Systems & Control Letters 97 (2016) 83–90

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Lyapunov and converse Lyapunov theorems for stochastic semistability☆

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ARTICLE INFO

Article history:
Received 14 October 2015
Received in revised form
10 May 2016
Accepted 29 August 2016

Keywords:
Stochastic semistability
Lyapunov theory
Converse Lyapunov theorems

ABSTRACT

This paper develops Lyapunov and converse Lyapunov theorems for stochastic semistable nonlinear dynamical systems. Semistability is the property whereby the solutions of a stochastic dynamical system almost surely converge to (not necessarily isolated) Lyapunov stable in probability equilibrium points determined by the system initial conditions. Specifically, we provide necessary and sufficient Lyapunov conditions for stochastic semistability and show that stochastic semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

1. Introduction

The aim of this paper is to develop Lyapunov and converse Lyapunov theorems for stochastic semistability. Semistability is the property of a dynamical system whereby its trajectories converge to (not necessarily isolated) Lyapunov stable equilibria. Semistability, rather than asymptotic stability, is the appropriate notion of stability for systems having a continuum of equilibria. Examples of such systems arise in chemical kinetics [1], adaptive control [2], compartmental modeling [3], thermodynamics [4] and, more recently, collaborative control of a network of autonomous agents [5,6]. In all these examples, the system trajectories converge to limit points that depend continuously on the system initial conditions.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [2] show. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

For linear deterministic systems, semistability was originally defined in [7] and applied to matrix second-order systems in [8]. Refs. [2,9] extended the notion of semistability to nonlinear deterministic systems and gave Lyapunov results for semistability. Semistability was also addressed in [5,6] for consensus protocols in nonlinear dynamical networks, with [6] giving new Lyapunov theorems as well as the first converse Lyapunov theorem for semistability which holds with a smooth (i.e., infinitely differentiable) Lyapunov function.

In numerous applications where dynamical models are used to describe the behavior of natural and engineering systems, stochastic components and random disturbances are incorporated into the models. The stochastic aspects of the models are used to quantify system uncertainty as well as the dynamic relationships of sequences of random events between system–environment interactions. For example, stochastic modeling can be used to capture communication uncertainty between agents in a network, wherein the evolution of each link of the random network communication topology follows a Markov process. In this case, the development of almost sure consensus of multiagent systems with nonlinear stochastic dynamics under distributed nonlinear consensus protocols is necessary. And from a practical viewpoint, it is not sufficient to only guarantee that the network almost surely converges to a state of consensus since steady-state convergence

☆ This work was supported in part by the Air Force Office of Scientific Research under Grant FA9550-16-1-0100.

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http://dx.doi.org/10.1016/j.sysconle.2016.08.010
0167-6911/

is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable in probability, and consequently, stochastically semistable.

Using a notion of stochastic semistability, almost sure consensus of multiagent systems under distributed nonlinear protocols over random networks is addressed in [10]. The authors in [10] consider stochastic systems driven by a discrete-valued, right-continuous strong Markov excitation process. In this paper, we extend the notion of semistability to nonlinear stochastic systems involving Markov diffusion processes that have a continuum of equilibrium solutions. In particular, we develop almost sure convergence and stochastic Lyapunov stability properties to address almost sure semistability requiring the trajectories of a nonlinear stochastic dynamical system to converge almost surely to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable. Furthermore, we provide necessary and sufficient Lyapunov conditions for semistability and show that semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and develop mathematical preliminaries necessary for developing the results in this paper. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, and $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, A^{T} for the transpose of the matrix A, and I_n or I for the $n\times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D}\subseteq\mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S}\subseteq\mathbb{R}^n$.

We model a stochastic dynamical system $\mathcal{G}$ generating a stochastic process $x:\overline{\mathbb{R}}_+\times\Omega\to\mathcal{D}$ on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra of subsets of Ω, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega)=1$ [11]. We equip the probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a standard d-dimensional Wiener process $w(t)$ up to time t inclusively and satisfying $\mathcal{F}_\tau\subset\mathcal{F}_t$, $0\leq\tau<t$, such that $\{\omega\in\Omega: x(t,\omega)\in\mathcal{B}\}\in\mathcal{F}_t$, $t\geq 0$, for all Borel sets $\mathcal{B}\subset\mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here we use the notation $x(t)$ to represent the stochastic process $x(t,\omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega,\mathcal{F},\mathbb{P})$ over the semi-infinite parameter space $[0,\infty)$ by $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, and $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n\times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega,\mathcal{F},\mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely.

Given $x\in\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$, $\{x=0\}$ denotes the set $\{\omega\in\Omega: x(t,\omega)=0\}$, and so on. Given $x\in\mathcal{L}^0(\Omega,\mathcal{F},\mathbb{P})$ and $\mathcal{E}\in\mathcal{F}$, we say x is nonzero on $\mathcal{E}$ if $\mathbb{P}(\{x=0\}\cap\mathcal{E})=0$. Furthermore, given $x\in\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ and a σ-algebra $\mathcal{E}\subseteq\mathcal{F}$, $\mathbb{E}^{\mathbb{P}}[x]$ and $\mathbb{E}^{\mathbb{P}}[x|\mathcal{E}]$ denote, respectively, the expectation of the random variable x and the conditional expectation of x given $\mathcal{E}$, with all moments taken under the measure $\mathbb{P}$. Here, for simplicity of exposition, we omit the symbol $\mathbb{P}$ in denoting expectation, and similarly for conditional expectation. Specifically, we denote the expectation with respect to the probability space $(\Omega,\mathcal{F},\mathbb{P})$ by $\mathbb{E}[\ \cdot\ ]$, and similarly for conditional expectation.

Finally, we write $\mathrm{tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x)\triangleq\frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x)\triangleq\frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x\in\mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n\triangleq\{x:\Omega\to\mathbb{R}^n:\mathbb{E}[x^{\mathrm{T}}x]<\infty\}$. For an open set $\mathcal{D}\subseteq\mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}}\triangleq\{x\in\mathcal{H}_n: x:\Omega\to\mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0\in\mathbb{R}^n$, $\mathcal{H}_n^{x_0}\triangleq\{x\in\mathcal{H}_n: x\overset{\text{a.s.}}{=}x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V:\mathcal{D}\to\mathbb{R}$ that are two-times continuously differentiable with respect to $x\in\mathcal{D}\subseteq\mathbb{R}^n$.

