^{2\alpha}\right\} \\
&= -c^2 \left[\min\left\{\frac{|x_1|^2}{c^2}, |x_1|^{2\alpha}\right\} + \min\left\{\frac{|x_2|^2}{c^2}, |x_2|^{2\alpha}\right\}\right] \\
&\leq -c^2 \min\left\{\frac{|x_1|^2 + |x_2|^2}{c^2}, \left[|x_1|^2 + |x_2|^2\right]^\alpha\right\} \\
&= -c^2 \min\left\{\frac{V(x)}{c^2}, V(x)^\alpha\right\} \qquad (57)
\end{aligned}
$$

where the last inequality in (57) follows from the Minkowski inequality, (38) is verified. Hence, with the feedback control law $\phi(x)$ given by (55), the closed-loop system (51) and (52) is globally finite-time stable with the settling-time function satisfying either

$$K(x_0) \leq \left\lceil \log_{[1-c^2 V(x_0)^{\alpha-1}]} \frac{c^{\frac{2}{1-\alpha}}}{V(x_0)} \right\rceil + 1, \quad x_0 \in \mathbb{R}^2$$
$$V(x_0) > c^{\frac{2}{1-\alpha}} \qquad (58)$$

or

$$K(x_0) = 1, \quad x_0 \in \mathbb{R}^2 \setminus \{0\}, \quad V(x_0) \leq c^{\frac{2}{1-\alpha}}. \qquad (59)$$

Next, consider the spacecraft with one axis of symmetry given by

$$\dot{\omega}_1(t) = I_{23}\omega_3\omega_2(t) + u_1(t), \quad \omega_1(0) = \omega_{10}, \quad t \geq 0 \qquad (60)$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3\omega_1(t) + u_2(t), \quad \omega_2(0) = \omega_{20} \qquad (61)$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moment of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3, \omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and u_1 and u_2 are the spacecraft control moments. In order to apply (55), we use a simple Euler discretization scheme to discretize (60) and (61), which yields

$$\omega_1(k+1) = \omega_1(k) + h\left[I_{23}\omega_3\omega_2(k) + u_1(k)\right]$$
$$\omega_1(0) = \omega_{10}, \quad k \in \overline{\mathbb{Z}}_+ \qquad (62)$$

$$\omega_2(k+1) = \omega_2(k) + h\left[-I_{23}\omega_3\omega_1(k) + u_2(k)\right]$$
$$\omega_2(0) = \omega_{20} \qquad (63)$$

where $h > 0$ denotes the sampling time.

Note that (62) and (63) have the form of (31), and hence, the control law

$$u_1(k) = -I_{23}\omega_3\omega_2(k) - \frac{c}{h}\operatorname{sign}(\omega_1(k)) \min\left\{\frac{|\omega_1(k)|}{c}, |\omega_1(k)|^\alpha\right\} \qquad (64)$$

$$u_2(k) = I_{23}\omega_3\omega_1(k) - \frac{c}{h}\operatorname{sign}(\omega_2(k)) \min\left\{\frac{|\omega_2(k)|}{c}, |\omega_2(k)|^\alpha\right\} \qquad (65)$$

where $c > 0$, guarantees global finite-time stability of the controlled spacecraft (60) and (61). Moreover, there exists a settling-time function $K : \mathbb{R}^2 \to \overline{\mathbb{Z}}_+$ such that either (58) or (59) hold, where $x_0 = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$, and

$$J(x_0, \phi(x(\cdot))) = \omega_{10}^2 + \omega_{20}^2, \quad x_0 \in \mathbb{R}^2. \qquad (66)$$

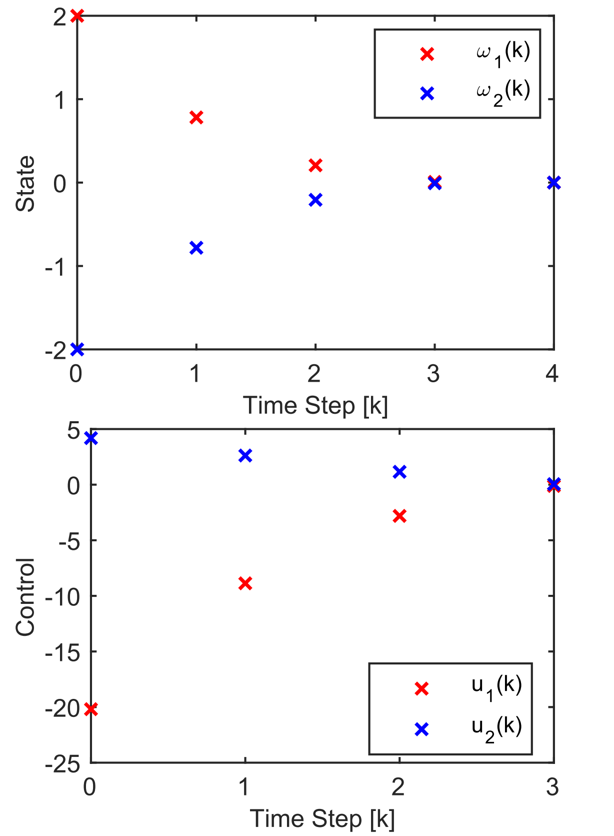

Fig. 1. Closed-loop system trajectory and control inputs versus time of (60) and (61) with control inputs (64) and (65).

Let $I_1 = I_2 = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = 2\,\mathrm{Hz}$, and $\omega_{20} = -2\,\mathrm{Hz}$. The controlled system trajectory and control profile, with $h = 0.1$, $c = 0.7$, and $\alpha = 0.8$, are shown in Fig. 1. Note that $(\omega_1(k), \omega_2(k)) = (0, 0)$ for $k = 4 < K(x_0) = 16$. Finally, $J(x_0, \phi(x(\cdot))) = 8\,\mathrm{Hz}^2$. It is clear from Fig. 1 that the inverse optimal controller (64) and (65) guarantees finite-time stabilization. The parameters c and α in (64) and (65) can be varied to reduce the conservatism between the *guaranteed* settling-time function $K(x_0)$ and the *achieved* finite-time convergence. However, achieving faster finite-time convergence comes at the expense of higher controller effort.

VI. Conclusion

In this article, an optimal control problem for finite-time stabilization for nonlinear discrete-time dynamical systems is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that stabilizes the closed-loop system in finite time. Specifically, we utilized a steady-state Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function satisfying a difference inequality involving fractional powers and a minimum operator. A numerical example was presented to show the efficacy of the proposed framework. Future extensions will focus on developing finite-time optimal controllers for stochastic discrete-time systems by building on the recent stochastic discrete-time stability analysis results in [18].

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