Stochastic Semistability for Nonlinear Dynamical Systems With Application to Consensus on Networks With Communication Uncertainty

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***Abstract*—This article focuses on semistability and finite time semistability analysis and synthesis of stochastic dynamical systems having a continuum of equilibria. Stochastic semistability is the property whereby the solutions of a stochastic dynamical system almost surely converge to Lyapunov stable in probability equilibrium points determined by the system initial conditions. In this article, we extend the theories of semistability and finite-time semistability for deterministic dynamical systems to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. Specifically, Lyapunov and converse Lyapunov theorems for stochastic semistability are developed for dynamical systems driven by Markov diffusion processes. These results are then used to develop a general framework for designing semistable consensus protocols for dynamical networks in the face of stochastic communication uncertainty for achieving multiagent coordination tasks in finite time. The proposed controller architectures involve the exchange of generalized charge or energy state information between agents guaranteeing that the closed-loop dynamical network is stochastically semistable to an equipartitioned equilibrium representing a state of almost sure consensus consistent with basic thermodynamic principles.**

***Index Terms*—Communication uncertainty, consensus protocols, distributed control, Markov processes, nonlinear networks, stochastic finite time semistability, stochastic semistability, thermodynamic protocols.**

I. Introduction

FOR deterministic dynamical systems the authors in [1]–[4] developed a unified stability analysis framework for systems having a continuum of equilibria. Since every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable nor finite time stable. Hence, asymptotic and finite time stability are not the appropriate notions of stability for systems having a continuum of equilibria. Two notions that are of particular relevance to such systems are convergence and semistability. Convergence is the property whereby every system solution converges (asymptotically or in finite time) to a limit point that may depend on the system initial condition. Semistability (resp., finite time semistability) is the additional requirement that all solutions converge asymptotically (resp., in finite time) to limit points that are Lyapunov stable. Semistability (resp., finite time semistability) for an equilibrium thus implies Lyapunov stability, and is implied by asymptotic (resp., finite time) stability.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point [1]. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the dynamical system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

In this article, we extend the theories of semistability and finite-time semistability for deterministic dynamical systems developed in [1]–[4] to develop a rigorous framework for stochastic semistability and stochastic finite-time semistability. First, in Section III, we extend the theory of stochastic semistability given in [5] by presenting new Lyapunov theorems as well as the first converse Lyapunov theorem for stochastic semistability, which holds with a continuous Lyapunov function whose infinitesimal generator decreases along the stochastic dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria. It is important to note here that stochastic semistability theory as developed in [5] involves a stronger set of stability in probability definitions that do not allow for a small probability of escape of the system sample trajectories for small deviations from the system equilibrium. While our stochastic semistability results developed in this article resemble the results in [5], the proofs of our results are rendered more difficult by the fact that the results in this article are predicated on a weaker set of stability in probability definitions, and hence, provide a stronger set of stochastic semistability results.

Next, in Section IV, we establish stochastic finite time semistability theory. In particular, we present the notions of

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finite time convergence in probability and finite time semistability in probability for nonlinear stochastic dynamical systems driven by Markov diffusion processes. Furthermore, we establish the continuity of a settling time operator and develop a sufficient Lyapunov stability theorem for finite time semistability in probability. Specifically, we develop almost sure finite time convergence and stochastic Lyapunov stability properties to address almost sure finite time semistability requiring that the sample trajectories of a nonlinear stochastic dynamical system converge almost surely in finite time to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable.

Next, in Sections V and VI, we use the results of Sections III and IV to develop a general, thermodynamically motivated framework for designing semistable and finite-time semistable protocols for stochastic dynamical networks for achieving coordination tasks asymptotically and in finite time. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles (UAV's) and autonomous underwater vehicles (AUV's) for combat, surveillance, and reconnaissance [6], distributed reconfigurable sensor networks for managing power levels of wireless networks [7], air and ground transportation systems for air traffic control and payload transport and traffic management [8], swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles [9], [10], and congestion control in communication networks for routing the flow of information through a network [11].

Even though convergence, semistability, finite time semistability, and optimality for deterministic multiagent network systems involving cooperative control tasks, such as formation control, rendezvous, flocking, cyclic pursuit, and consensus have received considerable attention in the literature (see, for example, [4], [6]–[36]), stochastic multiagent networks have not been as plethorically developed; notable contributions include [37]–[41]. These contributions address asymptotic convergence [39], time-varying network topologies [38], communication delays [41], asynchronous switchings [40], and optimality [37]; however, none of the aforementioned references address the problems of stochastic semistability and stochastic finite time semistability.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. Thus, from a practical viewpoint, it is not sufficient for a nonlinear control protocol to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

To capture network system uncertainty and communication uncertainty between the agents in a network, wherein the evolution of each link of the network communication topology follows a Markov process for modeling unknown communication noise and attenuations, we use the results of Sections III and IV to develop almost sure consensus protocols for multiagent systems with nonlinear stochastic dynamics. Specifically, we use our stochastic semistability and stochastic finite time semistability frameworks to design distributed asymptotic and finite time consensus control protocols for nonlinear bidirectional dynamical networks with stochastic communication uncertainty. The proposed controller architectures are predicated on the recently developed notion of stochastic dynamical thermodynamics [42], [43] resulting in controller architectures involving the exchange of generalized charge or energy state information between agents that guarantee that the closed-loop dynamical network is consistent with basic thermodynamic principles. Finally, we note that a preliminary conference version of this article appeared in [44]. The present article considerably expands on [44] by providing detailed proofs of all of the results in [44], a converse Lyapunov theorem for stochastic semistability, stochastic finite time Lyapunov semistability theorems, additional discussion, and several numerical examples.

We begin by establishing notation, definitions, and mathematical preliminaries in Section II.

II. Mathematical Preliminaries

In this section, we establish notation, definitions, and review some basic results on stability of nonlinear stochastic dynamical systems [45]–[49]. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm or an induced matrix norm (depending on context), $||\cdot||_{\mathrm{F}}$ for the Frobenius matrix norm, A^{T} for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [47]. Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure [48, p. 10], with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Wiener process $w(t)$ up to time t. We denote by $\mathcal{G}$ a stochastic dynamical system generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega \mid x(t, \omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here we use the notation $x(t)$ to represent the stochastic process $x(t, \omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ over the semi-infinite parameter space $[0, \infty)$ by $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n \times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely. Furthermore, $\mathbb{E}[\,\cdot\,]$ and $\mathbb{E}^{x_0}[\,\cdot\,]$ denote, respectively, the expectation with respect to the probability measure $\mathbb{P}$ and with respect to the classical Wiener measure $\mathbb{P}^{x_0}$.

Finally, we write tr$(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x) \triangleq \frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n \triangleq \{x : \Omega \to \mathbb{R}^n \mid \mathbb{E}[x^{\mathrm{T}} x] < \infty\}$. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n \mid x : \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n \mid x \overset{\text{a.s.}}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V : \mathcal{D} \to \mathbb{R}$ that are two-times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$. Finally, we write $x(t) \overset{\text{a.s.}}{\to} \mathcal{M}$ as $t \to \infty$ to denote that $x(t)$ approaches the set $\mathcal{M}$ almost surely, that is, for every $\varepsilon > 0$ there exists finite stopping time $T > 0$ such that $\mathrm{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$, where $\mathrm{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} ||p - x||$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t) \quad x(t_0) \overset{\text{a.s.}}{=} x_0 \quad t \geq t_0 \tag{1}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, and $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n \times d}$ are continuous functions and satisfy $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$ for some $x_{\mathrm{e}} \in \mathcal{D}$. An *equilibrium point* of (1) is a point $x_{\mathrm{e}} \in \mathcal{D}$ such that $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$. It is easy to see that x_{e} is an equilibrium point of (1) if and only if the constant stochastic process $x(\cdot) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$ is a solution of (1). We denote the set of equilibrium points of (1) by $\mathcal{E} \triangleq \{\omega \in \Omega \mid x(t, \omega) = x_{\mathrm{e}}\} = \{x_{\mathrm{e}} \in \mathcal{D} \mid f(x_{\mathrm{e}}) = 0 \text{ and } D(x_{\mathrm{e}}) = 0\}$.

The filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x : [t_0, \tau] \times \Omega \to \mathcal{D}$ is said to be a *solution* of (1) on the time interval $[t_0, \tau]$ with initial condition $x(t_0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t \geq t_0}$, $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\mathrm{d}s + \int_{t_0}^{t} D(x(s))\mathrm{d}w(s) \text{ a.s.} \qquad t \in [t_0, \tau] \tag{2}$$

where the integrals in (2) are Itô integrals.

Note that for each fixed $t \geq t_0$, the random variable $\omega \mapsto x(t, \omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t, \omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq t_0$. A pathwise solution $t \mapsto x(t)$ of (1) in $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal pathwise solutions to (1) $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume (1) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* of (1) are given in [47, Corollary 6.3.5].

Furthermore, we assume that $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n \times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\| \quad x, y \in \mathcal{D} \tag{3}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2) \quad x \in \mathcal{D} \tag{4}$$

for some Lipschitz constant $L > 0$, and hence, since $x(t_0) \in \mathcal{H}_n^{\mathcal{D}}$ and $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ to (1) in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}} \setminus \{0\}$ there exists $\tau_x > 0$ such that if $x_1 : [t_0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [t_0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (1); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths almost surely solve (1), then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_1(t) = x_2(t),\ t_0 \leq t \leq \tau_x\big) = 1$.

A weaker sufficient condition for the existence of a unique solution to (1) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (3) without the growth condition (4) is given in [50]. Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by ([49, Th. 3.4], [47, Th. 9.2.8])

$$\mathbb{P}(x(t) \in \mathcal{B} | x(t_0) \overset{\text{a.s.}}{=} x_0) = \mathbb{P}(t - t_0, x_0, 0, \mathcal{B}) \quad x_0 \in \mathbb{R}^n \tag{5}$$

for all $t \geq t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(s, x, t, \mathcal{B}), t \geq s$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Finally, recall that every continuous process with Feller transition probability function is also a strong Markov process [49, p.101].

Definition 2.1 ([48, Def. 7.7]): Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) \overset{\text{a.s.}}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t} \quad x_0 \in \mathcal{D}. \tag{6}$$

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq t_0$, satisfies (1), then the limit in (6) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq t_0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (1) and is given by [48, Th. 7.9]

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\, D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x) \quad x \in \mathcal{D}. \tag{7}$$

The following definition introduces the notions of Lyapunov and asymptotic stability in probability.

Definition 2.2 ([49]):

i) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (1) is *Lyapunov stable in probability* if, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\mathrm{e}})$

$$\mathbb{P}^{x_0}\left(\sup_{t \geq t_0} \|x(t) - x_{\mathrm{e}}\| > \varepsilon\right) \leq \rho. \tag{8}$$

ii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\mathrm{e}}$ to (1) is *asymptotically stable in probability* if it is Lyapunov stable in probability and there exists $\delta > 0$ such that, for all

$x_0 \in \mathcal{B}_\delta(x_e)$

$$\lim_{x_0 \to x_e} \mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x(t) - x_e\| = 0\right) = 1. \tag{9}$$

iii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e$ to (1) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$

$$\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x(t) - x_e\| = 0\right) = 1. \tag{10}$$

Next, we provide sufficient conditions for local and global asymptotic stability in probability for the nonlinear stochastic dynamical system (1).

