



Synchronization of biological neural network systems with stochastic perturbations and time delays[☆]

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Abstract

With advances in biochemistry, molecular biology, and neurochemistry there has been impressive progress in the understanding of the molecular properties of anesthetic agents. However, despite these advances, we still do not understand how anesthetic agents affect the properties of neurons that translate into the induction of general anesthesia at the macroscopic level. There is extensive experimental verification that collections of neurons may function as oscillators and the synchronization of oscillators may play a key role in the transmission of information within the central nervous system. This may be particularly relevant to understand the mechanism of action for general anesthesia. In this paper, we develop a stochastic synaptic drive firing rate model for an excitatory and inhibitory cortical neuronal network in the face of system time delays and stochastic input disturbances. In addition, we provide sufficient conditions for global asymptotic and exponential mean-square synchronization for this model.

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1. Introduction

Numerous complex large-scale dynamical networks often demonstrate a degree of synchronization. System synchronization typically involves coordination of events that allows a dynamical system to operate in unison resulting in system self-organization. The onset of synchronization in populations of coupled dynamical networks has been studied for various complex networks including network models for mathematical biology, statistical physics, kinetic theory, bifurcation theory, as well as plasma physics [1]. Synchronization of firing neural oscillator populations using probabilistic analysis has also been addressed in the neuroscience literature [2]. One of the most important questions in neuroscience is how do neurons, or collections of neurons, communicate. In other words, what is the neural code? There is extensive experimental verification that collections of neurons may function as oscillators [3–5] and the synchronization of oscillators may play a key role in the transmission of information within the central nervous system. This may be particularly relevant to understand the mechanism of action for general anesthesia [6].

It has been known for a long time that general anesthesia has profound effects on the spectrum of oscillations in the electroencephalograph [7,8]. More recently, the authors in [9] have suggested that thalamocortical circuits function as neural pacemakers and that alterations in the thalamic oscillations are associated with the induction of general anesthesia. Furthermore, it is well known that anesthetic drugs frequently induce epileptiform activity as part of the progression to the state of unconsciousness [10].

Multiple lines of evidence indicate that anesthetic agents impact neural oscillators. In addition, epileptiform activity implies synchronization of oscillators. This leads to the possibility that synchronization of these oscillators is involved in the transition to the anesthetic state. In this paper, we extend the synaptic drive model of a network of biological neurons developed in [6] to investigate the conditions that would lead to synchronization of neural oscillators. In particular, we develop an excitatory and inhibitory synaptic drive firing rate model with time-varying delays and stochastic input uncertainty, and its global synchronization is investigated. The system uncertainty model involves a Markov process wherein stochastic integration is interpreted in the sense of Itô.

The notation used in this paper is fairly standard. Specifically, \mathbb{R}^n denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $\|\cdot\|$ denotes the Euclidean vector norm, and $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\tau > 0$ denotes a family of continuous vector-valued functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^n with topology of uniform convergence and designated operator norm $\|\psi\| = \sup_{-\tau \leq \theta \leq 0} \|\psi(\theta)\|$ for $\psi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. We write I or I_n for the $n \times n$ identity matrix, $\mathbf{1}_n$ for the $n \times 1$ ones vector, $\text{tr}(\cdot)$ for the trace operator, $\text{rank } A$ for the rank of the matrix A , $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of a Hermitian matrix A , $\ker(A)$ for the kernel (nullspace) of the matrix A , $\text{span}(\mathbf{1}_n)$ for the span of $\mathbf{1}_n$, $\mathbb{E}[\cdot]$ for the expectation operator for a given probability space, and $\mathbb{E}[X|Y]$ for the conditional expectation of X with respect to Y .

2. Biological neural networks

The fundamental building block of the central nervous system, the *neuron*, can be divided into three functionally distinct parts, namely, the *dendrites*, *soma* (or cell body), and *axon*. The dendrites play the role of input devices that collect signals from other neurons and

transmit them to the soma; whereas the soma generates a signal that is transmitted to other neurons by the axon. The axons of other neurons connect to the dendrites and soma surfaces by means of connectors called *synapses*. The behavior of the neuron is best described in terms of the electrochemical potential gradient across the cell membrane. If the voltage gradient across the membrane increases to a critical threshold value, then there is a subsequent abrupt step-like increase in the potential gradient, the action potential. This action potential is transmitted from the soma along the axon to a dendrite of a receiving neuron. The action potential elicits the release of neurotransmitter molecules that diffuse to the dendrite of a “receiving” neuron. This alters the voltage gradient across the receiving neuron.