Definition 2.1 (*[12]*). Let $(S,\mathfrak{S})$ and $(T,\mathfrak{T})$ be measurable spaces, and let $\mu: S\times\mathfrak{T}\to\overline{\mathbb{R}}_+$. If the function $\mu(s,B)$ is $\mathfrak{S}$-measurable in $s\in S$ for a fixed $B\in\mathfrak{T}$ and $\mu(s,B)$ is a probability measure in $B\in\mathfrak{T}$ for a fixed $s\in S$, then μ is called a *(probability) kernel* from S to T. Furthermore, for $s\leq t$, the function $\mu_{s,t}:S\times\mathfrak{S}\to\mathbb{R}$ is called a *regular conditional probability measure* if $\mu_{s,t}(\cdot,\mathfrak{S})$ is measurable, $\mu_{s,t}(S,\cdot)$ is a probability measure, and $\mu_{s,t}(\cdot,\cdot)$ satisfies

$$\begin{aligned}\mu_{s,t}(x(s),B)&=\mathbb{P}(x(t)\in B|x(s))\\&=\mathbb{P}(x(t)\in B|\mathcal{F}_s)\quad\text{a.s.},\ x(\cdot)\in\mathcal{H}_n.\end{aligned}\tag{1}$$

Any family of regular conditional probability measures $\{\mu_{s,t}\}_{s\leq t}$ satisfying the Chapman–Kolmogorov equation [11] is called a *semigroup of Markov kernels*. The Markov kernels are called *time homogeneous* if and only if $\mu_{s,t}=\mu_{0,t-s}$ holds for all $s\leq t$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t)=f(x(t))\mathrm{d}t+D(x(t))\mathrm{d}w(t),\quad x(0)\overset{\text{a.s.}}{=}x_0,\ t\in\mathcal{I}_{x(0)},\tag{2}$$

where, for every $t\in\mathcal{I}_{x_0}$, $x(t)\in\mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(0)\in\mathcal{H}_n^{x_0}$, $\mathcal{D}\subseteq\mathbb{R}^n$ is an open set with $0\in\mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, $x(0)$ is independent of $(w(t)-w(0))$, $t\geq 0$, $f:\mathcal{D}\to\mathbb{R}^n$ and $D:\mathcal{D}\to\mathbb{R}^{n\times d}$ are continuous, $\mathcal{E}\triangleq f^{-1}(0)\cap D^{-1}(0)\triangleq\{x\in\mathcal{D}: f(x)=0\text{ and }D(x)=0\}$ is nonempty, and $\mathcal{I}_{x(0)}=[0,\tau_{x(0)})$, $0\leq\tau_{x(0)}\leq\infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (2). An *equilibrium point* of (2) is a point $x_{\mathrm{e}}\in\mathbb{R}^n$ such that $f(x_{\mathrm{e}})=0$ and $D(x_{\mathrm{e}})=0$. It is easy to see that x_{e} is an equilibrium point of (2) if and only if the constant stochastic process $x(\cdot)\overset{\text{a.s.}}{=}x_{\mathrm{e}}$ is a solution of (2). We denote the set of equilibrium points of (2) by $\mathcal{E}\triangleq\{\omega\in\Omega: x(t,\omega)=x_{\mathrm{e}}\}=\{x_{\mathrm{e}}\in\mathcal{D}: f(x_{\mathrm{e}})=0\text{ and }D(x_{\mathrm{e}})=0\}$.

The filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x:[0,\tau]\times\Omega\to\mathcal{D}$ is said to be a *solution* of (2) on the time interval $[0,\tau]$ with initial condition $x(0)\overset{\text{a.s.}}{=}x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, $f\in\mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, $D\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, and

$$\begin{aligned}x(t)=x_0&+\int_0^t f(x(\sigma))\mathrm{d}\sigma\\&+\int_0^t D(x(\sigma))\mathrm{d}w(\sigma)\quad\text{a.s.},\ t\in[0,\tau],\end{aligned}\tag{3}$$

where the integrals in (3) are Itô integrals.

Note that for each fixed $t \geq 0$, the random variable $\omega \mapsto x(t, \omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t, \omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq 0$. A solution $t \mapsto x(t)$ of (2) is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (2) exist on $[0, \infty)$, and hence, we assume (2) is *forward complete*. Sufficient conditions for forward completeness of (2) are given in [13].

Furthermore, we assume that $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\|, \quad x, y \in \mathcal{D}\backslash\{0\}, \tag{4}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2), \quad x \in \mathcal{D}\backslash\{0\}, \tag{5}$$

for some Lipschitz constant $L > 0$, and hence, since $x(0) \in \mathcal{H}_n^{\mathcal{D}}$ and $x(0)$ is independent of $(w(t) - w(0))$, $t \geq 0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ of (2) forward in time for all initial conditions in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}}\backslash\{0\}$ there exists $\tau_x > 0$ such that, if $x_1 : [0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (2); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, with continuous sample paths almost surely, and both solve (2), then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_1(t) = x_2(t),\ 0 \leq t \leq \tau_x\big) = 1$.

A weaker sufficient condition for the existence of a unique solution to (2) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (4) without the growth condition (5) is given in [14]. Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by ([15, Thm. 3.4], [11, Thm. 9.2.8])

$$\mathbb{P}(x(t) \in B | x(t_0) \overset{\text{a.s.}}{=} x_0) = \mathbb{P}(0, x_0, t - t_0, B), \quad x_0 \in \mathbb{R}^n, \tag{6}$$

for all $t \geq t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(\sigma, x, t, \mathcal{B})$, $t \geq \sigma$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Finally, we say that the dynamical system (2) is *convergent in probability* with respect to the closed set $\mathcal{H}_n^{\mathcal{D}_c} \subseteq \mathcal{H}_n^{\mathcal{D}}$ if and only if the pointwise $\lim_{t\to\infty} s(t, x)$ exists for every $x \in \mathcal{H}_n^{\mathcal{D}_c}$.

Definition 2.2. A point $p \in \mathcal{D}$ is a *limit point* of the trajectory $s(\cdot, x)$ of (2) if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x) \overset{\text{a.s.}}{\to} p$ as $n \to \infty$. The set of all limit points of $s(t, x)$, $t \geq 0$, is the *limit set* $\omega(x)$ of $s(\cdot, x)$ of (2).

Definition 2.3 (*[16, Def. 7.7]*). Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) \overset{\text{a.s.}}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t\to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D}, \tag{7}$$

where $\mathbb{E}^{x_0}$ denotes the expectation with respect to the probability measure $\mathbb{P}^{x_0}(x(t) \in \mathcal{B}) \triangleq \mathbb{P}(0, x_0, t, \mathcal{B})$.

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq 0$, satisfies (2), then the limit in (7) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq 0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (2) and is given by [16, Thm. 7.9]

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D}. \tag{8}$$

Definition 2.4 (*[17]*). An open set $\mathcal{D} \subset \mathbb{R}^n$ is said to be *invariant with respect to* (2) if, for all $x_0 \in \mathcal{D}$, $\mathbb{P}(x(t) \in \mathcal{D}) = 1$, $t \geq 0$.

The following definition introduces several notions of stability in probability.

Definition 2.5 (*[18]*).

(i) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is *Lyapunov stable in probability* if, for every $\varepsilon > 0$ and $\rho > 0$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$,

$$\mathbb{P}^{x_0}\left(\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \varepsilon\right) \leq \rho. \tag{9}$$

(ii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is *asymptotically stable in probability* if it is Lyapunov stable in probability and there exist $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$, then

$$\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x(t) - x_{\mathrm{e}}\| = 0\right) = 1. \tag{10}$$

(iii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and (10) holds for all $x_0 \in \mathbb{R}^n$.

The following lemma gives an equivalent characterization of stability in probability in terms of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions. For the definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions see [19, p.162].