Theorem 2.1 ([49, Th. 5.3 and Th. 5.11, Corollary 5.1]): Consider the nonlinear stochastic dynamical system (1) and assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(x_e) = 0 \tag{11}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq x_e \tag{12}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2} D(x) \leq 0, \qquad x \in \mathcal{D}. \tag{13}$$

Then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e$ to (1) is Lyapunov stable in probability. If, in addition

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2} D(x) < 0, \quad x \in \mathcal{D}, \quad x \neq x_e \tag{14}$$

then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e$ to (1) is asymptotically stable in probability. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e$ to (1) is globally asymptotically stable in probability.

III. Stochastic Semistability

In this section, we develop a stability analysis framework for stochastic systems having a continuum of equilibria. Specifically, we present necessary and sufficient conditions for *stochastic semistability*. To develop stochastic semistability theory, we need some additional notation and definitions.

The measurable map $s : [0, \tau_x) \times \mathcal{D} \times \Omega \to \mathcal{D}$ denotes the *dynamic* or *flow* of the stochastic dynamical system (1) and, for all $t, \tau \in [0, \tau_x)$, satisfies the *cocycle* property $s(\tau, s(t, x), \omega) = s(t + \tau, x, \omega)$ and the *identity* (on $\mathcal{D}$) property $s(0, x, \omega) = x$ for all $x \in \mathcal{D}$ and $\omega \in \Omega$. The measurable map $s_t \triangleq s(t, \cdot, \omega) : \mathcal{D} \to \mathcal{D}$ is continuously differentiable for all $t \in [0, \tau_x)$ outside a $\mathbb{P}$-nullset and the sample path trajectory $s^x \triangleq s(\cdot, x, \omega) : [0, \tau_x) \to \mathcal{D}$ is continuous in $\mathcal{D}$ for all $t \in [0, \tau_x)$. Thus, for every $x \in \mathcal{D}$, there exists a trajectory of measures defined for all $t \in [0, \tau_x)$ satisfying the dynamical processes (1) with initial condition $x(0) \overset{\text{a.s.}}{=} x_0$. For simplicity of exposition we write $s(t, x)$ for $s(t, x, \omega)$ omitting its dependence on ω.

Next, the following definitions for limit sets and stochastic invariance are needed.

Definition 3.1: A point $p \in \mathcal{D}$ is a *limit point* of the trajectory $s(\cdot, x)$ of (1) if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x) \overset{\text{a.s.}}{\to} p$ as $n \to \infty$. The set of all limit points of $s(t, x)$, $t \geq 0$, is the *limit set* $\omega(x)$ of $s(\cdot, x)$ of (1).

Definition 3.2 ([51]): An open set $\mathcal{D} \subset \mathbb{R}^n$ is said to be *positively invariant with respect to* (1) if $\mathcal{D}$ is Borel and, for all $x_0 \in \mathcal{D}$, $\mathbb{P}^{x_0}(x(t) \in \mathcal{D}) = 1$, $t \geq t_0$.

It is important to note that the ω-limit set of a stochastic dynamical system is a ω-limit set of a trajectory of measures, that is, $p \in \omega(x)$ is a weak limit of a sequence of measures taken along every sample continuous bounded trajectory of (1). It can be shown that the ω-limit set of a stationary stochastic dynamical system attracts bounded sets and is measurable with respect to the σ-algebra of invariant sets. Thus, the measures of the stochastic process $x(\cdot)$ tend to an invariant set of measures and $x(t)$ asymptotically tends to the closure of the support set (i.e., kernel) of this set of measures almost surely.

However, unlike deterministic dynamical systems, wherein ω-limit sets serve as global attractors, in stochastic dynamical systems stochastic invariance (see Definition 3.2) leads to ω-limit sets being defined for each fixed sample $\omega \in \Omega$ of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and hence, are pathwise attractors. This is due to the fact that a cocycle property rather than a semigroup property holds for stochastic dynamical systems. For details, see [52]–[54].

The following proposition gives a sufficient condition for a trajectory of (1) to converge almost surely to a limit point. For this result, $\mathcal{D}_c \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ denotes a positively invariant set with respect to (1) and $s_t(\mathcal{H}_n^{\mathcal{D}_c})$ denotes the image of $\mathcal{H}_n^{\mathcal{D}_c} \subset \mathcal{H}_n^{\mathcal{D}}$ under the flow $s_t : \mathcal{H}_n^{\mathcal{D}_c} \to \mathcal{H}_n^{\mathcal{D}}$; that is, $s_t(\mathcal{H}_n^{\mathcal{D}_c}) \triangleq \{y \mid y = s_t(x_0) \text{ for some } x(0) \overset{\text{a.s.}}{=} x_0 \in \mathcal{H}_n^{\mathcal{D}_c}\}$.

Proposition 3.1: Consider the nonlinear stochastic dynamical system (1) and let $x \in \mathcal{D}_c$. If the limit set $\omega(x)$ of (1) contains a Lyapunov stable in probability equilibrium point y, then $\lim_{x\to y} \mathbb{P}^x\big(\|\lim_{t\to\infty} s(t, x) - y\| = 0\big) = 1$, that is, $\omega(x) \overset{\text{a.s.}}{=} \{y\}$ as $x \to y$.

Proof: Suppose $y \in \omega(x)$ is Lyapunov stable in probability and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ be an open neighborhood of y. Since y is Lyapunov stable in probability, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_c$ of y such that $s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ as $x \to y$ for every $t \geq 0$. Now, since $y \in \omega(x)$, it follows that there exists $\tau \geq 0$ such that $s(\tau, x) \in \mathcal{H}_n^{\mathcal{N}_\delta}$. Hence, $s(t + \tau, x) = s_t(s(\tau, x)) \in s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ for every $t > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ is arbitrary, it follows that $y \overset{\text{a.s.}}{=} \lim_{t\to\infty} s(t, x)$. Thus, $\lim_{n\to\infty} s(t_n, x) \overset{\text{a.s.}}{=} y$ as $x \to y$ for every sequence $\{t_n\}_{n=1}^{\infty}$, and hence, $\omega(x) \overset{\text{a.s.}}{=} \{y\}$ as $x \to y$. ■

The following definition introduces the notion of stochastic semistability.

Definition 3.3: An equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (1) is *stochastically semistable* if the following statements hold.

i) For every $\varepsilon > 0$

$$\lim_{x_0 \to x_e} \mathbb{P}^{x_0}\left(\sup_{0 \leq t < \infty} \|x(t) - x_e\| > \varepsilon\right) = 0.$$

Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$

$$\mathbb{P}^{x_0}\left(\sup_{0 \leq t < \infty} \|x(t) - x_e\| > \varepsilon\right) \leq \rho.$$

ii) $\lim_{\mathrm{dist}(x_0, \mathcal{E}) \to 0} \mathbb{P}^{x_0}(\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{E}) = 0) = 1$. Equivalently, for every $\rho \in (0, 1)$, there exists

$\delta = \delta(\rho) > 0$ such that if $\mathrm{dist}(x_0, \mathcal{E}) \le \delta$, then $\mathbb{P}^{x_0}(\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{E}) = 0) \ge 1 - \rho$.

The dynamical system (1) is *stochastically semistable* if every equilibrium solution of (1) is stochastically semistable. Finally, the dynamical system (1) is *globally stochastically semistable* if i) holds and $\mathbb{P}^{x_0}(\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{E}) = 0) = 1$ for all $x_0 \in \mathbb{R}^n$.

Remark 3.1: Note that if $x(t) \overset{\text{a.s.}}{\equiv} x_\mathrm{e} \in \mathcal{E}$ only satisfies i) in Definition 3.3, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_\mathrm{e} \in \mathcal{E}$ of (1) is Lyapunov stable in probability.

Next, we present sufficient conditions for stochastic semistability.

Theorem 3.1: Consider the nonlinear stochastic dynamical system (1). Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be an open neighborhood of $\mathcal{E}$ and assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ such that

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^\mathrm{T}(x)V''(x)D(x) < 0, \quad x \in \mathcal{Q}\backslash\mathcal{E}. \quad (15)$$

If every equilibrium point of (1) is Lyapunov stable in probability, then (1) is stochastically semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (1) is globally stochastically semistable.

Proof: Since every equilibrium point of (1) is Lyapunov stable in probability by assumption, for every $z \in \mathcal{E}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z))$, $\varepsilon > 0$, is bounded and contained in $\mathcal{Q}$ as $\varepsilon \to 0$. The set $\mathcal{V}_\varepsilon \triangleq \bigcup_{z\in\mathcal{E}} \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$, $\varepsilon > 0$, is an open neighborhood of $\mathcal{E}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}_\varepsilon$ so that there exists $z \in \mathcal{E}$ such that $x \in \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)}$, $t \ge 0$, as $\varepsilon \to 0$. Since $\mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ is bounded and invariant with respect to the solution of (1) as $\varepsilon \to 0$, it follows that $\mathcal{V}_\varepsilon$ is invariant with respect to the solution of (1) as $\varepsilon \to 0$. Furthermore, it follows from (15) that $\mathcal{L}V(s(t,x)) < 0$, $t \ge 0$, and hence, since $\mathcal{V}_\varepsilon$ is bounded it follows from [51, Corollary 4.1] that $\lim_{t\to\infty} \mathcal{L}V(s(t,x)) \overset{\text{a.s.}}{=} 0$ as $\varepsilon \to 0$.

It is easy to see that $\mathcal{L}V(x) \ne 0$ by assumption and $\mathcal{L}V(x_\mathrm{e}) = 0$, $x_\mathrm{e} \in \mathcal{E}$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{E}$ as $t \to \infty$ and $\varepsilon \to 0$, which implies that $\lim_{\mathrm{dist}(x,\mathcal{E})\to 0} \mathbb{P}^x(\lim_{t\to\infty} \mathrm{dist}(s(t,x), \mathcal{E}) = 0) = 1$. Finally, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 3.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 3.3, (1) is semistable.

Finally, for $\mathcal{Q} = \mathbb{R}^n$ global stochastic semistability follows from identical arguments using the radially unbounded condition on $V(\cdot)$. ■

Finally, we provide a partial converse to Theorem 3.1. For this result, recall that $\mathcal{L}V(x_\mathrm{e}) = 0$ for every $x_\mathrm{e} \in \mathcal{E}$. Also note that it follows from (6) that $\mathcal{L}V(x) = \mathcal{L}V(s(0,x))$. In addition, the following definition is required.

Definition 3.4: For a given $\rho \in (0,1)$, the *ρ-domain of semistability* is the set of points $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ such that if $x(t)$, $t \ge 0$, is a solution to (1) with $x(0) \overset{\text{a.s.}}{=} x_0$, then $x(t)$ converges to a Lyapunov stable in probability equilibrium point in $\mathcal{D}$ with probability greater than or equal to $1 - \rho$.

Theorem 3.2: Consider the nonlinear stochastic dynamical system (1). Suppose (1) is stochastically semistable with a ρ-domain of semistability $\mathcal{D}_0$. Then there exist a continuous nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that i) $V(x) = 0$, $x \in \mathcal{E}$, ii) $V(x) \ge \alpha(\mathrm{dist}(x, \mathcal{E}))$, $x \in \mathcal{D}_0$, and iii) $\mathcal{L}V(x) < 0$, $x \in \mathcal{D}_0\backslash\mathcal{E}$.