The electrochemical potential for a neuron can be described by a nonlinear four-state system [11]. Coupling these system equations for each neuron in a large neural population is computationally prohibitive. To simplify the mathematical modeling, it has been common to use phenomenological firing rate models for studying neural coding, memory, and network dynamics [11]. Firing rate models involve the averaged behavior of the spiking rates of groups of neurons rather than tracking the spike rate of each individual neuron cell. In such population models, the activity of a neuron, that is, the rate at which the neuron generates an action potential (fires) is modeled as a function of the voltage (across the membrane). The “firing” of a neuron evokes voltage changes, postsynaptic potentials on receiving neurons; that is, neurons electrically connected to the firing neurons via axon–dendrite connections. In general, neurons are either excitatory or inhibitory depending on whether the postsynaptic potential increases or decreases the potential of the receiving neuron. In particular, excitatory neurotransmitters *depolarize* postsynaptic membranes by increasing membrane potentials and can collectively generate an action potential. Inhibitory neurotransmitters *hyperpolarize* the postsynaptic membrane by decreasing membrane potentials, thereby nullifying the actions of excitatory neurotransmitters and in certain cases prevent the generation of action potentials.

Biological neural network models predict a voltage in the receiving or postsynaptic neuron given by

$$V(t) = \sum_{i=1}^{n_E} \sum_j \alpha_i^E(t-t_j) + \sum_{i'=1}^{n_I} \sum_{j'} \alpha_{i'}^I(t-t_{j'}) \tag{1}$$

where $i \in \{1, \dots, n_E\}$ and $i' \in \{1, \dots, n_I\}$ enumerate the action potential or firings of the excitatory and inhibitory transmitting (presynaptic) neurons at firing times t_j and $t_{j'}$, respectively, n_E and n_I denote the number of the excitatory and inhibitory transmitting neurons, respectively, and $\alpha_i^E(\cdot)$ and $\alpha_{i'}^I(\cdot)$ are the functions (in volts) describing the evolution of the excitatory and inhibitory postsynaptic potentials, respectively.

Using a (possibly discontinuous) function $f_i(\cdot)$ to represent the firing rate (in Hz) of the i th neuron and assuming that the firing rate is a function of the voltage $v_i^E(\cdot)$ (resp., $v_i^I(\cdot)$) across the membrane of the i th neuron given by $f_i(v_i^E)$ (resp., $f_i(v_i^I)$), it follows that

$$v_i^E(t) = \sum_{j=1, j \neq i}^{n_E} A_{ij}^{EE} \int_{-\infty}^t \alpha_j^E(t-\tau) f_j(v_j^E(\tau)) d\tau + \sum_{j'=1}^m A_{ij'}^{EI} \int_{-\infty}^t \alpha_{j'}^I(t-\tau) f_{j'}(v_{j'}^I(\tau)) d\tau + v_{th_i}^E(t), \quad i = 1, \dots, n_E, \tag{2}$$

$$v_i^I(t) = \sum_{j=1}^{n_E} A_{ij}^{IE} \int_{-\infty}^t \alpha_j^E(t-\tau) f_j(v_j^E(\tau)) d\tau + \sum_{j=1, j \neq i}^{n_I} A_{ij}^{II} \int_{-\infty}^t \alpha_j^I(t-\tau) f_j(v_j^I(\tau)) d\tau + v_{th_i}^I(t), \quad i = 1, \dots, n_I, \tag{3}$$

where the *neuronal connectivity matrix* A^{XY} , with units of volts \times synapse, contains entries A_{ij}^{XY} , $X, Y \in \{E, I\}$, representing the coupling strength of the j th neuron on the i th neuron such that either $A_{ij}^{XE} > 0$ or $A_{ij}^{XI} < 0$, $X \in \{E, I\}$, if the j th neuron is connected (i.e., contributes a postsynaptic potential) to the i th neuron, and $A_{ij}^{XY} = 0$, otherwise. Furthermore, $v_{th_i}^E(\cdot)$ and $v_{th_i}^I(\cdot)$ are continuous threshold input voltages. Note that $A_{ii}^{EE} \triangleq A_{ii}^{II} \triangleq 0$ by definition.