Lemma 2.1. (i) *The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is Lyapunov stable in probability if and only if for every $\rho > 0$ there exist a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ and a constant $c = c(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_c(x_{\mathrm{e}})$,*

$$\mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \alpha_\rho(\|x_0 - x_{\mathrm{e}}\|)\right] \leq \rho. \tag{11}$$

(ii) *The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is asymptotically stable in probability if and only if for every $\rho > 0$ there exist a class $\mathcal{KL}$ function $\beta_\rho(\cdot, \cdot)$ and a constant $c = c(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_c(x_{\mathrm{e}})$,*

$$\mathbb{P}^{x_0}\left[\|x(t) - x_{\mathrm{e}}\| > \beta_\rho(\|x_0 - x_{\mathrm{e}}\|, t)\right] \leq \rho, \quad t \geq 0. \tag{12}$$

(iii) *The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (2) is globally asymptotically stable in probability if and only if (12) is satisfied for every initial condition $x_0 \in \mathbb{R}^n$.*

Proof. (i) Suppose there exist a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ and a constant $c = c(\rho) > 0$ such that, for every $\rho > 0$ and $x_0 \in \mathcal{B}_c(x_{\mathrm{e}})$,

$$\mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \alpha_\rho(\|x_0 - x_{\mathrm{e}}\|)\right] \leq \rho. \tag{13}$$

Now, given $\varepsilon > 0$, let $\delta(\rho, \varepsilon) = \min\{c(\rho), \alpha_\rho^{-1}(\varepsilon)\}$. Then, for $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$,

$$\begin{aligned}
&\mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \alpha_\rho(\|x_0 - x_{\mathrm{e}}\|)\right] \\
&\quad\geq \mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \alpha_\rho(\delta)\right] \\
&\quad\geq \mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \alpha_\rho(\alpha_\rho^{-1}(\varepsilon))\right] \\
&\quad\geq \mathbb{P}^{x_0}\left[\sup_{t\geq 0} \|x(t) - x_{\mathrm{e}}\| > \varepsilon\right].
\end{aligned}$$

Therefore, for every given $\varepsilon > 0$ and $\rho > 0$, there exists $\delta > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$,

$$\mathbb{P}^{x_0}\left[\sup_{t\geq 0}\|x(t)-x_e\| > \varepsilon\right] \leq \rho,$$

which proves that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is Lyapunov stable in probability.

Conversely, for every given ε and ρ, let $\bar{\delta}(\varepsilon,\rho)$ be the supremum of all admissible $\delta(\varepsilon,\rho)$. Note that the function $\delta(\cdot,\cdot)$ is positive and nondecreasing in its first argument, but not necessarily continuous. For every $\rho > 0$ choose a class $\mathcal{K}$ function $\gamma_\rho(r)$ such that $\gamma_\rho(r) \leq k\bar{\delta}(r,\rho)$, $0 < k < 1$. Let $c(\rho) = \lim_{r\to\infty}\gamma_\rho(r)$ and $\alpha_\rho(r) = \gamma_\rho^{-1}(r)$, and note that $\alpha_\rho(\cdot)$ is class $\mathcal{K}$ [20, Lemma 4.2]. Next, for every $\rho > 0$ and $x_0 \in \mathcal{B}_{c(\rho)}(x_e)$, let $\varepsilon = \alpha_\rho(\|x_0 - x_e\|)$. Then, $\|x_0 - x_e\| < \bar{\delta}(\varepsilon,\rho)$ and

$$\mathbb{P}^{x_0}\left[\sup_{t\geq 0}\|x(t)-x_e\| > \varepsilon = \alpha_\rho(\|x_0-x_e\|)\right] \leq \rho. \tag{14}$$

(ii) Suppose there exists a class $\mathcal{KL}$ function $\beta(r,s)$ such that (12) is satisfied. Then,

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \beta_\rho(\|x_0-x_e\|,0)\right] \leq \rho, \quad t \geq 0,$$

which implies that equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is Lyapunov stable in probability. Moreover, for $x_0 \in \mathcal{B}_{c(\rho)}(x_e)$, the solution to (2) satisfies

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \beta_\rho(\|c(\rho)\|,t)\right] \leq \rho, \quad t \geq 0.$$

Now, letting $t\to\infty$ yields $\mathbb{P}^{x_0}[\lim_{t\to\infty}\|x(t)-x_e\| > 0] \leq \rho$ for every $\rho > 0$, and hence, $\mathbb{P}^{x_0}[\lim_{t\to\infty}\|x(t)-x_e\| = 0] = 1$, which implies that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is asymptotically stable in probability.

Conversely, suppose that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is asymptotically stable in probability. In this case, for every $\rho > 0$ there exist a constant $c(\rho) > 0$ and a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ such that, for every $r \in (0, c(\rho)]$, the solution $x(t)$, $t \geq 0$, to (2) satisfies

$$\mathbb{P}^{x_0}\left[\sup_{t\geq 0}\|x(t)-x_e\| > \alpha_\rho(r)\right] \leq \mathbb{P}^{x_0}\left[\sup_{t\geq 0}\|x(t)-x_e\| > \alpha_\rho(\|x_0-x_e\|)\right] \leq \rho \tag{15}$$

for all $\|x_0 - x_e\| < r$. Moreover, given $\eta > 0$ there exists $T = T_\rho(\eta,r) \geq 0$ such that

$$\mathbb{P}^{x_0}\left[\sup_{t\geq T_\rho(\eta,r)}\|x(t)-x_e\| > \eta\right] \leq \rho.$$

Let $\overline{T}_\rho(\eta,r)$ be the infimum of all admissible $T_\rho(\eta,r)$ and note that $\overline{T}_\rho(\eta,r)$ is nonnegative and nonincreasing in η, nondecreasing in r, and $\overline{T}_\rho(\eta,r) = 0$ for all $\eta \geq \alpha(r)$. Now, let

$$W_{r,\rho}(\eta) = \frac{2}{\eta}\int_{\frac{\eta}{2}}^{\eta}\overline{T}_\rho(s,r)\mathrm{d}s + \frac{r}{\eta} \geq \overline{T}_\rho(\eta,r) + \frac{r}{\eta}$$

and note that $W_{r,\rho}(\eta)$ is positive and has the following properties: (i) For every fixed r and ρ, $W_{r,\rho}(\eta)$ is continuous, strictly decreasing, and $\lim_{\eta\to\infty}W_{r,\rho}(\eta) = 0$; and (ii) for every fixed η and ρ, $W_{r,\rho}(\eta)$ is strictly increasing in r.

Next, let $U_{r,\rho} = W_{r,\rho}^{-1}$ and note that $U_{r,\rho}$ satisfies properties (i) and (ii) of $W_{r,\rho}$, and $\overline{T}_\rho(U_{r,\rho}(\sigma),r) < W_{r,\rho}(U_{r,\rho}(\sigma)) = \sigma$. Therefore,

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > U_{r,\rho}(t)\right] \leq \rho, \quad t \geq 0, \tag{16}$$

for all $\|x_0\| < r$. Now, using (15) and (16) it follows that

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \sqrt{\alpha_\rho(\|x_0-x_e\|)U_{c(\rho),\rho}(t)}\right] \leq \rho, \quad \|x_0-x_e\| < c(\rho), \ t \geq 0.$$

Thus, inequality (12) is satisfied with $\beta_\rho(\|x_0-x_e\|,t) = \sqrt{\alpha_\rho(\|x_0-x_e\|)U_{c(\rho),\rho}(t)}$.