Proof: Let $\mathfrak{B}^{x_0}$ denote the set of all sample trajectories of (1) for which $\lim_{t\to\infty} \mathrm{dist}(x(t,\omega), \mathcal{E}) = 0$ and $x(\{t \ge 0\}, \omega) \in \mathfrak{B}^{x_0}$, $\omega \in \Omega$, and let $\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega)$, $\omega \in \Omega$, denote the indicator function defined on the set $\mathfrak{B}^{x_0}$, that is

$$\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \ge 0\}, \omega) \in \mathfrak{B}^{x_0} \\ 0, & \text{otherwise.} \end{cases}$$

Note that by definition $\mathbb{P}^{x_0}(\mathfrak{B}^{x_0}) \ge 1 - \rho$ for all $x_0 \in \mathcal{D}_0$. Define the function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{t\ge 0}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\mathrm{dist}(s(t,x), \mathcal{E})\mathbb{1}_{\mathfrak{B}^x}(\omega)\right]\right\} \quad x \in \mathcal{D}_0 \quad (16)$$

and note that $V(\cdot)$ is well defined since (1) is stochastically semistable. Clearly, (i) holds. Furthermore, since $V(x) \ge \mathrm{dist}(x, \mathcal{E})$, $x \in \mathcal{D}_0$, it follows that (ii) holds with $\alpha(r) = r$.

To show that $V(\cdot)$ is continuous on $\mathcal{D}_0\backslash\mathcal{E}$, define $T : \mathcal{D}_0\backslash\mathcal{E} \to [0,\infty)$ by $T(z) \triangleq \inf\{h : \mathbb{E}\left[\mathrm{dist}(s(h,z), \mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] < \mathrm{dist}(z, \mathcal{E})/2 \text{ for all } t \ge h > 0\}$, and denote

$$\mathcal{W}_\varepsilon \triangleq \left\{x \in \mathcal{D}_0 \mid \mathbb{P}^x\left(\sup_{t\ge 0} \mathrm{dist}(s(t,x), \mathcal{E}) \le \varepsilon\right) \ge 1 - \rho\right\}. \quad (17)$$

Note that $\mathcal{W}_\varepsilon \supset \mathcal{E}$ is open and contains an open neighborhood of $\mathcal{E}$. Consider $z \in \mathcal{D}_0\backslash\mathcal{E}$ and define $\lambda \triangleq \mathrm{dist}(z, \mathcal{E}) > 0$. Then it follows from stochastic semistability of (1) that there exists $h > 0$ such that $\mathbb{P}^z\left(s(h,z) \in \mathcal{W}_{\lambda/2}\right) \ge 1 - \rho$. Consequently, $\mathbb{P}^z\left(s(h+t,z) \in \mathcal{W}_{\lambda/2}\right) \ge 1 - \rho$ for all $t \ge 0$, and hence, it follows that $T(z)$ is well defined. Since $\mathcal{W}_{\lambda/2}$ is open, there exists a neighborhood $\mathcal{B}_\sigma(s(T(z), z)$ such that $\mathbb{P}^z\left(\mathcal{B}_\sigma(s(T(z),z)) \subset \mathcal{W}_{\lambda/2}\right) \ge 1 - \rho$. Hence, $\mathcal{N} \subset \mathcal{D}_0$ is a neighborhood of z such that $s_{T(z)}(\mathcal{H}_n^{\mathcal{N}}) \triangleq \mathcal{B}_\sigma(s(T(z), z))$.

Next, choose $\eta > 0$ such that $\eta < \lambda/2$ and $\mathcal{B}_\eta(z) \subset \mathcal{N}$. Then, for every $t > T(z)$ and $y \in \mathcal{B}_\eta(z)$

$$\begin{aligned}&[(1+2t)/(1+t)]\mathbb{E}\left[\mathrm{dist}(s(t,y), \mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \\ &\quad \le 2\mathbb{E}\left[\mathrm{dist}(s(t,y), \mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \le \lambda.\end{aligned}$$

Therefore, for every $y \in \mathcal{B}_\eta(z)$

$$\begin{aligned} V(z) - V(y) &= \sup_{t\ge 0}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\mathrm{dist}(s(t,z), \mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\right\} \\ &\quad - \sup_{t\ge 0}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\mathrm{dist}(s(t,y), \mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right\} \\ &= \sup_{0\le t\le T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\mathrm{dist}(s(t,z), \mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\right\} \\ &\quad - \sup_{0\le t\le T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\mathrm{dist}(s(t,y), \mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right\}. \end{aligned} \quad (18)$$

Hence

$$\begin{aligned}|V(z)-V(y)| &\le \sup_{0\le t\le T(z)}\left|\frac{1+2t}{1+t}\big(\mathbb{E}\left[\mathrm{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\right.\\ &\qquad\left. -\,\mathbb{E}\left[\mathrm{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\big)\right|\\ &\le 2\sup_{0\le t\le T(z)}\Big|\mathbb{E}\left[\mathrm{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\\ &\qquad -\,\mathbb{E}\left[\mathrm{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\Big|\\ &\le 2\sup_{0\le t\le T(z)}\mathbb{E}\left[\mathrm{dist}(s(t,z),s(t,y))\right]\\ &\qquad z\in\mathcal{D}_0\backslash\mathcal{E},\quad y\in\mathcal{B}_\eta(z).\end{aligned}\tag{19}$$

Now, since $f(\cdot)$ and $D(\cdot)$ satisfy (3) and (4), it follows from continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [47, Th. 7.3.1] and (19) that $V(\cdot)$ is continuous on $\mathcal{D}_0\backslash\mathcal{E}$.

To show that $V(\cdot)$ is continuous on $\mathcal{E}$, consider $x_{\rm e}\in\mathcal{E}$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}_0\backslash\mathcal{E}$ that converges to $x_{\rm e}$. Since $x_{\rm e}$ is Lyapunov stable in probability, it follows that $x(t)\overset{\text{a.s.}}{\equiv}x_{\rm e}$ is the unique solution to (1) with $x(0)\overset{\text{a.s.}}{=}x_{\rm e}$. By continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [47, Th. 7.3.1], $s(t,x_n)\overset{\text{a.s.}}{\to}s(t,x_{\rm e})\overset{\text{a.s.}}{=}x_{\rm e}$ as $n\to\infty$, $t\ge 0$.

Let $\varepsilon>0$ and note that it follows from *ii*) of Proposition 2.2 in [5] that there exists $\delta=\delta(x_{\rm e})>0$ such that for every solution of (1) in $\mathcal{B}_\delta(x_{\rm e})$ there exists $\hat{T}=\hat{T}(x_{\rm e},\varepsilon)>0$ such that $\mathbb{P}\left(s_t(\mathcal{H}_n^{\mathcal{B}_\delta(x_{\rm e})})\subset\mathcal{W}_\varepsilon\right)\ge 1-\rho$ for all $t\ge\hat{T}$. Next, note that there exists a positive integer N_1 such that $x_n\in\mathcal{B}_\delta(x_{\rm e})$ for all $n\ge N_1$. Now, it follows from (16) that:

$$V(x_n)\le 2\sup_{0\le t\le\hat{T}}\mathbb{E}[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)]+2\varepsilon\qquad n\ge N_1.\tag{20}$$

Next, it follows from [47, Th. 7.3.1], that $\mathbb{E}[|s(\cdot,x_n)|]$ converges to $\mathbb{E}[|s(\cdot,x_{\rm e})|]$ uniformly on $[0,\hat{T}]$. Hence,

$$\begin{aligned}\lim_{n\to\infty}\sup_{0\le t\le\hat{T}}&\mathbb{E}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right]\\ &=\sup_{0\le t\le\hat{T}}\mathbb{E}\left[\lim_{n\to\infty}\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right]\\ &\le\sup_{0\le t\le\hat{T}}\mathrm{dist}(x_{\rm e},\mathcal{E})\\ &=0\end{aligned}\tag{21}$$

which implies that there exists a positive integer $N_2=N_2(x_{\rm e},\varepsilon)\ge N_1$ such that

$$\sup_{0\le t\le\hat{T}}\mathbb{E}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right]<\varepsilon$$

for all $n\ge N_2$. Combining (20) with the above result yields $V(x_n)<4\varepsilon$ for all $n\ge N_2$, which implies that $\lim_{n\to\infty}V(x_n)=0=V(x_{\rm e})$.

Finally, we show that $\mathcal{L}V(x(t))$ is negative along the solution of (1) on $\mathcal{D}_0\backslash\mathcal{E}$. Note that for every $x\in\mathcal{D}_0\backslash\mathcal{E}$ and $0<h\le 1/2$ such that $\mathbb{P}\left(s(h,x)\in\mathcal{D}_0\backslash\mathcal{E}\right)\ge 1-\rho$, it follows from the definition of $T(\cdot)$ that $\mathbb{E}\left[V(s(h,x))\right]$ is reached at some time $\hat{t}$ such that $0\le\hat{t}\le T(x)$. Hence, it follows from the law of iterated expectation that

$$\begin{aligned}\mathbb{E}&\left[V(s(h,x))\right]\\ &=\mathbb{E}\left[\mathbb{E}\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{s(h,x)}}(\omega)\right]\frac{1+2\hat{t}}{1+\hat{t}}\right]\\ &=\mathbb{E}\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\mathbb{1}_{\mathfrak{B}^x}(\omega)\right]\frac{1+2\hat{t}+2h}{1+\hat{t}+h}\\ &\quad\cdot\left[1-\frac{h}{(1+2\hat{t}+2h)(1+\hat{t})}\right]\\ &\le V(x)\left[1-\frac{h}{2(1+T(x))^2}\right]\end{aligned}\tag{22}$$

which implies that

$$\begin{aligned}\mathcal{L}V(x)&=\lim_{h\to 0^+}\frac{\mathbb{E}\left[V(s(h,x))\right]-V(x)}{h}\\ &\le-\frac{1}{2}V(x)(1+T(x))^{-2}<0,\quad x\in\mathcal{D}_0\backslash\mathcal{E}\end{aligned}$$

and hence, (iii) holds. ■

IV. Stochastic Finite Time Semistability

In this section, we extend the results of Section III to address *stochastic finite-time semistability*. Here we assume that the uniform Lipschitz continuity condition (3) and the growth condition (4) are satisfied for all $x,y\in\mathcal{D}\backslash\mathcal{E}$. Furthermore, we assume that for every initial condition $x_0\in\mathcal{D}\backslash\mathcal{E}$, (1) has a unique solution in forward time.

The notion of stochastic finite time semistability involves finite time almost sure convergence along with stochastic semistability.

Definition 4.1: An equilibrium solution $x(t)\overset{\text{a.s.}}{\equiv}x_{\rm e}\in\mathcal{E}$ of (1) is (*globally*) *stochastically finite-time semistable* if there exists an operator $T:\mathcal{H}_n\to\mathcal{H}_1^{[0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold.

i) *Finite-time convergence in probability:* For every $x(0)\in\mathcal{H}_n/\mathcal{E}$, $s^{x(0)}(t)$ is defined on $[0,T(x(0)))$, $s^{x(0)}(t)\in\mathcal{H}_n/\mathcal{E}$ for all $t\in[0,T(x(0)))$, and

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(x(0))}\mathrm{dist}(s^{x(0)}(t),\mathcal{E})=0\right)=1.$$

ii) *Lyapunov stability in probability:* For every $\varepsilon>0$

$$\lim_{x_0\to x_{\rm e}}\mathbb{P}^{x_0}\left(\sup_{0\le t<\infty}\|s^{x(0)}(t)-x_{\rm e}\|>\varepsilon\right)=0.$$

Equivalently, for every $\varepsilon>0$ and $\rho\in(0,1)$, there exist $\delta=\delta(\varepsilon,\rho)>0$ such that, for all $x_0\in\mathcal{B}_\delta(x_{\rm e})$, $\mathbb{P}^{x_0}\left(\sup_{0\le t<\infty}\|s^{x(0)}(t)-x_{\rm e}\|>\varepsilon\right)\le\rho$.

iii) *Finiteness of the stochastic settling-time operator:* For every $x\in\mathcal{H}_n\backslash\mathcal{E}$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x\left[T(x)\right]<\infty$.