Next, defining the *synaptic drive*—a dimensionless quantity per synapse—of each (excitatory or inhibitory) neuron by

$$S_i^{(E,I)}(t) \triangleq \int_{-\infty}^t \alpha_i^{(E,I)}(t-\tau) f_i(v_i^{(E,I)}(\tau)) d\tau, \tag{4}$$

and assuming

$$\alpha_i^{(E,I)}(t) = B^{(E,I)} e^{-t/\lambda_i^{(E,I)}}, \tag{5}$$

where the dimensionless gain $B^{(E,I)}$ is equal to B^E if the i th neuron is excitatory and B^I if the i th neuron is inhibitory, and similarly for $S_i^{(E,I)}$, $v_i^{(E,I)}$, $\alpha_i^{(E,I)}$, and $\lambda_i^{(E,I)}$, it follows from Eqs. (4) and (5) that

$$\frac{dS_i^{(E,I)}(t)}{dt} = -\frac{1}{\lambda_i^{(E,I)}} S_i^{(E,I)}(t) + B^{(E,I)} f_i(v_i^{(E,I)}(t)). \tag{6}$$

Now, using the expressions for the excitatory and inhibitory voltage given by Eqs. (2) and (3), respectively, it follows that

$$\frac{dS_i^E(t)}{dt} = -\frac{1}{\lambda_i^E} S_i^E(t) + B^E f_i \left(\sum_{j=1, j \neq i}^{n_E} A_{ij}^{EE} S_j^E(t) + \sum_{j=1}^{n_I} A_{ij}^{EI} S_j^I(t) + v_{th_i}^E(t) \right), \quad i = 1, \dots, n_E, \tag{7}$$

$$\frac{dS_i^I(t)}{dt} = -\frac{1}{\lambda_i^I} S_i^I(t) + B^I f_i \left(\sum_{j=1}^{n_E} A_{ij}^{IE} S_j^E(t) + \sum_{j=1, j \neq i}^{n_I} A_{ij}^{II} S_j^I(t) + v_{th_i}^I(t) \right), \quad i = 1, \dots, n_I. \tag{8}$$

The above analysis reveals that a form for capturing the neuroelectronic behavior of biological excitatory or inhibitory neuronal networks can be written as

$$\frac{dS_i(t)}{dt} = -\tau_i S_i(t) + B_i f_i \left(\sum_{j=1, j \neq i}^n A_{ij} S_j(t) + v_{th_i}(t) \right), \quad S_i(0) = S_{i0}, \quad t \geq 0, \quad i = 1, \dots, n, \tag{9}$$

where $S_i(t) \in \mathcal{D} \subseteq \mathbb{R}$, $t \geq 0$, is the i th synaptic drive, $v_{th_i}(t) \in \mathbb{R}$, $t \geq 0$, denotes the threshold input voltage of the i th neuron, A_{ij} is a constant representing the coupling strength of the j th neuron on the i th neuron, $\tau_i \triangleq 1/\lambda_i$ is a time constant, B_i is a constant gain for the firing rate of the i th neuron, and $f_i(\cdot)$ is a nonlinear activation function describing the relationship between the synaptic drive and the firing rate of the i th neuron.

In this paper, we assume that $f_i(\cdot)$ is a continuous function such as a half-wave rectification function. Specifically, for a typical neuron [12]

$$f_i(x) = [x]_+, \tag{10}$$

where $i \in \{1, \dots, n\}$ and $[x]_+ = x$ if $x \geq 0$, and $[x]_+ = 0$ otherwise. Alternatively, we can approximate $f_i(x)$ by the smooth (i.e., infinitely differentiable) half-wave rectification function

$$f_i(x) = \frac{x e^{\gamma x}}{1 + e^{\gamma x}}, \tag{11}$$

where $i \in \{1, \dots, n\}$ and $\gamma \gg 0$. Note that $f'_i(x) \approx 1$ for $x > 0$ and $f''_i(x) \approx 0, x \neq 0$. In addition, note that Eqs. (10) and (11) reflect the fact that as the voltage increases across the membrane of the i th neuron, the firing rate increases as well. Often, the membrane potential-firing rate curve exhibits a linear characteristic for a given range of voltages. At higher voltages, however, a saturation phenomenon appears, indicating that the full effect of the firing rate has been reached. To capture this effect, $f_i(\cdot)$ can be modeled as

$$f_i(x) = \frac{f_{\max} e^{\gamma x}}{1 + e^{\gamma x}}, \tag{12}$$

where $i \in \{1, \dots, n\}, \gamma \gg 0$, and $f_{\max} = \lim_{\gamma \rightarrow \infty} f_i(x), x > 0$, denotes the maximum firing rate.