(iii) If (12) holds for all $x_0 \in \mathbb{R}^n$, then using similar arguments as in the proof of (ii) it can be shown that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ to (2) is globally asymptotically stable. Conversely, let the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ to (2) be globally asymptotically stable in probability. In this case, $\lim_{\varepsilon\to\infty}\bar{\delta}(\varepsilon,\rho) = \infty$, and hence, $\alpha_\rho(\cdot)$ can be chosen to belong to class $\mathcal{K}_\infty$; furthermore (14) holds for all $x_0 \in \mathbb{R}^n$. Moreover, (16) holds for every $r > 0$. Now, let

$$\psi_\rho(r,\sigma) \triangleq \min\left\{\alpha_\rho(r), \inf_{\gamma\in(r,\infty]}U_{\gamma,\rho}(\sigma)\right\}$$

and note that

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \psi_\rho(\|x_0-x_e\|,t)\right] \leq \rho, \quad t \geq 0, \ x_0 \in \mathbb{R}^n.$$

If $\psi(\cdot,\cdot)$ is not class $\mathcal{KL}$, then define

$$\phi_\rho(r,\sigma) \triangleq \int_r^{r+1}\psi_\rho(\lambda,\sigma)\mathrm{d}\lambda + \frac{r}{(r+1)(\sigma+1)}$$

and note that $\phi_\rho(r,\sigma)$ is positive and has the following properties: (i) For each fixed $\sigma \geq 0$, $\phi_\rho(r,\sigma)$ is continuous and strictly increasing in r; (ii) for each fixed $r \geq 0$, $\phi_\rho(r,\sigma)$ is strictly decreasing in σ and $\lim_{\sigma\to\infty}\phi_\rho(r,\sigma) = 0$; and (iii) $\phi_\rho(r,\sigma) \geq \psi_\rho(r,\sigma)$. Thus,

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \phi_\rho(\|x_0-x_e\|,t)\right] \leq \rho, \quad x_0 \in \mathbb{R}^n, \ t \geq 0, \tag{17}$$

and hence, using the global version of (14) it follows that

$$\mathbb{P}^{x_0}\left[\|x(t)-x_e\| > \sqrt{\alpha_\rho(\|x_0-x_e\|)\phi_\rho(\|x_0-x_e\|,t)}\right] \leq \rho, \quad x_0 \in \mathbb{R}^n, \ t \geq 0.$$

Thus, (12) is satisfied globally with $\beta_\rho(\|x_0-x_e\|,t) = \sqrt{\alpha_\rho(\|x_0-x_e\|)\phi_\rho(\|x_0-x_e\|,t)}$. □

The following proposition gives a sufficient condition for a trajectory of (2) to converge to a limit point. For this result, $\mathcal{D}_c \subseteq \mathcal{D}$ denotes a positively invariant set with respect to (2) and $s_t(\mathcal{H}_n^{\mathcal{D}_c})$ denotes the image of $\mathcal{H}_n^{\mathcal{D}_c} \subset \mathcal{H}_n^{\mathcal{D}}$ under the flow $s_t : \mathcal{H}_n^{\mathcal{D}_c} \to \mathcal{H}_n^{\mathcal{D}}$; that is, $s_t(\mathcal{H}_n^{\mathcal{D}_c}) \triangleq \{y : y = s_t(x_0) \text{ for some } x_0 \in \mathcal{H}_n^{\mathcal{D}_c}\}$.

Proposition 2.1. *Consider the nonlinear stochastic dynamical system (2) and let $x \in \mathcal{H}_n^{\mathcal{D}_c}$. If the limit set $\omega(x)$ of (2) contains a Lyapunov stable in probability equilibrium point y, then $y \stackrel{\text{a.s.}}{=} \lim_{t\to\infty}s(t,x)$, that is, $\omega(x) \stackrel{\text{a.s.}}{=} \{y\}$.*

Proof. Suppose $y \in \omega(x)$ is Lyapunov stable in probability and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ be an open neighborhood of y. Since y is Lyapunov stable in probability, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_c$ of y such that $s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ for every $t \geq 0$. Now, since $y \in \omega(x)$, it follows that there exists $\tau \geq 0$ such that $s(\tau,x) \in \mathcal{H}_n^{\mathcal{N}_\delta}$. Hence, $s(t+\tau,x) = s_t(s(\tau,x)) \in s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ for every $t > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ is arbitrary, it follows that $y \stackrel{\text{a.s.}}{=} \lim_{t\to\infty}s(t,x)$. Thus, $\lim_{n\to\infty}s(t_n,x) \stackrel{\text{a.s.}}{=} y$ for every sequence $\{t_n\}_{n=1}^\infty$, and hence, $\omega(x) \stackrel{\text{a.s.}}{=} \{y\}$. □

The following definition introduces the notion of stochastic semistability. For the statement of this definition define $\mathrm{dist}(x,\mathcal{E}) \triangleq \inf_{y\in\mathcal{E}}\|x-y\|$.

Definition 2.6. An equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (2) is *stochastically semistable* if the following statements hold.

(i) For every $\varepsilon > 0$ and $\rho > 0$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$, $\mathbb{P}^{x_0}\left[\sup_{0\le t<\infty} \|x(t) - x_e\| > \varepsilon\right] \le \rho$.
(ii) $\mathbb{P}^{x_0}\left[\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0\right] = 1$.

The dynamical system (2) is *stochastically semistable* if every equilibrium solution of (2) is stochastically semistable. Finally, the dynamical system (2) is *globally stochastically semistable* if it is stochastically semistable for every initial condition $x(0) \in \mathcal{H}_n$.

Remark 2.1. If $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ only satisfies (i) in Definition 2.6, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (2) is Lyapunov stable in probability.

Definition 2.7. The *domain of semistability* is the set of points $x_0 \in \mathcal{D}$ such that if $x(t)$, $t \ge 0$, is a solution to (2) with $x(0) \overset{\text{a.s.}}{=} x_0$, then $x(t)$ converges in probability to a Lyapunov stable in probability equilibrium point in $\mathcal{D}$.

Note that if (2) is stochastically semistable, then its domain of semistability contains the set of equilibria in its interior. Next, we present alternative equivalent characterizations for stochastic semistability of (2).