The dynamical system (1) is (*globally*) *stochastically finite time semistable* if every equilibrium solution of (1) is globally stochastically finite time semistable.

It is easy to see from Definition 4.1 that

$$T(x(0)) = \inf\{t \in \overline{\mathbb{R}}_+ : s(t, x(0)) = 0\}, \quad x(0) \in \mathcal{H}_n^{\mathbb{R}^n \backslash \mathcal{E}}.$$

Proposition 4.1: Suppose (1) is stochastically finite time semistable. Let $x_{\rm e} \in \mathcal{E}$ be an equilibrium point of (1) and let $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty]}$ be the stochastic finite time operator. Then the following statements hold.

i) If $\tau \geq 0$ and $x(0) \in \mathcal{H}_n$, then $T(s(\tau, x(0))) \overset{\text{a.s.}}{=} \max\{T(x(0)) - \tau, 0\}$.

ii) $T(\cdot)$ is sample continuous on $\mathcal{H}_n$ if and only if $T(\cdot)$ is sample continuous at every $z_{\rm e} \in \mathcal{H}_n \cap \mathcal{E}$.

Proof: The proof is similar to the proof of Proposition 3.2 of [55] and, hence, is omitted. ■

Next, we present a sufficient condition for global stochastic finite time semistability.

Theorem 4.1: Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (1) with $\mathcal{D} = \mathbb{R}^n$ and assume that there exist a radially unbounded nonnegative function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ and a function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $V^{-1}(0) = \mathcal{E}$, $V(x)$ is two-times continuously differentiable for all $x \in \mathbb{R}^n \backslash \mathcal{E}$, $\eta(\cdot)$ is continuously differentiable, and, for all $x \in \mathbb{R}^n \backslash \mathcal{E}$

$$V'(x)f(x) + \frac{1}{2}\text{tr}\, D^{\rm T}(x)V''(x)D(x) \leq -\eta(V(x)) \qquad (23)$$

$$\int_0^\varepsilon \frac{\mathrm{d}v}{\eta(v)} < \infty, \quad \varepsilon \in [0, \infty) \qquad (24)$$

$$\eta'(v) > 0, \qquad v \geq 0. \qquad (25)$$

If every point in the set $\mathcal{M} \triangleq \{x \in \mathcal{Q} : \eta(V(x)) = 0\}$ is Lyapunov stable in probability, then $\mathcal{G}$ is globally stochastically finite time semistable. Moreover, there exists a settling-time operator $T : \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_0)] \leq \int_0^{V(x_0)} \frac{\mathrm{d}v}{\eta(v)}, \quad x_0 \in \mathbb{R}^n. \qquad (26)$$

Proof: It follows from (23) and [51, Corollary 4.2] that $\lim_{t\to\infty} V(x(t))$ exists and is finite almost surely, and $\lim_{t\to\infty} \eta(V(s(t,x))) \overset{\text{a.s.}}{=} 0$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{M}$ as $t \to \infty$, which implies that $\lim_{\text{dist}(x,\mathcal{M})\to 0} \mathbb{P}^x(\lim_{t\to\infty} \text{dist}(s(t,x), \mathcal{M}) = 0) = 1$. Now, since every point in $\mathcal{M}$ is Lyapunov stable in probability, it follows from Proposition 3.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{M}$ is Lyapunov stable in probability. Hence, by definition, (1) is globally stochastically semistable. This further implies that the stochastic settling time operator $T(x)$ exists with probability one for all $x \in \mathcal{H}_n \backslash \mathcal{E}$.

Next, we show that $T(x(0))$ is finite with probability one and satisfies (26), and hence, $\mathbb{E}^{x_0}[T(x(0))] < \infty$. Define $T_0 \triangleq T(x(0))$ and $\alpha(V) \triangleq \int_0^V \frac{\mathrm{d}v}{\eta(v)}$, $V \in \overline{\mathbb{R}}_+$. Now, using Itô's (chain rule) formula the stochastic differential of $V(x(t))$ along the system sample trajectories $x(t)$, $t \geq 0$, is given by

$$\mathrm{d}V(x(t)) = \mathcal{L}V(x(t))\mathrm{d}t + \frac{\partial V}{\partial x}D(x(t))\mathrm{d}w(t).$$

Next, using (23) it follows that:

$$\begin{aligned} T_0 &= \int_0^{T_0} \frac{\eta(V(x(\tau)))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\ &\leq \int_0^{T_0} -\frac{\mathcal{L}V(x(\tau))}{\eta(V(x(\tau)))}\mathrm{d}\tau \\ &\leq \int_0^{T_0} -\frac{\mathrm{d}V(x(t))}{\eta(V(x(\tau)))} \\ &\quad + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) \\ &= \int_0^{T_0} -\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) \\ &\quad + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau). \end{aligned} \qquad (27)$$

Once again, using Ito's (chain rule) formula it follows that:

$$\begin{aligned} &\mathrm{d}\alpha(V(x(t))) \\ &= \bigg[\frac{\partial\alpha(V(x))}{\partial x}f(x(t)) + \frac{1}{2}\text{tr}\, D^{\rm T}(x(t))\frac{\partial^2\alpha(V(x))}{\partial x^2} \\ &\quad \cdot D(x(t))\bigg]\mathrm{d}t + \frac{\partial\alpha(V(x))}{\partial x}\mathrm{d}w(t) \\ &= \bigg[\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\text{tr}\, D^{\rm T}(x(t)) \\ &\quad \cdot \frac{\partial}{\partial x}\left(\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\right)D(x(t))\bigg]\mathrm{d}t + \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\mathrm{d}w(t) \\ &= \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\bigg[\bigg(\frac{\partial V(x)}{\partial x}f(x(t)) + \frac{1}{2}\text{tr}\, D^{\rm T}(x(t))\frac{\partial^2(V(x))}{\partial x^2} \\ &\quad \cdot D(x(t))\bigg)\mathrm{d}t + \frac{\partial V(x)}{\partial x}\mathrm{d}w(t)\bigg] + \frac{1}{2}\text{tr}\, D^{\rm T}(x(t)) \\ &\quad \cdot \left(\frac{\partial V(x)}{\partial x}\right)^{\rm T}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t \\ &= \frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\mathrm{d}V(x(t)) + \frac{1}{2}\text{tr}\, D^{\rm T}(x(t))\left(\frac{\partial V(x)}{\partial x}\right)^{\rm T}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2} \\ &\quad \cdot \left(\frac{\partial V(x)}{\partial x}\right)D(x(t))\mathrm{d}t. \end{aligned} \qquad (28)$$

Hence, it follows from (27) and (25) that:

$$\begin{aligned} T_0 &\leq \int_0^{T_0} -\mathrm{d}\alpha(V(x(\tau))) \\ &\quad + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau) \\ &\quad + \int_0^{T_0} \frac{1}{2}\text{tr}\, D^{\rm T}(x(\tau))\left(\frac{\partial V(x)}{\partial x}\right)^{\rm T}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2} \\ &\quad \cdot \left(\frac{\partial V(x)}{\partial x}\right)D(x(\tau))\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&= \alpha(V(x(0))) - \alpha(V(x(T_0))) \\
&\quad + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))} \frac{\partial V}{\partial x} D(x(\tau)) \mathrm{d}w(\tau) \\
&\quad - \int_0^{T_0} \frac{\eta'(V)}{\eta^2(V)} \frac{1}{2} \mathrm{tr} \left(\frac{\partial V(x)}{\partial x} D^{\mathrm{T}}(x(\tau)) \right)^{\mathrm{T}} \\
&\quad \cdot \left(\frac{\partial V(x)}{\partial x} D(x(\tau)) \right) \mathrm{d}\tau \\
&\le \int_0^{V(x(0))} \frac{\mathrm{d}v}{\eta(v)} - \int_0^{V(x(T_0))} \frac{\mathrm{d}v}{\eta(v)} \\
&\quad + \int_0^{T_0} \frac{1}{\eta(V(x(\tau)))} \frac{\partial V}{\partial x} D(x(\tau)) \mathrm{d}w(\tau). \qquad (29)
\end{aligned}
$$

Taking the expectation on both sides of (29) and using the fact that $x(0) \overset{\text{a.s.}}{=} x_0$ and $\mathbb{P}^{x_0}(x(T_0) \in \mathcal{E}) = 1$ implies $V(x(T_0)) \overset{\text{a.s.}}{=} 0$, (26) follows. ■

Remark 4.1: If $\eta(V) = cV^\theta$, where $c > 0$ and $\theta \in (0, 1)$, then $\eta(\cdot)$ satisfies (24) and (25). In this case, (26) becomes

$$\mathbb{E}^{x_0}[T(x(0))] \le \frac{V(x_0)^{1-\theta}}{c(1-\theta)}.$$

V. Almost Sure Asymptotic Consensus for Stochastic Dynamical Networks

In this section, we use the results of Section III to develop a thermodynamically motivated consensus framework for multiagent nonlinear stochastic systems that achieve stochastic semistability and almost sure state equipartition. Here we use graph-theoretic notions to represent a dynamical network and present solutions to the consensus problem for networks with undirected graph topologies (or information flows).

We begin by establishing some notion and definitions. Specifically, let $\mathfrak{G}(\mathcal{C}) = (\mathcal{V}, \mathcal{E})$ be a *directed graph* (or digraph) denoting the dynamical network (or dynamic graph) with the set of *nodes* (or vertices) $\mathcal{V} = \{1, \ldots, q\}$ involving a finite nonempty set denoting the agents, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ involving a set of ordered pairs denoting the direction of information flow, and a *connectivity matrix* $\mathcal{C} \in \mathbb{R}^{q\times q}$ such that $\mathcal{C}_{(i,j)} = 1$, $i, j = 1, \ldots, q$, if $(j, i) \in \mathcal{E}$, while $\mathcal{C}_{(i,j)} = 0$ if $(j, i) \notin \mathcal{E}$. The edge $(j, i) \in \mathcal{E}$ denotes that agent j can obtain information from agent i, but not necessarily vice versa. Moreover, we assume $\mathcal{C}_{(i,i)} = 0$ for all $i \in \mathcal{V}$. A *graph* or *undirected graph* $\mathfrak{G}$ associated with the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ is a directed graph for which the *arc set* is symmetric, that is, $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$. Weighted graphs can also be considered here; however, since this extension does not alter any of the conceptual results in the article we do not consider this extension for simplicity of exposition.