3. Synaptic drive firing model with time-varying delay and stochastic multiplicative uncertainty

Let $S_t \in \mathcal{C}((-\infty, +\infty), \mathbb{R}^n)$, where $S_t = S(t + \theta), \theta \in (-\infty, 0], t \geq 0$, and assume $v_{th_i}(t) \equiv 0$. To capture communication delays in our biological neural network model (9), define $S(t) \triangleq [S_1(t), S_2(t), \dots, S_n(t)]^T, f(S) \triangleq [f_1(S_1), f_2(S_2), \dots, f_n(S_n)]^T$, where $f_i(\cdot)$ is defined by Eq. (11) or Eq. (12), $L \triangleq \text{diag}[\tau_1, \tau_2, \dots, \tau_n]$ and $B \triangleq \text{diag}[B_1, B_2, \dots, B_n]$. Furthermore, define

$$\hat{S}(t) \triangleq \begin{bmatrix} \sum_{j=2}^n A_{1j} S_j(t - \delta_{1j}(t)) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \sum_{j=1, j \neq 2}^n A_{2j} S_j(t - \delta_{2j}(t)) \\ \vdots \\ 0 \end{bmatrix} + \dots \\ + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{n-1} A_{nj} S_j(t - \delta_{nj}(t)) \end{bmatrix}, \tag{13}$$

where $\delta_{ij}(t)$ denotes the continuous or discontinuous, time-varying time delay of the transmission signal from the j th neuron to the i th neuron at time $t, \delta_{ij}(t) \geq 0, t \geq 0$, and $S_j(t)$ denotes the j th component of $S(t)$. The system delays $\delta_{ij}(t)$ correspond to the times of the spike hitting the synapse and t is the time after the spike, and hence, these delays account for the distance traveled by the voltage spikes down the axon.

We modify the biological neural network system (9) to include the effects of stochastic perturbations as well as time delays. Specifically, we consider the model

$$dS(t) = (-LS(t) + Bf(\hat{S}(t))) dt + \sigma(S(t)) dw(t), \\ S(\theta) = \phi(\theta), \quad -\infty < \theta \leq 0, \quad t > 0, \tag{14}$$

where $\phi(\cdot) \in \mathcal{C} \triangleq \mathcal{C}((-\infty, 0], \mathbb{R}^n)$ is a continuous vector-valued function specifying the initial state of the system (14), $w(t) = [w_1(t), w_2(t), \dots, w_n(t)]^T$ captures noise in the input voltage and is represented as Brownian motion, that is, an n -dimensional mutually independent standard Wiener process, and $\sigma(S) = \text{diag}[\sigma_1(S), \sigma_2(S), \dots, \sigma_n(S)]$ represents the state-dependent noise intensity matrix for the Gaussian white noise process $dw(t)$. Henceforth, we consider Eq. (14) as the model of the perturbed biological neural network.

Next, since $\hat{S}(t)$ defined by Eq. (13) contains $n(n-1)$ terms with different time delays, each term can be written as the product of an $n \times n$ -dimensional matrix and an n -dimensional vector. Specifically, for $i' = 1, 2, \dots, n, j = 1, 2, \dots, n, i' \neq j$, define $i \triangleq i'(n-1) + j, i' > j$, and $i \triangleq i'(n-1) + j - 1, i' < j$, where $i = 1, 2, \dots, n(n-1)$, define $\delta_i(t) \triangleq \delta_{i'j}(t)$, and define the matrix $A_i \in \mathbb{R}^{n \times n}$ whose (i', j) th entry is $A_{i'j}$ and all the other entries are 0. Thus, the i th term in Eq. (13) can be replaced by $A_i S(t - \delta_i(t)), i \in \{1, 2, \dots, n(n-1)\}$. Hence, setting $N = n(n-1), \hat{S}(t)$ can be written as