Proposition 2.2. *Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (2). Then the following statements are equivalent:*

(i) *$\mathcal{G}$ is stochastically semistable.*
(ii) *For every $x_e \in \mathcal{E}$ and $\rho > 0$, there exist class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha_\rho(\cdot)$ and $\beta_\rho(\cdot)$, respectively, and $\delta = \delta(x_e, \rho) > 0$ such that, if $x_0 \in \mathcal{B}_\delta(x_e)$, then*

$$\mathbb{P}^{x_0}\left[\sup_{0\le t<\infty} \|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right] \le \rho,$$

and $\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \beta_\rho(t)\right] \le \rho$, $t \ge 0$.
(iii) *For every $x_e \in \mathcal{E}$ and $\rho > 0$, there exist class $\mathcal{K}$ functions $\alpha_{1\rho}(\cdot)$ and $\alpha_{2\rho}(\cdot)$, a class $\mathcal{L}$ function $\beta_\rho(\cdot)$, and $\delta = \delta(x_e, \rho) > 0$ such that, if $x_0 \in \mathcal{B}_\delta(x_e)$, then*

$$\begin{aligned}&\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \alpha_{2\rho}(\|x_0 - x_e\|)\beta_\rho(t)\right] \\ &\quad \le \mathbb{P}^{x_0}\left[\alpha_{1\rho}(\|x(t) - x_e\|) > \alpha_{2\rho}(\|x_0 - x_e\|)\right] \le \rho, \quad t \ge 0.\end{aligned}$$

Proof. To show that (i) implies (ii), suppose (2) is stochastically semistable and let $x_e \in \mathcal{E}$. It follows from Lemma 2.1 that for every $\rho > 0$ there exists $\delta = \delta(x_e, \rho) > 0$ and a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ such that if $\|x_0 - x_e\| \le \delta$, then $\mathbb{P}^{x_0}\left[\sup_{0\le t<\infty} \|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right] \le \rho$, $t \ge 0$. Without loss of generality, we may assume that δ is such that $\overline{\mathcal{B}_\delta(x_e)}$ is contained in the domain of semistability of (2). Hence, for every $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$, $\lim_{t\to\infty} x(t) \overset{\text{a.s.}}{=} x^* \in \mathcal{E}$ and, consequently, $\mathbb{P}^{x_0}\left[\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0\right] = 1$.

For every $\varepsilon > 0$, $\rho > 0$, and $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$, define $T_{x_0}(\varepsilon, \rho)$ to be the infimum of T with the property that $\mathbb{P}^{x_0}\left[\sup_{t\ge T} \operatorname{dist}(x(t), \mathcal{E}) > \varepsilon\right] \le \rho$, that is,

$$T_{x_0}(\varepsilon, \rho) \triangleq \inf\left\{T : \mathbb{P}^{x_0}\left[\sup_{t\ge T} \operatorname{dist}(x(t), \mathcal{E}) > \varepsilon\right] \le \rho\right\}.$$

For each $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$ and ρ, the function $T_{x_0}(\varepsilon, \rho)$ is nonnegative and nonincreasing in ε, and $T_{x_0}(\varepsilon, \rho) = 0$ for sufficiently large ε.

Next, let $T(\varepsilon, \rho) \triangleq \sup\{T_{x_0}(\varepsilon, \rho) : x_0 \in \overline{\mathcal{B}_\delta(x_e)}\}$. We claim that T is well defined. To show this, consider $\varepsilon > 0$, $\rho > 0$, and $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$. Since $\mathbb{P}^{x_0}\left[\sup_{t\ge T_{x_0}(\varepsilon,\rho)} \operatorname{dist}(x(t), \mathcal{E}) > \varepsilon\right] \le \rho$, it follows from the sample continuity of s that, for every $\varepsilon > 0$ and $\rho > 0$, there exists an open neighborhood $\mathcal{U}$ of x_0 such that $\mathbb{P}^{x_0}\left[\sup_{t\ge T_z(\varepsilon,\rho)} \operatorname{dist}(s(t, z), \mathcal{E}) > \varepsilon\right] \le \rho$ for every $z \in \mathcal{U}$. Hence, $\limsup_{z\to x_0} T_z(\varepsilon, \rho) \le T_{x_0}(\varepsilon, \rho)$ implying that the function $x_0 \mapsto T_{x_0}(\varepsilon, \rho)$ is upper semicontinuous at the arbitrarily chosen point x_0, and hence on $\overline{\mathcal{B}_\delta(x_e)}$. Since an upper semicontinuous function defined on a compact set achieves its supremum, it follows that $T(\varepsilon, \rho)$ is well defined. The function $T(\cdot)$ is the pointwise supremum of a collection of nonnegative and nonincreasing functions, and hence is nonnegative and nonincreasing. Moreover, $T(\varepsilon, \rho) = 0$ for every $\varepsilon > \max\{\alpha_\rho(\|x_0 - x_e\|) : x_0 \in \overline{\mathcal{B}_\delta(x_e)}\}$.

Let $\psi_\rho(\varepsilon) \triangleq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} T(\sigma, \rho)\mathrm{d}\sigma + \frac{1}{\varepsilon} \ge T(\varepsilon, \rho) + \frac{1}{\varepsilon}$. The function $\psi_\rho(\varepsilon)$ is positive, continuous, strictly decreasing, and $\psi_\rho(\varepsilon) \to 0$ as $\varepsilon \to \infty$. Choose $\beta_\rho(\cdot) = \psi^{-1}(\cdot)$. Then $\beta_\rho(\cdot)$ is positive, continuous, strictly decreasing, and $\lim_{\sigma\to\infty} \beta_\rho(\sigma) = 0$. Furthermore, $T(\beta_\rho(\sigma), \rho) < \psi_\rho(\beta_\rho(\sigma)) = \sigma$. Hence, $\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \beta_\rho(t)\right] \le \rho$, $t \ge 0$.

Next, to show that (ii) implies (iii), suppose (ii) holds and let $x_e \in \mathcal{E}$. Then it follows from (i) of Lemma 2.1 that x_e is Lyapunov stable in probability. For every $\rho > 0$, choosing x_0 sufficiently close to x_e, it follows from the inequality $\mathbb{P}^{x_0}\left[\sup_{t\ge 0} \|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right] \le \rho$ that trajectories of (2) starting sufficiently close to x_e are bounded, and hence, the positive limit set of (2) is nonempty. Since $\mathbb{P}^{x_0}\left[\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0\right] = 1$, it follows that the positive limit set is contained in $\mathcal{E}$. Now, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} x(t) \overset{\text{a.s.}}{=} x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. If $x^* = x_e$, then it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\hat{\beta}_\rho(\cdot)$ such that $\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \hat{\beta}_\rho(t)\right] \le \mathbb{P}^{x_0}\left[\|x(t) - x_e\| > \hat{\beta}_\rho(t)\right] \le \rho$ for every x_0 satisfying $\|x_0 - x_e\| < \delta$ and $t \ge 0$. Hence, $\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \sqrt{\|x(t) - x_e\|}\sqrt{\hat{\beta}_\rho(t)}\right] \le \rho$, $t \ge 0$. Next, consider the case where $x^* \ne x_e$ and let $\alpha_{1\rho}(\cdot)$ be a class $\mathcal{K}$ function. In this case, note that $\mathbb{P}^{x_0}\left[\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E})/\alpha_{1\rho}(\|x(t) - x_e\|) = 0\right] = 1$, and hence, it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\beta_\rho(\cdot)$ such that $\mathbb{P}^{x_0}\left[\operatorname{dist}(x(t), \mathcal{E}) > \alpha_{1\rho}(\|x(t) - x_e\|)\beta_\rho(t)\right] \le \rho$, $t \ge 0$. Finally, note that $\alpha_{1\rho} \circ \alpha_\rho$ is of class $\mathcal{K}$ (by [20, Lemma 4.2]), and hence, (iii) follows immediately.