To address the consensus problem, consider q continuous-time agents with dynamics

$$
\begin{aligned}
&\mathrm{d}x_i(t) = u_i(t)\mathrm{d}t + \mathrm{row}_i(D(x(t)))\mathrm{d}w(t) \\
&\qquad i = 1, \ldots, q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \ge 0 \qquad (30)
\end{aligned}
$$

where $q \ge 2$ is the number of agents in the network with a communication graph topology $\mathfrak{G}(\mathcal{C})$, $D(x)\mathrm{d}w$, where $D(x) = [\mathrm{row}_1(D(x)), \ldots, \mathrm{row}_q(D(x))]^{\mathrm{T}} : \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^d$, captures probabilistic variations in the information transfer rates between agents, and, for every $i \in \{1, \ldots, q\}$, $x_i(t) \in \mathcal{H}_1$ denotes the information state of the ith agent and $u_i(t) \in \mathcal{H}_1$ denotes the control input of the ith agent. For a general distributed control architecture resulting in a network consensus action corresponding to an underlying conservation law, we assume $\mathbf{e}_q^{\mathrm{T}} D(x) = 0$, $x \in \mathbb{R}^q$, where $\mathbf{e}_q \triangleq [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^q$, and where the agent state $x_i(t) \in \mathcal{H}_1$ denotes the generalized charge (i.e., Nöether charge or simply charge) state and the control input $u_i(t) \in \mathcal{H}_1$ denotes the conserved current input for all $t \ge 0$.

The nonlinear consensus protocol is given by

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))] \qquad (31)$$

where $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, are Lipschitz continuous. Here we assume that the control process $u_i(\cdot)$ in (31) is restricted to a class of admissible control protocols consisting of measurable functions adapted to the filtration $\{\mathcal{F}_t\}_{t\ge0}$ such that, for every $i \in \{1, \ldots, q\}$, $u_i(\cdot) \in \mathcal{H}_1$, $t \ge 0$, and, for all $t \ge s$, $w_i(t) - w_i(s)$ is independent of $u_i(\tau), w_i(\tau), \tau \le s$, and $x_i(0)$, and hence, $u_i(\cdot)$ is nonanticipative. Furthermore, we assume $u_i(\cdot)$ takes values in a compact metrizable set, and hence, it follows from Theorem 2.2.4 of [56] that there exists a unique pathwise solution to (30) and (31) in $(\Omega, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}^{x_{i0}})$ for every $i \in \{1, \ldots, q\}$. Finally, note that the closed-loop system (30) and (31) is given by

$$
\begin{aligned}
\mathrm{d}x_i(t) &= \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t \\
&\quad + \mathrm{row}_i(D(x(t)))\mathrm{d}w(t) \\
&\quad i = 1, \ldots, q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \ge 0. \qquad (32)
\end{aligned}
$$

Equation (32) represents the collective dynamics of q agents which interact by exchanging charge. The coefficients scaling the functions $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, appearing in (32) represent the topology of the charge exchange between the agents. More specifically, given $i, j \in \{1, \ldots, q\}$, $i \neq j$, a coefficient of $\mathcal{C}_{(i,j)} = 1$ denotes that subsystem j receives charge from subsystem i, and a coefficient of zero denotes that subsystem i and j are disconnected, and hence, cannot share any charge.

Remark 5.1: Although our results can be directly extended to the case where (30) and (31) describe the dynamics of an aggregate multiagent system with an aggregate state vector $x(t) = [x_1^{\mathrm{T}}(t), \ldots, x_q^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{H}_{Nq}$, where $x_i(t) \in \mathcal{H}_N$ and $u_i(t) \in \mathcal{H}_N$, $i = 1, \ldots, q$, by using Kronecker algebra, for simplicity of exposition we focus on individual agent states evolving in $\mathcal{H}_1$ (i.e., $N = 1$).

Next, note that since

$$
\begin{aligned}
&\mathbf{e}_q^{\mathrm{T}}\mathrm{d}x(t) = \mathbf{e}_q^{\mathrm{T}} f(x(t))\mathrm{d}t + \mathbf{e}_q^{\mathrm{T}} D(x(t))\mathrm{d}w(t) = 0 \\
&\qquad\qquad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \ge 0 \qquad (33)
\end{aligned}
$$

it follows that $\sum_{i=1}^{q} \mathrm{d}x_i(t) \overset{\text{a.s.}}{=} 0$, $t \ge 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law. Now, it follows from Nöether's theorem [57] that to every conservation law there corresponds a symmetry. To show this for our multiagent network, the following definition and assumptions are needed.

Definition 5.1 ([58]): A directed graph $\mathfrak{G}(\mathcal{C})$ is *strongly connected* if for every ordered pair of vertices (i,j), $i \neq j$, there exists a *path* (i.e., a sequence of arcs) leading from i to j.

Recall that the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ is *irreducible*, that is, there does not exist a permutation matrix such that $\mathcal{C}$ is cogredient to a lower-block triangular matrix, if and only if $\mathfrak{G}(\mathcal{C})$ is strongly connected (see [58, Th. 2.7]).

Assumption 5.1: For the connectivity matrix $\mathcal{C} \in \mathbb{R}^{q\times q}$ associated with the multiagent stochastic dynamical system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases} 0, & \text{if } \sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0 \\ 1, & \text{otherwise} \end{cases}$$
$$i \neq j, \quad i,j = 1,\ldots,q \tag{34}$$

and

$$\mathcal{C}_{(i,i)} \triangleq -\sum_{k=1,\, k\neq i}^{q} \mathcal{C}_{(k,i)}, \quad i=j, \quad i=1,\ldots,q \tag{35}$$

rank $\mathcal{C} = q-1$, and for $\mathcal{C}_{(i,j)} = 1$, $i\neq j$, $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$.

Assumption 5.2: For $i,j = 1,\ldots,q, \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}(x_i - x_j)\,[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -\mathrm{row}_i(D(x))\mathrm{row}_i^{\mathrm{T}}(D(x))$.

The information connectivity between the agents can be represented by the network communication graph topology $\mathfrak{G}(\mathcal{C})$ having q nodes such that $\mathfrak{G}(\mathcal{C})$ has an undirected edge from node i to node j if and only if agent j can receive charge from agent i. Since the coefficients scaling $\sigma_{ij}(\cdot)$, $i,j \in \{1,\ldots,q\}$, $i \neq j$, are constants, the communication graph topology of the network $\mathfrak{G}(\mathcal{C})$ is fixed. Furthermore, note that the graph $\mathfrak{G}$ is *weakly connected* since the underlying undirected graph is connected; that is, every agent receives charge from, or delivers charge to, at least one other agent.

The fact that $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$, $i \neq j$, implies that agent i and j are *connected*, and hence, can share information; alternatively, $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) \equiv 0$ implies that agent i and j are *disconnected*, and hence, cannot share information. Assumption 5.1 thus implies that if the charge (or generalized energies) in the connected agents i and j are equal, then charge exchange between the agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Furthermore, if $\mathcal{C} = \mathcal{C}^{\mathrm{T}}$ and rank $\mathcal{C} = q-1$, then it follows that the connectivity matrix $\mathcal{C}$ is irreducible, which implies that for any pair of i and j, $i \neq j$, of $\mathcal{G}$ there exists a sequence information connectors (information arcs) of $\mathcal{G}$ that connect agents i and j.

Assumption 5.2 implies that charge flows from charge rich agents to charge poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (i.e., energy in transition) must flow in the direction of lower temperatures. It is important to note here that due to the stochastic term $D(x)\mathrm{d}w$ capturing probabilistic variations in the charge transfer (i.e., generalized current) between the agents, the second assumption requires that the scaled net charge flow $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]$ is bounded by the negative intensity of the diffusion coefficient given by $\frac{1}{2}\mathrm{tr}\, D(x)D^{\mathrm{T}}(x)$. For further details on Assumptions 5.1 and 5.2, see [42], [43].

The intensity $D(x)$ of the general probabilistic variations $D(x)\mathrm{d}w$ in the agent communication can take different forms to capture communication measurement noise or errors in the information transfer rates between agents. For example, we can consider $D(x) = M\hat{D}(x)$, where

$$M \triangleq [m_{(1,2)},\ldots,m_{(1,q)},m_{(2,3)},\ldots,m_{(2,q)},\ldots,$$
$$m_{(q-1,q)}] \in \mathbb{R}^{q\times\frac{1}{2}q(q-1)}$$
$$\hat{D}(x) \triangleq \mathrm{diag}[d_{(1,2)}(x),\ldots,d_{(1,q)}(x),d_{(2,3)}(x),\ldots$$
$$d_{(2,q)}(x),\ldots,d_{(q-1,q)}(x)] \in \mathbb{R}^{\frac{1}{2}q(q-1)\times\frac{1}{2}q(q-1)}$$

and $m_{(i,j)}d_{(i,j)}(x_i,x_j)\mathrm{d}w_i$ represents stochastic variations in the information flow between the ith and jth agent. Furthermore, considering

$$d_{(i,j)}(x_i,x_j) = \mathcal{C}_{(i,j)}(x_j - x_i)^p \tag{36}$$

where $p > 0$ and $m_{(i,j)} \in \mathbb{R}^q$ satisfies $m_{(i,j)_i} \geq 0$, $m_{(i,j)_j} \leq 0$, $m_{(i,j)_j} = -m_{(i,j)_i}$, $m_{(i,j)_k} = 0$, $k \neq i$, $k \neq j$, where $m_{(i,j)_i}$ denotes the ith component of $m_{(i,j)}$, it follows that $\mathrm{e}_q^{\mathrm{T}} m_{(i,j)} = 0$, and hence, it can be shown that (33) holds. Note that (36) captures nonlinear relative uncertainty between interagent communication. Of course, more general nonlinear uncertainties can also be considered.

For simplicity of exposition, in the reminder of the article we let $d = 1$ and $p = 1$, and consider q continuous-time agents with dynamics

$$\mathrm{d}x_i(t) = u_i(t)\mathrm{d}t + \sum_{j=1, j\neq i}^{q} \gamma\mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t)$$
$$i = 1,\ldots,q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0 \tag{37}$$

where $\gamma \in \mathbb{R}$, so that the closed-loop system (37) and (31) is given by

$$\mathrm{d}x_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t$$
$$+ \sum_{j=1, j\neq i}^{q} \gamma\mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t)$$
$$i = 1,\ldots,q, \quad x_i(0) \overset{\text{a.s.}}{=} x_{i0}, \quad t \geq 0. \tag{38}$$

In this case, (32) can be cast in the form of (1) with

$$f(x) = \begin{bmatrix} \sum_{j=1, j\neq 1}^{q} \mathcal{C}_{(1,j)}[\sigma_{1j}(x_j) - \sigma_{j1}(x_1)] \\ \vdots \\ \sum_{j=1, j\neq q}^{q} \mathcal{C}_{(q,j)}[\sigma_{qj}(x_j) - \sigma_{jq}(x_q)] \end{bmatrix} \tag{39}$$

$$D(x) = \begin{bmatrix} \sum_{j=1, j\neq 1}^{q} \gamma\mathcal{C}_{(1,j)}[x_j(t) - x_1(t)] \\ \vdots \\ \sum_{j=1, j\neq q}^{q} \gamma\mathcal{C}_{(q,j)}[x_j(t) - x_q(t)] \end{bmatrix} \tag{40}$$

where the stochastic term $D(x)\mathrm{d}w$ represents probabilistic variations in the charge transfer rate (i.e., generalized currents) between the agents. Furthermore, Assumption 5.2 now takes the following form.

Assumption 5.2′. For $i,j=1,\ldots,q$, $\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\le -(q-1)\gamma^2\mathcal{C}_{(i,j)}^2(x_i-x_j)^2$.