$$\hat{S}(t) = \sum_{i=1}^N A_i S(t - \delta_i(t)). \tag{15}$$

For the statement of the results in this paper, we require some additional notation and definitions. Specifically, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space associated with Eq. (14), where Ω denotes the sample space, \mathcal{F} denotes a σ -algebra, and \mathbb{P} defines a probability measure on the σ -algebra \mathcal{F} , that is, \mathbb{P} is a nonnegative countably additive set function on \mathcal{F} such that $\mathbb{P}(\Omega) = 1$ [13]. Note that Eq. (14) is a Markov process, and hence, there exists a filtration $\{\mathcal{F}_t\}$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t \subset \mathcal{F}, 0 \leq \tau < t$, such that $\{\omega \in \Omega : S(t) \in \mathcal{B}\} \in \mathcal{F}_t, t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ -algebra \mathcal{B} . The infinitesimal operator $\mathcal{L} : [0, \infty) \times \mathcal{C}((-\infty, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$ associated with the stochastic process (14), acting on the functional $V : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}$, is defined by

$$\mathcal{L}V(t, S_t) \triangleq \limsup_{h \rightarrow 0^+} \frac{\mathbb{E}[V(t+h, S_{t+h}) | S_t] - V(t, S_t)}{h}. \tag{16}$$

For a two-times continuously differentiable function $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ of the random variable S , the infinitesimal operator $\mathcal{L}V(t, S)$ is defined as [14,15]

$$\begin{aligned} \mathcal{L}V(t, S) &\triangleq \lim_{h \rightarrow 0^+} \frac{\mathbb{E}[V(t+h, S(t+h))] - V(t, S)}{h} \\ &= \frac{\partial V(t, S)}{\partial t} + V'(t, S)(-LS + Bf(\hat{S})) + \frac{1}{2} \sigma^T(S) V''(t, S) \sigma(S), \end{aligned} \tag{17}$$

where $V'(t, S)$ denotes the Fréchet derivative of V and $V''(t, S)$ denotes the Hessian matrix of V with respect to S at (t, S) . The following lemma provides an explicit formula for the infinitesimal operator on two kinds of functionals using the ideas from [16, Lemma 3.1].

Lemma 3.1. Consider the biological neural network given by Eq. (14) and let

$$V_1(t, \psi) = \int_{-d(t)}^0 \psi^T(\theta) H \psi(\theta) d\theta, \tag{18}$$

$$V_2(t, \psi) = \int_{-d(t)}^0 e^{\varepsilon(t+\theta)} \psi^T(\theta) H \psi(\theta) d\theta, \tag{19}$$

where $t \geq 0$, $\psi \in C((-\infty, 0], \mathbb{R}^n)$, $\varepsilon > 0$, $H \in \mathbb{R}^{n \times n}$, $d : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, and $d(t) \geq 0$, $t \geq 0$. Then, the infinitesimal operator acting on $V_1 : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}$ and $V_2 : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}$ is given by

$$\mathcal{L}V_1(t, S_t) = S^T(t)HS(t) - (1 - \dot{d}(t))S^T(t - d(t))HS(t - d(t)), \tag{20}$$

$$\mathcal{L}V_2(t, S_t) = e^{\varepsilon t}S^T(t)HS(t) - e^{\varepsilon(t-d(t))}(1 - \dot{d}(t))S^T(t - d(t))HS(t - d(t)). \tag{21}$$

Proof. For sufficiently small $h > 0$

$$\begin{aligned} \mathbb{E}[V_1(t+h, S_{t+h})|S_t] &= \mathbb{E}\left[\int_{-d(t+h)}^0 S^T(t+h+\theta)HS(t+h+\theta) d\theta \middle| S_t\right] \\ &= \mathbb{E}\left[\int_{h-d(t+h)}^h S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] \\ &= \mathbb{E}\left[\int_0^h S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] + \int_{-d(t)}^0 S^T(t+\theta)HS(t+\theta) d\theta \\ &\quad + \mathbb{E}\left[\int_{-d(t+h)}^{-d(t)} S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] + \mathbb{E}\left[\int_{h-d(t+h)}^{-d(t+h)} S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] \\ &= \mathbb{E}\left[\int_0^h S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] + \int_{-d(t)}^0 S^T(t+\theta)HS(t+\theta) d\theta \\ &\quad - \mathbb{E}\left[\int_{-d(t)}^{-(d(t)+h\dot{d}(t)+O(h))} S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] \\ &\quad - \mathbb{E}\left[\int_{-d(t+h)}^{h-d(t+h)} S^T(t+\theta)HS(t+\theta) d\theta \middle| S_t\right] \\ &= hS^T(t)HS(t) + V_1(t, S_t) - h(1 - \dot{d}(t))S^T(t - d(t))HS(t - d(t)) + O(h), \quad t \geq 0, \end{aligned} \tag{22}$$

where $O(h)$ denotes higher-order terms in h . Substituting Eq. (22) into Eq. (16) yields Eq. (20). The proof of Eq. (21) is similar to the proof of Eq. (20) and, hence, is omitted. \square

To develop a global synchronization property for the biological neural network system (14), we introduce the notion of stochastic synchronization. Here, we focus on mean-square synchronization.