Finally, to show that (iii) implies (i), suppose (iii) holds and let $x_e \in \mathcal{E}$. Then it follows that for every $\rho > 0$, $\mathbb{P}^{x_0}\left[\alpha_{1\rho}(\|x(t) - x_e\|) > \alpha_{2\rho}(\|x(0) - x_e\|)\right] \le \rho$, $t \ge 0$, that is, $\mathbb{P}^{x_0}[\|x(t) - x_e\| > \alpha_\rho(\|x(0) - x_e\|)] \le \rho$, where $t \ge 0$ and $\alpha_\rho = {\alpha_{1\rho}}^{-1} \circ \alpha_{2\rho}$ is of class $\mathcal{K}$ (by [20, Lemma 4.2]). It now follows from (i) of Lemma 2.1 that x_e is Lyapunov stable in probability. Since x_e was chosen arbitrarily, it follows that every equilibrium point is Lyapunov stable in probability. Furthermore, $\mathbb{P}^{x_0}\left[\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0\right] = 1$. Choosing x_0 sufficiently close to x_e, it follows from the inequality $\mathbb{P}^{x_0}\left[\|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right] \le \rho$, $t \ge 0$, that trajectories of (2) starting sufficiently close to x_e are bounded, and hence, the positive limit set of (2) is nonempty. Since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} x(t) \overset{\text{a.s.}}{=} x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 2.6, (2) is stochastically semistable. □

3. Stochastic semistability of nonlinear dynamical systems

In this section, we develop necessary and sufficient conditions for stochastic semistability. First, we present a sufficient condition for stochastic semistability.

Theorem 3.1. *Consider the nonlinear stochastic dynamical system (2). Let $\mathcal{Q}$ be an open neighborhood of $\mathcal{E}$ and assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ such that*

$$V'(x)f(x) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{Q}\backslash\mathcal{E}. \tag{18}$$

If every equilibrium point of (2) is Lyapunov stable in probability, then (2) is stochastically semistable.

Proof. Since every equilibrium point of (2) is Lyapunov stable in probability by assumption, for every $z \in \mathcal{E}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z)$ is bounded and contained in $\mathcal{Q}$. The set $\mathcal{V} \triangleq \bigcup_{z\in\mathcal{E}} \mathcal{V}_z$ is an open neighborhood of $\mathcal{E}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}$ so that there exists $z \in \mathcal{E}$ such that $x \in \mathcal{V}_z$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z}, t \geq 0$. Since $\mathcal{V}_z$ is bounded and invariant with respect to the solution of (2), it follows that $\mathcal{V}$ is invariant with respect to the solution of (2). Furthermore, it follows from (18) that $\mathcal{L}V(s(t,x)) < 0, t \geq 0$, and hence, since $\mathcal{V}$ is bounded it follows from [17, Cor. 4.1] that $\lim_{t\to\infty} \mathcal{L}V(s(t,x)) \overset{\text{a.s.}}{=} 0$. It is easy to see that $\mathcal{L}V(x) \neq 0$ by assumption and $\mathcal{L}V(x_{\mathrm{e}}) = 0, x_{\mathrm{e}} \in \mathcal{E}$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{E}$ as $t \to \infty$, which implies that $\mathbb{P}^x[\lim_{t\to\infty} \operatorname{dist}(s(t,x),\mathcal{E}) = 0] = 1$. Finally, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 2.6, (2) is semistable. □

Next, we present a slightly more general theorem for stochastic semistability wherein we do not assume that all points in $\mathcal{L}V^{-1}(0)$ are Lyapunov stable in probability but rather we assume that all points in $(\eta \circ V)^{-1}(0)$ are Lyapunov stable in probability for some continuous function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$.

Theorem 3.2. *Consider the nonlinear stochastic dynamical system (2) and let $\mathcal{Q}$ be an open neighborhood of $\mathcal{E}$. Assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ and a continuous function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that*

$$V'(x)f(x) + \frac{1}{2}\operatorname{tr} D^{\mathrm{T}}(x)V''(x)D(x) \leq -\eta(V(x)), \quad x \in \mathcal{Q}. \tag{19}$$

If every point in the set $\mathcal{M} \triangleq \{x \in \mathcal{Q} : \eta(V(x)) = 0\}$ is Lyapunov stable in probability, then (2) is stochastically semistable.

Proof. Since, by assumption, (2) is Lyapunov stable in probability for all $z \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z)$ is bounded and contained in $\mathcal{Q}$. The set $\mathcal{V} \triangleq \bigcup_{z\in\mathcal{M}} \mathcal{V}_z$ is an open neighborhood of $\mathcal{M}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}$ so that there exists $z \in \mathcal{M}$ such that $x \in \mathcal{V}_z$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z}, t \geq 0$. Since $\mathcal{V}_z$ is bounded it follows that $\mathcal{V}$ is invariant with respect to the solution of (2). Furthermore, it follows from (19) that $\mathcal{L}V(s(t,x)) \leq -\eta(V(s(t,x))), t \geq 0$, and hence, since $\mathcal{V}$ is bounded and invariant with respect to the solution of (2), it follows from [17, Cor. 4.2] that $\lim_{t\to\infty} \eta(V(s(t,x))) \overset{\text{a.s.}}{=} 0$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{M}$ as $t \to \infty$, which implies that $\mathbb{P}^x[\lim_{t\to\infty} \operatorname{dist}(s(t,x),\mathcal{M}) = 0] = 1$. Finally, since every point in $\mathcal{M}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$, where $x^* \in \mathcal{M}$ is Lyapunov stable in probability. Hence, by definition, (2) is semistable. □

Example 3.1. Consider the nonlinear stochastic dynamical system on $\mathcal{H}_2$ given by

$$\mathrm{d}x_1(t) = [\sigma_{12}(x_2(t)) - \sigma_{21}(x_1(t))]\mathrm{d}t + \gamma(x_2(t) - x_1(t))\mathrm{d}w(t),$$
$$x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{20}$$

$$\mathrm{d}x_2(t) = [\sigma_{21}(x_1(t)) - \sigma_{12}(x_2(t))]\mathrm{d}t + \gamma(x_1(t) - x_2(t))\mathrm{d}w(t),$$
$$x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{21}$$

where $\sigma_{ij}(\cdot), i,j = 1,2, i \neq j$, are Lipschitz continuous and $\gamma > 0$. Eqs. (20) and (21) represent the collective dynamics of two agents which interact by exchanging information. The information states of the agents are described by the scalar random variables x_1 and x_2. The unity coefficients scaling $\sigma_{ij}(\cdot), i,j \in \{1,2\}, i \neq j$, appearing in (20) and (21) represent the topology of the information exchange between the agents. More specifically, given $i,j \in \{1,2\}, i \neq j$, a coefficient of 1 denotes that agent j receives information from agent i, and a coefficient of zero denotes that agents i and j are disconnected, and hence, cannot share any information. The communication topology between the agents can be represented by a graph $\mathfrak{G}$ having two nodes such that $\mathfrak{G}$ has a directed edge from node i to node j if and only if agent j can receive information from agent i. Since the coefficients scaling $\sigma_{ij}(\cdot), i,j \in \{1,2\}, i \neq j$, are constants, the communication topology is fixed. Furthermore, note that the directed communication graph $\mathfrak{G}$ is *weakly connected* since the underlying undirected graph is connected; that is, every agent receives information from, or delivers information to, at least one other agent.