Theorem 5.1: Consider the nonlinear stochastic multiagent system given by (38) and assume that Assumptions 5.1 and 5.2′ hold. Then, for every $\alpha\in\mathbb{R}$, $\alpha\mathbf{e}_q$ is a stochastically semistable equilibrium state of (38). Furthermore, $x(t)\overset{\text{a.s.}}{\to}\frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x(0)$ as $t\to\infty$ and $\frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x(0)$ is a stochastically semistable equilibrium state.

Proof: To show that (32) is stochastically semistable, first note that if $x_i=x_j$, $i,j\in\{1,\ldots,q\}$, then $f_i(x)=0$ and $D_i(x)=0$ for all $i=1,\ldots,q$ is immediate from Assumption 5.1. Next, we show that $f_i(x)=0$ and $D_i(x)=0$ for all $i=1,\ldots,q$ implies $x_1=\cdots=x_q$. If $f_i(x)=0$ for all $i=1,\ldots,q$, then it follows from Assumption 5.2′ that:

$$\begin{aligned}0&=\sum_{i=1}^q x_i f_i(x)\\&=\sum_{i=1}^q x_i\left(\sum_{j=1,j\ne i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\right)\\&=\sum_{i=1}^q\sum_{j=1,j\ne i}^q\frac{1}{2}\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\\&\le\sum_{i=1}^q\sum_{j=1,j\ne i}^q-\frac{q-1}{2}\gamma^2\mathcal{C}_{(i,j)}^2(x_i-x_j)^2\\&\le 0\end{aligned}\tag{41}$$

and, by Assumption 5.2′, $\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\le -(q-1)\gamma^2\mathcal{C}_{(i,j)}^2(x_i-x_j)^2\le 0$ for $i,j=1,\ldots,q$. Hence, $\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]=0$ for $i,j=1,\ldots,q$, which implies $x_i=x_j$, $i,j=1,\ldots,q$. Therefore, $\mathcal{E}\overset{\triangle}{=}f^{-1}(0)\cap D^{-1}(0)=\{(x_1,\ldots,x_q)\in\mathbb{R}^q: x_1=\cdots=x_q=\alpha,\alpha\in\mathbb{R}\}$.

Next, consider the Lyapunov function candidate

$$V(x_1,\ldots,x_q)=\sum_{i=1}^q\frac{1}{2}(x_i-\alpha)^2,\quad (x_1,\ldots,x_q)\in\mathbb{R}^q\tag{42}$$

where $\alpha\in\mathbb{R}$. Now, the infinitesimal generator of the closed-loop system (32) is given by

$$\begin{aligned}&\mathcal{L}V(x_1,\ldots,x_q)\\&=\sum_{i=1}^q(x_i-\alpha)\left(\sum_{j=1,j\ne i}^q\mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\right)\\&\quad+\frac{1}{2}\sum_{i=1}^q\left(\sum_{j=1,j\ne i}^q\gamma\mathcal{C}_{(i,j)}(x_j-x_i)\right)^2,\\&\qquad(x_1,\ldots,x_q)\in\mathbb{R}^q.\end{aligned}\tag{43}$$

Note that since $\mathcal{C}_{(i,j)}=\mathcal{C}_{(j,i)}$, $i,j\in\{1,\ldots,q\}$, $i\ne j$, and $\mathcal{C}_{(i,i)}=0$, $i\in\{1,\ldots,q\}$, it follows that:

$$-\alpha\sum_{i=1}^q\sum_{j=1,j\ne i}^q\mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]=0\tag{44}$$

and hence

$$\begin{aligned}&\sum_{i=1}^q(x_i-\alpha)\left(\sum_{j=1,j\ne i}^q\mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\right)\\&=\sum_{i=1}^q x_i\left(\sum_{j=1,j\ne i}^q\mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\right)\\&=\sum_{i=1}^q\sum_{j=1,j\ne i}^q\frac{1}{2}\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)].\end{aligned}\tag{45}$$

Next, note that

$$\begin{aligned}&\sum_{i=1}^q\left(\sum_{j=1,j\ne i}^q\gamma\mathcal{C}_{(i,j)}(x_j-x_i)\right)^2\\&\le\sum_{i=1}^q(q-1)\sum_{j=1,j\ne i}^q\gamma^2\mathcal{C}_{(i,j)}^2(x_j-x_i)^2,\\&\qquad(x_1,\ldots,x_q)\in\mathbb{R}^q\end{aligned}$$

and hence, it follows from (43) that:

$$\begin{aligned}&\mathcal{L}V(x_1,\ldots,x_q)\\&\le\sum_{i=1}^q\sum_{j=1,j\ne i}^q\frac{1}{2}\mathcal{C}_{(i,j)}(x_i-x_j)[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\\&\quad+\frac{q-1}{2}\sum_{i=1}^q\sum_{j=1,j\ne i}^q\gamma^2\mathcal{C}_{(i,j)}^2(x_j-x_i)^2\\&=\sum_{i=1}^q\sum_{j=1,j\ne i}^q\frac{1}{2}\mathcal{C}_{(i,j)}(x_i-x_j)\Big([\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\\&\quad+(q-1)\gamma^2\mathcal{C}_{(i,j)}(x_i-x_j)\Big)\\&\le 0,\quad(x_1,\ldots,x_q)\in\mathbb{R}^q\end{aligned}\tag{46}$$

which, by Theorem 2.1, implies that $x_1=\cdots=x_q=\alpha$ is Lyapunov stable in probability.

Finally, note that $\mathcal{L}V(x_1,\ldots,x_q)\ne 0$ when $x_i\ne x_j$, $i,j\in\{1,\ldots,q\}$, $i\ne j$, and hence, $\mathcal{L}V(x_1,\ldots,x_q)<0$, $(x_1,\ldots,x_q)\in\mathbb{R}^q\setminus\mathcal{E}$. Therefore, it follows from Theorem 3.1 that $x_1=\cdots=x_q=\alpha$ is stochastically semistable for all $\alpha\in\mathbb{R}$. Furthermore, note that $\mathbf{e}_q^{\mathrm{T}}\mathrm{d}x(t)\overset{\text{a.s.}}{=}0$, $t\ge 0$, implies that

$$x(t)\overset{\text{a.s.}}{\to}\frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x(0)\overset{\text{a.s.}}{=}\frac{1}{q}[x_1(0)+\cdots+x_q(0)]\mathbf{e}_q\quad\text{as } t\to\infty$$

which proves the result. ■

Example 5.1: Consider the five mobile agents with the communication topology shown in Fig. 1 and dynamics on $\mathcal{H}_5$ given

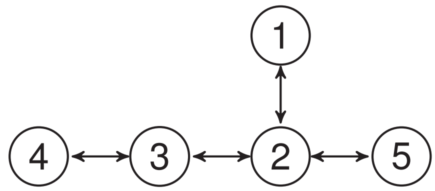

Fig. 1. Communication topology for the five mobile agents.

by

$$\mathrm{d}x_1(t) = u_1(t)\mathrm{d}t + \gamma[x_2(t) - x_1(t)]\mathrm{d}w(t),\quad x_1(0) \overset{\text{a.s.}}{=} x_{10},\quad t \geq 0 \tag{47}$$

$$\mathrm{d}x_2(t) = u_2(t)\mathrm{d}t + \gamma[x_1(t) - x_2(t) + x_3(t) - x_2(t) + x_5(t) - x_2(t)]\mathrm{d}w(t),\quad x_2(0) \overset{\text{a.s.}}{=} x_{20} \tag{48}$$

$$\mathrm{d}x_3(t) = u_3(t)\mathrm{d}t + \gamma[x_2(t) - x_3(t) + x_4(t) - x_3(t)]\mathrm{d}w(t)\quad x_3(0) \overset{\text{a.s.}}{=} x_{30} \tag{49}$$

$$\mathrm{d}x_4(t) = u_4(t)\mathrm{d}t + \gamma[x_3(t) - x_4(t)]\mathrm{d}w(t),\quad x_4(0) \overset{\text{a.s.}}{=} x_{40} \tag{50}$$

$$\mathrm{d}x_5(t) = u_5(t)\mathrm{d}t + \gamma[x_2(t) - x_5(t)]\mathrm{d}w(t),\quad x_5(0) \overset{\text{a.s.}}{=} x_{50} \tag{51}$$

with controls

$$u_1(t) = x_2(t) - x_1(t) \tag{52}$$

$$u_2(t) = x_1(t) - x_2(t) + x_3(t) - x_2(t) + x_5(t) - x_2(t) \tag{53}$$

$$u_3(t) = x_2(t) - x_3(t) + x_4(t) - x_3(t) \tag{54}$$

$$u_4(t) = x_3(t) - x_4(t) \tag{55}$$

$$u_5(t) = x_2(t) - x_5(t). \tag{56}$$

Note that (52)–(56) are of the form of (31) with $\sigma_{ij}(x_j) = x_j$, $i, j \in \{1, 2, 3, 4, 5\}$, $i \neq j$. For our simulation we take $x_{10} = 0$, $x_{20} = 10$, $x_{30} = 20$, $x_{40} = 30$, $x_{50} = 40$, and $\gamma = 0.2$. Fig. 2 shows the sample trajectories along with the standard deviation of the states of each agent versus time for 10 sample paths. The mean control profile is also plotted in Fig. 2. △

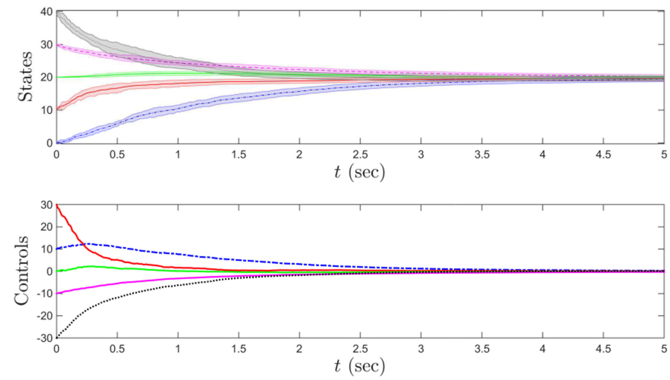

Fig. 2. Sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $x_1(t)$ in blue, $x_2(t)$ in red, $x_3(t)$ in green, $x_4(t)$ in magenta, and $x_5(t)$ in black. The control profile is plotted as the mean of the ten sample runs. (See color figure online.)

VI. Finite Time Consensus for Stochastic Dynamical Networks

Since in many consensus control protocol applications it is desirable for the closed-loop dynamical system that exhibits semistability to also possess the property that the system trajectories that almost surely converge to a Lyapunov stable in probability system state do so in finite time rather than merely asymptotically, in this section we build on the deterministic results of [4], [59] and use Theorem 4.1 to develop a thermodynamically motivated finite time consensus framework for multiagent nonlinear stochastic systems that achieve finite time stochastic semistability and almost sure state equipartition.