Note that (20) and (21) can be cast in the form of (2) with

$$f(x) = \begin{bmatrix} \sigma_{12}(x_2) - \sigma_{21}(x_1) \\ \sigma_{21}(x_1) - \sigma_{12}(x_2) \end{bmatrix}, \qquad D(x) = \begin{bmatrix} \gamma(x_2 - x_1) \\ \gamma(x_1 - x_2) \end{bmatrix},$$

where the stochastic term $D(x)\mathrm{d}w$ represents probabilistic variations in the information transfer between the agents. Furthermore, note that since

$$\mathbf{e}_2^{\mathrm{T}}\mathrm{d}x(t) = \mathbf{e}_2^{\mathrm{T}}f(x(t))\mathrm{d}t + \mathbf{e}_2^{\mathrm{T}}D(x(t))\mathrm{d}w(t) = 0, \quad x(0) \overset{\text{a.s.}}{=} x_0, \ t \geq 0,$$

where $\mathbf{e}_2 \triangleq [1\ 1]^{\mathrm{T}}$, it follows that $\mathrm{d}x_1(t) + \mathrm{d}x_2(t) = 0$, which implies that the total information is conserved.

In this example, we use Theorem 3.1 to analyze the collective behavior of (20) and (21). Specifically, we are interested in the consensus (i.e., state equipartitioning) behavior of the agents. For this purpose, we make the assumptions $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j, i \neq j$, and $(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -\gamma^2(x_1 - x_2)$ for $i,j \in \{1,2\}$. The first assumption implies that if the information (or energies) in the connected agents i and j are equal, then information exchange between the agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. The second assumption implies that information flows from information rich agents to information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. It is important to note that due to the stochastic term $D(x)\mathrm{d}w$ capturing probabilistic variations in the information transfer between the agents, the second assumption requires that the scaled net information flow $(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]$ is bounded by the negative intensity of the diffusion coefficient$-\frac{1}{2}\operatorname{tr} D(x)D^{\mathrm{T}}(x)$.

To show that (20) and (21) are stochastically semistable, note that $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$ and consider the Lyapunov function candidate $V(x_1,x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that

$$\begin{aligned}
\mathcal{L}V(x_1,x_2) &= (x_1 - \alpha)[\sigma_{12}(x_2) - \sigma_{21}(x_1)] \\
&\quad + (x_2 - \alpha)[\sigma_{21}(x_1)\sigma_{12}(x_2)] \\
&\quad + \frac{1}{2}[(\gamma(x_2 - x_1))^2 + (\gamma(x_1 - x_2))^2] \\
&= x_1[\sigma_{12}(x_2) - \sigma_{21}(x_1)] \\
&\quad + x_2[\sigma_{21}(x_1) - \sigma_{12}(x_2)] + (\gamma(x_1 - x_2))^2 \\
&= (x_1 - x_2)[\sigma_{12}(x_2) - \sigma_{21}(x_1) + \gamma^2(x_1 - x_2)] \\
&\leq 0, \quad (x_1,x_2) \in \mathbb{R} \times \mathbb{R},
\end{aligned} \tag{22}$$

which implies that $x_1 = x_2 = \alpha$ is Lyapunov stable in probability.

Next, it is easy to see that $\mathcal{L}V(x_1,x_2) \neq 0$ when $x_1 \neq x_2$, and hence, $\mathcal{L}V(x_1,x_2) < 0$, $(x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{E}$. Therefore, it follows from Theorem 3.1 that $x_1 = x_2 = \alpha$ is stochastically semistable for all $\alpha \in \mathbb{R}$. Furthermore, $x(t) \xrightarrow{\text{a.s.}} \frac{1}{2}\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}}x(0) \overset{\text{a.s.}}{=} \frac{1}{2}[x_1(0)+x_2(0)]\mathbf{e}_2$ as $t \to \infty$. Finally, note that an identical assertion holds for the collective dynamics of n agents with a connected undirected communication graph topology. △

Finally, we provide a converse Lyapunov theorem for stochastic semistability. For this result, recall that $\mathcal{L}V(x_e) = 0$ for every $x_e \in \mathcal{E}$. Also note that it follows from (7) that $\mathcal{L}V(x) = \mathcal{L}V(s(0,x))$.

Theorem 3.3. *Consider the nonlinear stochastic dynamical system (2). Suppose (2) is stochastically semistable with the domain of semistability $\mathcal{D}_0$. Then there exist a continuous nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that (i) $V(x) = 0$, $x \in \mathcal{E}$, ii) $V(x) \geq \alpha(\mathrm{dist}(x,\mathcal{E}))$, $x \in \mathcal{D}_0$, and (iii) $\mathcal{L}V(x) < 0$, $x \in \mathcal{D}_0\setminus\mathcal{E}$.*

Proof. Define the function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathbb{E}^x\left[\mathrm{dist}(s(t,x),\mathcal{E})\right]\right\}, \quad x \in \mathcal{D}_0. \tag{23}$$

Note that $V(\cdot)$ is well defined since (2) is stochastically semistable. Clearly, (i) holds. Furthermore, since $V(x) \geq \mathrm{dist}(x,\mathcal{E})$, $x \in \mathcal{D}_0$, it follows that (ii) holds with $\alpha(r) = r$.

To show that $V(\cdot)$ is continuous on $\mathcal{D}_0\setminus\mathcal{E}$, define $T : \mathcal{D}_0\setminus\mathcal{E} \to [0,\infty)$ by $T(z) \triangleq \inf\{h : \mathbb{E}^z[\mathrm{dist}(s(t,z),\mathcal{E})] < \mathrm{dist}(z,\mathcal{E})/2$ for all $t \geq h > 0\}$, and denote

$$\mathcal{W}_\varepsilon \triangleq \left\{x \in \mathcal{D}_0 : \mathbb{P}^x\left[\sup_{t\geq 0}\mathrm{dist}(s(t,x),\mathcal{E}) > \varepsilon\right] = 0\right\}. \tag{24}$$

Note that $\mathcal{W}_\varepsilon \supset \mathcal{E}$ is open and positively invariant, and contains an open neighborhood of $\mathcal{E}$. Consider $z \in \mathcal{D}_0\setminus\mathcal{E}$ and define $\lambda \triangleq \mathrm{dist}(z,\mathcal{E}) > 0$. Then it follows from stochastic semistability of (2) that there exists $h > 0$ such that $s(h,z) \in \mathcal{H}_n^{\mathcal{W}_{\lambda/2}}$. Consequently, $s(h+t,z) \in \mathcal{H}_n^{\mathcal{W}_{\lambda/2}}$ for all $t \geq 0$, and hence, it follows that $T(z)$ is well defined. Since $\mathcal{W}_{\lambda/2}$ is open, there exists a neighborhood $\mathcal{B}_\sigma(s(T(z),z)) \subset \mathcal{H}_n^{\mathcal{W}_{\lambda/2}}$. Hence, $\mathcal{N} \subset \mathcal{D}_0$ is a neighborhood of z such that $s_{T(z)}(\mathcal{H}_n^{\mathcal{N}}) \triangleq \mathcal{B}_\sigma(s(T(z),z))$.