Specifically, let $d = 1$ and consider the q continuous-time agents with dynamics given by (37) with the nonlinear consensus protocol

$$u_i(t) = \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))] + c \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_j(t) - x_i(t))|x_j(t) - x_i(t)|^{\theta} \tag{57}$$

where $c > 0$ is a design constant, $0 < \theta < 1$, $\mathrm{sign}(y) \triangleq y/|y|$, $y \neq 0$, with $\mathrm{sign}(0) \triangleq 0$, and $\sigma_{ij}(\cdot)$, $i, j \in \{1, \ldots, q\}$, $i \neq j$, are as in (31). Note that the closed-loop system (37) and (57) is given by

$$\begin{aligned}\mathrm{d}x_i(t) = &\sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j(t)) - \sigma_{ji}(x_i(t))]\mathrm{d}t \\ &+ c \sum_{j=1, j\neq i}^{q} \mathcal{C}_{(i,j)}\mathrm{sign}(x_j(t) - x_i(t))|x_j(t) - x_i(t)|^{\theta} \\ &+ \sum_{j=1, j\neq i}^{q} \gamma\mathcal{C}_{(i,j)}[x_j(t) - x_i(t)]\mathrm{d}w(t) \\ &i = 1, \ldots, q,\quad x_i(0) \overset{\text{a.s.}}{=} x_{i0},\quad t \geq 0.\end{aligned} \tag{58}$$

Note that with $n = q$ and $d = 1$, (58) can be cast in the form of (1) with

$$f(x) = \begin{bmatrix} \sum_{j=1, j\neq 1}^{q} \mathcal{C}_{(1,j)}[\sigma_{1j}(x_j) - \sigma_{j1}(x_1)] \\ \quad + c\sum_{j=1, j\neq 1}^{q} \mathcal{C}_{(1,j)}\mathrm{sign}(x_j - x_1)|x_j - x_1|^{\theta} \\ \vdots \\ \sum_{j=1, j\neq q}^{q} \mathcal{C}_{(q,j)}[\sigma_{qj}(x_j) - \sigma_{jq}(x_q)] \\ \quad + c\sum_{j=1, j\neq q}^{q} \mathcal{C}_{(q,j)}\mathrm{sign}(x_j - x_q)|x_j - x_q|^{\theta} \end{bmatrix}$$

and $D(x)$ given by (40). Furthermore, note that since

$$\mathbf{e}_q^{\mathrm{T}}\mathrm{d}x(t) = \mathbf{e}_q^{\mathrm{T}}f(x(t))\mathrm{d}t + \mathbf{e}_q^{\mathrm{T}}D(x(t))\mathrm{d}w(t) = 0$$
$$x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0$$

it follows that $\sum_{i=1}^q \mathrm{d}x_i(t) \overset{\text{a.s.}}{=} 0$, $t \geq 0$, which implies that the total system charge is conserved, and hence, the controlled network satisfies an underlying conservation law.

The following proposition is necessary for the main result in this section. For the statement of this result and the main result of this section, let $L(\mathcal{C}) = [L_{(i,j)}]$ denote the graph Laplacian of $\mathfrak{G}(\mathcal{C})$, where $\mathcal{C} = [\mathcal{C}_{(i,j)}]$ and

$$L_{(i,j)} \triangleq \begin{cases} \sum_{k=1,k\neq i}^q \mathcal{C}_{(i,k)}, & j = i \\ -\mathcal{C}_{(i,j)}, & j \neq i \end{cases}. \tag{59}$$

Furthermore, let $\lambda_i(L(\mathcal{C}))$, $i \in \{1, \ldots, q\}$, denote the ith eigenvalue of $L(\mathcal{C})$ with $\lambda_{\min}(L(\mathcal{C})) \triangleq \lambda_1(L(\mathcal{C})) \leq \lambda_2(L(\mathcal{C})) \leq \cdots \leq \lambda_q(L(\mathcal{C})) \triangleq \lambda_{\max}(L(\mathcal{C}))$.

Proposition 6.1 ([15]): Consider the nonlinear stochastic multiagent system (58) with communication graph topology $\mathfrak{G}(\mathcal{C})$. Then the following statements hold.

i) $\lambda_1(L(\mathcal{C})) = 0$ with associated eigenvector $\mathbf{e}_q$.

ii) $x^{\mathrm{T}}L(\mathcal{C})x = \frac{1}{2}\sum_{i=1}^q \sum_{j=1,j\neq i}^q (x_j - x_i)^2$ for every $x = [x_1, \ldots, x_q]^{\mathrm{T}}$, and hence, $L(\mathcal{C})$ is nonnegative definite.

iii) $\lambda_2(L(\mathcal{C})) > 0$ and

$$\lambda_2(L(\mathcal{C})) = \min_{x\neq 0,\ \mathbf{e}_q^{\mathrm{T}}x=0} \frac{x^{\mathrm{T}}L(\mathcal{C})x}{x^{\mathrm{T}}x}. \tag{60}$$

Hence, if $\mathbf{e}_q^{\mathrm{T}}x = 0$, then

$$x^{\mathrm{T}}L(\mathcal{C})x \geq \lambda_2(L(\mathcal{C}))x^{\mathrm{T}}x, \quad x \in \mathbb{R}^q. \tag{61}$$

Theorem 6.1: Consider the nonlinear stochastic multiagent system given by (58) with $c > 0$ and $\theta \in (0, 1)$, and assume that Assumptions 5.1 and $5.2'$ hold. Then, for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{e}_q$ is a stochastically finite time semistable equilibrium state of (58). Moreover, $x(t) = \frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x(0)$ for all $t \geq T(x(0))$, where

$$\mathbb{E}^{x_0}[T(x(0))] \leq \frac{4V(x_0)^{\frac{1-\theta}{2}}}{c(1-\theta)(4\lambda_2(L(\mathcal{C})))^{\frac{1+\theta}{2}}}$$

and

$$V(x_0) = \frac{1}{2}\left(x_0 - \frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x_0\right)^{\mathrm{T}}\left(x_0 - \frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x_0\right).$$

Proof: To show that (58) is stochastically semistable, first note that if $x_i = x_j,\ i, j \in \{1, \ldots, q\}$, then $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ is immediate from Assumption 5.1. Next, we show that $f_i(x) = 0$ and $D_i(x) = 0$ for all $i = 1, \ldots, q$ implies $x_1 = \cdots = x_q$. If $f_i(x) = 0$ for all $i = 1, \ldots, q$, then it follows that from Assumption $5.2'$ that

$$\begin{aligned} 0 &= \sum_{i=1}^q x_i f_i(x) \\ &= \sum_{i=1}^q x_i \left(\sum_{j=1,j\neq i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]\right) \\ &\quad + c\sum_{i=1}^q x_i \left(\sum_{j=1,j\neq i}^q \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i)|x_j - x_i|^\theta\right) \\ &\leq \sum_{i=1}^q x_i \left(\sum_{j=1,j\neq i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]\right) \\ &= \sum_{i=1}^q \sum_{j=1,j\neq i}^q \frac{1}{2}\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \\ &\leq \sum_{i=1}^q \sum_{j=1,j\neq i}^q -\frac{q-1}{2}\gamma^2\mathcal{C}_{(i,j)}^2(x_i - x_j)^2 \\ &\leq 0 \end{aligned} \tag{62}$$

and, by Assumption $5.2'$, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -(q-1)\gamma^2\mathcal{C}_{(i,j)}^2(x_i - x_j)^2 \leq 0$ for $i, j = 1, \ldots, q$. Hence, $\mathcal{C}_{(i,j)}(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] = 0$ for $i, j = 1, \ldots, q$, which implies $x_i = x_j,\ i, j = 1, \ldots, q$. Therefore, $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) = \{(x_1, \ldots, x_q) \in \mathbb{R}^q : x_1 = \cdots = x_q = \alpha, \alpha \in \mathbb{R}\}$. Furthermore, since $\sum_{i=1}^q \mathrm{d}x_i(t) \overset{\text{a.s.}}{=} 0$, $t \geq 0$, it follows that $\sum_{i=1}^q x_i(t) \overset{\text{a.s.}}{=} \sum_{i=1}^q x_i(0)$, $t \geq 0$, and hence, $\alpha = \frac{1}{q}\mathbf{e}_q^{\mathrm{T}}x(0)$.

Next, consider the Lyapunov function candidate

$$V(x_1, \ldots, x_q) = \sum_{i=1}^q \frac{1}{2}(x_i - \alpha)^2, \quad (x_1, \ldots, x_q) \in \mathbb{R}^q \tag{63}$$

where $\alpha = \frac{1}{q}\mathbf{e}_q^{\mathrm{T}}x_0$, and note that $V^{-1}(0) = \mathcal{E}$. Now, the infinitesimal generator of the closed-loop system (58) is given by

$$\begin{aligned} &\mathcal{L}V(x_1, \ldots, x_q) \\ &= \sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1,j\neq i}^q \mathcal{C}_{(i,j)}[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]\right) \\ &\quad + c\sum_{i=1}^q (x_i - \alpha)\left(\sum_{j=1,j\neq i}^q \mathcal{C}_{(i,j)}\mathrm{sign}(x_j - x_i)|x_j - x_i|^\theta\right) \\ &\quad + \frac{1}{2}\sum_{i=1}^q \left(\sum_{j=1,j\neq i}^q \gamma\mathcal{C}_{(i,j)}(x_j - x_i)\right)^2, \\ &\qquad (x_1, \ldots, x_q) \in \mathbb{R}^q. \end{aligned} \tag{64}$$

Using identical arguments as in the proof of Theorem 5.1, the first and last terms in (64) give

$$\begin{aligned}&\sum_{i=1}^{q}(x_i-\alpha)\left(\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}[\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\right)\\&\quad+\frac{1}{2}\sum_{i=1}^{q}\left(\sum_{j=1,j\neq i}^{q}\gamma\mathcal{C}_{(i,j)}(x_j-x_i)\right)^2\\&\leq\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\frac{1}{2}\mathcal{C}_{(i,j)}(x_i-x_j)\Big([\sigma_{ij}(x_j)-\sigma_{ji}(x_i)]\\&\quad+(q-1)\gamma^2\mathcal{C}_{(i,j)}(x_i-x_j)\Big)\\&\leq 0,\quad (x_1,\ldots,x_q)\in\mathbb{R}^q.\end{aligned}\tag{65}$$

Next, the second term in (64) gives

$$\begin{aligned}&c\sum_{i=1}^{q}(x_i-\alpha)\left(\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(x_j-x_i)|x_j-x_i|^{\theta}\right)\\&=c\sum_{i=1}^{q}x_i\left(\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}\mathrm{sign}(x_j-x_i)|x_j-x_i|^{\theta}\right)\\&=\frac{c}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_i-x_j)\mathrm{sign}(x_j-x_i)|x_j-x_i|^{\theta}\\&=-\frac{c}{2}\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\left(\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j-x_i)^2\right)^{\frac{1+\theta}{2}}\\&\leq-\frac{c}{2}\left(\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j-x_i)^2\right)^{\frac{1+\theta}{2}},\\&\qquad (x_1,\ldots,x_q)\in\mathbb{R}^q\end{aligned}\tag{66}$$

where the last inequality in (66) follows from [60, Fact 2.11.130]. Now, note that the last term in (66) satisfies

$$\begin{aligned}&-\frac{c}{2}\left(\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j-x_i)^2\right)^{\frac{1+\theta}{2}}\\&=-\frac{c}{2}\left(\frac{\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j-x_i)^2}{V(x_1,\ldots,x_q)}V(x_1,\ldots,x_q)\right)^{\frac{1+\theta}{2}}.\end{aligned}\tag{67}$$

Next, define $x_{\mathrm{s}i}\triangleq x_i-\alpha$ and note that $x_{\mathrm{s}j}-x_{\mathrm{s}i}=x_j-x_i$. Furthermore, note that $\mathbf{e}_q^{\mathrm{T}}x_{\mathrm{s}}(t)\overset{\text{a.s.}}{=}0$, $t\geq 0$, where $x_{\mathrm{s}}=[x_{\mathrm{s}1},\ldots,x_{\mathrm{s}q}]^{\mathrm{T}}$. Now, since $\mathcal{C}_{(i,j)}=1$ or 0, clearly $\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}=\mathcal{C}_{(i,j)}$ for every $0<\theta<1$. Thus, by Proposition 6.1

$$\begin{aligned}&\frac{\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}^{\frac{2}{1+\theta}}(x_j-x_i)^2}{V(x_1,\ldots,x_q)}\\&\quad=\frac{\sum_{i=1}^{q}\sum_{j=1,j\neq i}^{q}\mathcal{C}_{(i,j)}(x_{\mathrm{s}j}-x_{\mathrm{s}i})^2}{V(x_1,\ldots,x_q)}\\&\quad=\frac{2x_{\mathrm{s}}^{\mathrm{T}}L(\mathcal{C})x_{\mathrm{s}}}{\frac{1}{2}x_{\mathrm{s}}^{\mathrm{T}}x_{\mathrm{s}}}\\&\quad\geq 4\lambda_2(L(\mathcal{C}))\\&\quad>0,\quad (x_1,\ldots,x_q)\in\mathbb{R}^q\ \mathcal{E}.\end{aligned}\tag{68}$$

Hence, using (65)–(68) it follows from (64) that:

$$\mathcal{L}V(x_1,\ldots,x_q)\leq-\frac{c}{2}\Big(4\lambda_2(L(\mathcal{C}))\Big)^{\frac{1+\theta}{2}}V(x_1,\ldots,x_q)^{\frac{1+\theta}{2}},\quad (x_1,\ldots,x_q)\in\mathbb{R}^q\backslash\mathcal{E}.\tag{69}$$

Now, by Theorem 4.1 and Remark 4.1, $x(t)=\frac{1}{q}\mathbf{e}_q\mathbf{e}_q^{\mathrm{T}}x(0)$, $t\geq T(x(0))$, where

$$\mathbb{E}^{x_0}[T(x(0))]\leq\frac{4V(x_0)^{\frac{1-\theta}{2}}}{c(1-\theta)(4\lambda_2(L(\mathcal{C})))^{\frac{1+\theta}{2}}}.$$

■

Example 6.1: Consider the five mobile agents with the communication topology shown in Fig. 1 and dynamics on $\mathcal{H}_5$ given by (47)–(51). Furthermore, let

$$\begin{aligned}u_1(t)&=c\mathrm{sign}(x_2(t)-x_1(t))|x_2(t)-x_1(t)|^{0.5}\\&\quad+x_2(t)-x_1(t)\end{aligned}\tag{70}$$

$$\begin{aligned}u_2(t)&=c\sum_{j=1,3,5}\mathrm{sign}(x_j(t)-x_2(t))|x_j(t)-x_2(t)|^{0.5}\\&\quad+\sum_{j=1,3,5}(x_j(t)-x_2(t))\end{aligned}\tag{71}$$

$$\begin{aligned}u_3(t)&=c\sum_{j=2,4}\mathrm{sign}(x_j(t)-x_3(t))|x_j(t)-x_3(t)|^{0.5}\\&\quad+\sum_{j=2,4}(x_j(t)-x_3(t))\end{aligned}\tag{72}$$

$$\begin{aligned}u_4(t)&=c\mathrm{sign}(x_3(t)-x_4(t))|x_3(t)-x_4(t)|^{0.5}\\&\quad+x_3(t)-x_4(t)\end{aligned}\tag{73}$$

$$\begin{aligned}u_5(t)&=c\mathrm{sign}(x_2(t)-x_5(t))|x_2(t)-x_5(t)|^{0.5}\\&\quad+x_2(t)-x_5(t)\end{aligned}\tag{74}$$

where $c=5$. Note that (70)–(74) are of the form of (57) with $\sigma_{ij}(x_j)=x_j$, $i,j\in\{1,2,3,4,5\}$, $i\neq j$. Let $x_{10}=0$, $x_{20}=10$, $x_{30}=20$, $x_{40}=30$, $x_{50}=40$, and $\gamma=0.2$. Fig. 3 shows the sample trajectories along with the standard deviation of the states of each agent versus time for ten sample paths. The mean control profile is also plotted in Fig. 3. △

VII. Illustrative Numerical Example

In this section, we demonstrate the proposed distributed stochastic consensus framework on a set of control commanded

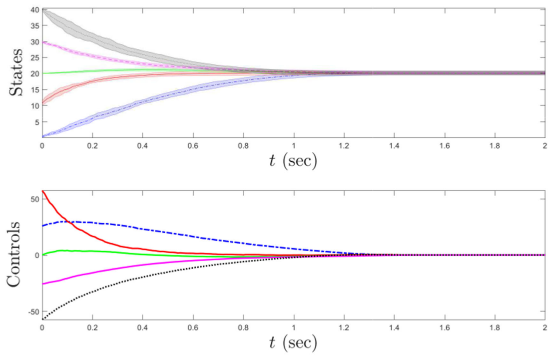

Fig. 3. Sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $x_1(t)$ in blue, $x_2(t)$ in red, $x_3(t)$ in green, $x_4(t)$ in magenta, and $x_5(t)$ in black. The control profile is plotted as the mean of the ten sample runs. (See color figure online.)

aircrafts achieving asymptotic pitch rate consensus. Specifically, consider the multiagent system comprised of the controlled longitudinal motion of three Boeing 747 aircrafts [61] linearized at an altitude of 40 kft and a velocity of 774 ft/s given by

$$\dot{z}_i(t) = Az_i(t) + B\delta_i(t), \quad z_i(0) = z_{i_0} \quad i = 1, 2, 3, \quad t \geq 0 \tag{75}$$

where $z_i(t) = [v_{x_i}(t), v_{z_i}(t), q_i(t), \theta_{e_i}(t)]^{\mathrm{T}} \in \mathbb{R}^4$, $t \geq 0$, is state vector of agent $i \in \{1, 2, 3\}$, with $v_{x_i}(t)$, $t \geq 0$, representing the x–body–axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/s), $v_{z_i}(t)$, $t \geq 0$, representing the z–body–axis component of the velocity of the aircraft center of mass with respect to the reference axes (in ft/s), $q_i(t)$, $t \geq 0$, representing the y–body–axis component of the angular velocity of the aircraft (pitch rate) with respect to the reference axes (in crad/s), $\theta_{e_i}(t)$, $t \geq 0$, representing the pitch Euler angle of the aircraft body axes with respect to the reference axes (in crad), $\delta(t)$, $t \geq 0$, representing the elevator control input (in crad), and

$$A = \begin{bmatrix} -0.003 & 0.039 & 0 & -0.332 \\ -0.065 & -0.319 & 7.74 & 0 \\ 0.020 & -0.101 & -0.429 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.010 \\ -0.180 \\ -1.16 \\ 0 \end{bmatrix}. \tag{76}$$

We propose a two-level control hierarchy composed of a lower-level controller for command following of the three aircrafts and a higher-level consensus controller for pitch rate consensus in the face of an uncertain triangular communication topology between the aircrafts. To address the lower-level controller design, let $x_i(t)$, $i = 1, 2, 3$, $t \geq 0$, denote a command generated by (32) (i.e., the guidance command) and let $s_i(t)$, $i = 1, 2, 3$, $t \geq 0$, denote the integrator state satisfying

$$\dot{s}_i(t) = Ez_i(t) - x_i(t), \quad s_i(0) = s_{i_0} \quad i = 1, 2, 3, \quad t \geq 0 \tag{77}$$

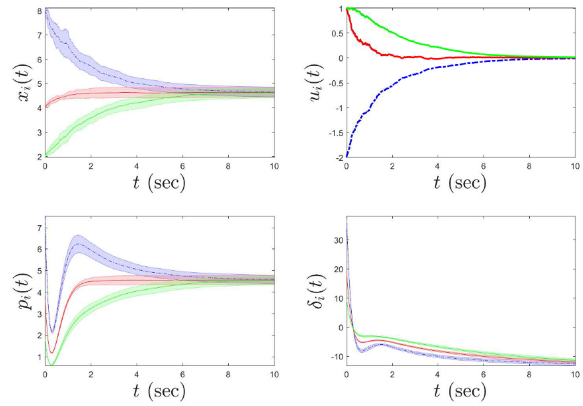

Fig. 4. Sample average along with the sample standard deviation versus time for agent guidance state ($x_i(t)$, $t \geq 0$), guidance input ($u_i(t)$, $t \geq 0$), pitch rate ($q_i(t)$, $t \geq 0$), and elevator control ($\delta_i(t)$, $t \geq 0$) responses for the standard consensus protocol given by (31) with $k_1 = 1$. The control profile is plotted as the mean of the 10 sample runs. (See color figure online.)

where $E = [0, 0, 1, 0]$. Now, defining the augmented state $\hat{z}(t) \triangleq [z^{\mathrm{T}}(t), s_i(t)]^{\mathrm{T}}$, (75) and (77) give

$$\dot{\hat{z}}_i(t) = \hat{A}\hat{z}_i(t) + \hat{B}_1\delta_i(t) + \hat{B}_2 x_i(t), \quad \hat{z}_i(0) = \hat{z}_{i_0}$$
$$i = 1, 2, 3, \quad t \geq 0 \tag{78}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & 0 \\ E & 0 \end{bmatrix}, \quad \hat{B}_1 \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{B}_2 \triangleq \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{79}$$

Furthermore, let the elevator control input be given by

$$\delta(t) = -K\hat{z}(t)$$
$$K = [-0.0157, 0.0831, -4.7557, -0.1400, -9.8603] \tag{80}$$

which is designed using an optimal linear-quadratic regulator.

For the higher level communication consensus controller design, we use (31) with $\sigma_{ij}(x_j) = x_j$ and $\sigma_{ji}(x_i) = x_i$ to generate $x_i(t)$, $t \geq 0$, that has a direct effect on the lower level controller design to achieve pitch rate consensus. Fig. 4 presents the sample trajectories along with the standard deviation of the states of each agent versus time for ten sample paths for all initial conditions set to zero and $x_1(0) \overset{\text{a.s.}}{=} 8$, $x_2(0) \overset{\text{a.s.}}{=} 4$, and $x_3(0) \overset{\text{a.s.}}{=} 2$. The mean control profile is also plotted in Fig. 4.

VIII. Conclusion

This article extends the notions semistability and finite time semistability to nonlinear stochastic dynamical systems having a continuum of equilibria. In particular, Lyapunov and converse Lyapunov theorems for stochastic semistability are established, as well as sufficient conditions for stochastic finite time semistability are presented. These results are then used to develop a thermodynamic-based framework for addressing consensus problems for multiagent dynamical systems with stochastic communication uncertainty between agents in the network. Specifically, nonlinear network protocols are designed that guarantee almost sure asymptotic and finite time convergence to Lyapunov stable in probability equilibria over a network of dynamic agents in the face of uncertain information flows. Our analysis relies on several tools from algebraic graph theory, stochastic semistability, stochastic finite time semistability, and dynamical thermodynamics [42]. Future extensions will focus on robustness properties of the proposed protocols, as well as asynchronism, system time delays, and dynamic network topologies for addressing possible information asynchronizing between agents, message transmission and processing delays, and communication link failures and communication dropouts.

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