Next, choose $\eta > 0$ such that $\eta < \lambda/2$ and $\mathcal{B}_\eta(z) \subset \mathcal{N}$. Then, for every $t > T(z)$ and $y \in \mathcal{B}_\eta(z)$, $[(1+2t)/(1+t)]\mathbb{E}^y[\mathrm{dist}(s(t,y),\mathcal{E})] \leq 2\mathbb{E}^y[\mathrm{dist}(s(t,y),\mathcal{E})] \leq \lambda$. Therefore, for each $y \in \mathcal{B}_\eta(z)$,

$$\begin{aligned}
V(z) - V(y) &= \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathbb{E}^z\left[\mathrm{dist}(s(t,z),\mathcal{E})\right]\right\} \\
&\quad - \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathbb{E}^y\left[\mathrm{dist}(s(t,y),\mathcal{E})\right]\right\} \\
&= \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}^z\left[\mathrm{dist}(s(t,z),\mathcal{E})\right]\right\} \\
&\quad - \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}^y\left[\mathrm{dist}(s(t,y),\mathcal{E})\right]\right\}.
\end{aligned} \tag{25}$$

Hence,

$$\begin{aligned}
&|V(z)-V(y)| \\
&\quad\leq \sup_{0\leq t\leq T(z)}\left|\frac{1+2t}{1+t}\left(\mathbb{E}^z\left[\mathrm{dist}(s(t,z),\mathcal{E})\right] - \mathbb{E}^y\left[\mathrm{dist}(s(t,y),\mathcal{E})\right]\right)\right| \\
&\quad\leq 2\sup_{0\leq t\leq T(z)}\left|\mathbb{E}^z\left[\mathrm{dist}(s(t,z),\mathcal{E})\right] - \mathbb{E}^y\left[\mathrm{dist}(s(t,y),\mathcal{E})\right]\right| \\
&\quad\leq 2\sup_{0\leq t\leq T(z)}\mathbb{E}\left[\mathrm{dist}(s(t,z),s(t,y))\right], \quad z \in \mathcal{D}_0\setminus\mathcal{E},\ y \in \mathcal{B}_\eta(z).
\end{aligned} \tag{26}$$

Now, since $f(\cdot)$ and $D(\cdot)$ satisfy (4) and (5), it follows from continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [11, Thm. 7.3.1] and (26) that $V(\cdot)$ is continuous on $\mathcal{D}_0\setminus\mathcal{E}$.

To show that $V(\cdot)$ is continuous on $\mathcal{E}$, consider $x_e \in \mathcal{E}$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}_0\setminus\mathcal{E}$ that converges to x_e. Since x_e is Lyapunov stable in probability, it follows that $x(t) \overset{\text{a.s.}}{\equiv} x_e$ is the unique solution to (2) with $x(0) \overset{\text{a.s.}}{=} x_e$. By continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [11, Thm. 7.3.1], $s(t,x_n) \xrightarrow{\text{a.s.}} s(t,x_e) \overset{\text{a.s.}}{=} x_e$ as $n \to \infty$, $t \geq 0$.

Let $\varepsilon > 0$ and note that it follows from (ii) of Proposition 2.2 that there exists $\delta = \delta(x_e) > 0$ such that for every solution of (2) in $\mathcal{B}_\delta(x_e)$ there exists $\hat{T} = \hat{T}(x_e,\varepsilon) > 0$ such that $s_t(\mathcal{H}_n^{\mathcal{B}_\delta(x_e)}) \subset \mathcal{H}_n^{\mathcal{W}_\varepsilon}$ for all $t \geq \hat{T}$. Next, note that there exists a positive integer N_1 such that $x_n \in \mathcal{B}_\delta(x_e)$ for all $n \geq N_1$. Now, it follows from (23) that

$$V(x_n) \leq 2\sup_{0\leq t\leq\hat{T}}\mathbb{E}^{x_n}[\mathrm{dist}(s(t,x_n),\mathcal{E})] + 2\varepsilon, \quad n \geq N_1. \tag{27}$$

Next, it follows from [11, Thm. 7.3.1] that $\mathbb{E}[|s(\cdot,x_n)|]$ converges to $\mathbb{E}[|s(\cdot,x_e)|]$ uniformly on $[0,\hat{T}]$. Hence,

$$\begin{aligned}
\lim_{n\to\infty}\sup_{0\leq t\leq\hat{T}}\mathbb{E}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\right] &= \sup_{0\leq t\leq\hat{T}}\mathbb{E}\left[\mathrm{dist}\Big(\lim_{n\to\infty}s(t,x_n),\mathcal{E}\Big)\right] \\
&= \sup_{0\leq t\leq\hat{T}}\mathrm{dist}(x_e,\mathcal{E}) \\
&= 0,
\end{aligned} \tag{28}$$

which implies that there exists a positive integer $N_2 = N_2(x_e,\varepsilon) \geq N_1$ such that

$$\sup_{0\leq t\leq\hat{T}}\mathbb{E}^{x_n}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\right] < \varepsilon$$

for all $n \geq N_2$. Combining (27) with the above result yields $V(x_n) < 4\varepsilon$ for all $n \geq N_2$, which implies that $\lim_{n\to\infty}V(x_n) = 0 = V(x_e)$.

Next, we show that $\mathcal{L}V(x(t))$ is negative along the solution of (2) on $\mathcal{D}\setminus\mathcal{E}$. Note that for every $x \in \mathcal{D}_0\setminus\mathcal{E}$ and $0 < h \leq 1/2$ such that $s(h,x) \in \mathcal{H}_n^{\mathcal{D}_0\setminus\mathcal{E}}$, it follows from the definition of $T(\cdot)$ that $\mathbb{E}^x[V(s(h,x))]$ is reached at some time $\hat{t}$ such that $0 \leq \hat{t} \leq T(x)$. Hence,

$$\begin{aligned}
\mathbb{E}^x\left[V(s(h,x))\right] &= \mathbb{E}^x\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\right]\frac{1+2\hat{t}}{1+\hat{t}} \\
&= \mathbb{E}^x\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\right]\frac{1+2\hat{t}+2h}{1+\hat{t}+h} \\
&\quad\times\left[1-\frac{h}{(1+2\hat{t}+2h)(1+\hat{t})}\right] \\
&\leq V(x)\left[1-\frac{h}{2(1+T(x))^2}\right],
\end{aligned} \tag{29}$$

which implies that

$$\begin{aligned}
\mathcal{L}V(x) &= \lim_{h\to 0^+}\frac{\mathbb{E}^x\left[V(s(h,x))\right]-V(x)}{h} \\
&\leq -\frac{1}{2}V(x)(1+T(x))^{-2} < 0, \quad x \in \mathcal{D}_0\setminus\mathcal{E},
\end{aligned}$$

and hence, (iii) holds. □

4. Conclusion

In this paper, we developed Lyapunov theorems for semistability of nonlinear stochastic dynamical systems. In addition, a converse theorem for stochastic semistability is developed using continuous Lyapunov functions. Future extensions will involve the development of arc-length-based Lyapunov tests [9] for stochastic semistability. However, since the sample paths of a stochastic dynamical system may not have an arc-length in the classical sense – due to lack of differentiability of solutions and unbounded variation of sample Wiener paths – stochastic integrals involving non-differentiable curves as the limiting value of polygonal curves can be used to approximate the arc length of the stochastic system.

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