Definition 3.1. The biological neural network given by Eq. (14) is said to be *globally asymptotically mean-square synchronized* if

$$\lim_{t \rightarrow \infty} \mathbb{E}[\|S_{it} - S_{jt}\|^2] = 0, \tag{23}$$

for all $\phi(\cdot) \in C((-\infty, 0], \mathbb{R}^n)$ and $i, j = 1, 2, \dots, n$, $i \neq j$, where $S_{it} \triangleq S_i(t + \theta)$, $\theta \in (-\infty, 0]$, $t \geq 0$, and $\|S_{it} - S_{jt}\| = \sup_{-\tau \leq \theta \leq 0} |S_i(t + \theta) - S_j(t + \theta)|$, $\tau > 0$.

Definition 3.2. The biological neural network given by Eq. (14) is said to be *globally exponentially mean-square synchronized* if there exist constants $\rho > 0$ and $p > 0$ such that

$$\mathbb{E}[\|S_{it} - S_{jt}\|^2] \leq \rho e^{-pt} \int_{-\infty}^0 |\phi_i(\theta) - \phi_j(\theta)|^2 d\theta, \quad t > 0, \quad p > 0, \tag{24}$$

for all $\phi(\cdot) = [\phi_1(\cdot), \dots, \phi_n(\cdot)]^T \in C((-\infty, 0], \mathbb{R}^n)$ and $i, j = 1, 2, \dots, n$, $i \neq j$.

4. Synchronization of stochastic biological neural networks with differentiable delays

In this section, we develop sufficient conditions for global mean-square synchronization for the biological neural network (14) with differentiable time delays using Barbalat's lemma and linear matrix inequalities (LMIs). In this paper, we assume that the noise intensity matrix function $\sigma(S)$ has a linear growth rate, that is, there exists $r > 0$ such that

$$\text{tr}[\sigma^2(S)] \leq rS^T M^T M S, \quad S \in \mathbb{R}^n, \tag{25}$$

where M is defined by

$$M \triangleq \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(n-1) \times n}. \tag{26}$$

The following theorem provides a sufficient condition for global mean-square asymptotic synchronization of the biological neural network system (14).

Theorem 4.1. *Consider the biological neural network given by Eq. (14) with $f_i(\cdot), i = 1, 2, \dots, n$, given by either Eq. (11) or Eq. (12), and assume that $\delta_i(t) \leq h_1 < 1$, and $\delta_i(t) \geq 0, t \geq 0, i = 1, 2, \dots, N$, hold. If there exist a positive-definite matrix $P \in \mathbb{R}^{n \times n}$, nonnegative-definite matrices $Q_i \in \mathbb{R}^{n \times n}, i = 1, 2, \dots, N$, and $R \in \mathbb{R}^{n \times n}$, and a nonnegative-definite diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that*

$$\begin{bmatrix} R & -PB \\ -BP & \Lambda \end{bmatrix} \geq 0, \tag{27}$$

$$\Omega_1 \triangleq \begin{bmatrix} -(1-h_1)Q_1 + A_1^T \Lambda A_1 & A_1^T \Lambda A_2 & \dots & A_1^T \Lambda A_N \\ A_2^T \Lambda A_1 & -(1-h_1)Q_2 + A_2^T \Lambda A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{N-1}^T \Lambda A_N \\ A_N^T \Lambda A_1 & \dots & A_N^T \Lambda A_{N-1} & -(1-h_1)Q_N + A_N^T \Lambda A_N \end{bmatrix} \leq 0, \tag{28}$$

and either $\Omega_2 < 0$ or both $\Omega_2 \leq 0$ and $\ker(\Omega_2) = \text{span}(\mathbf{1}_n)$ hold, where

$$\Omega_2 \triangleq -PL - LP + k_1 r M^T M + R + \sum_{i=1}^N Q_i, \tag{29}$$

$k_1 \triangleq \lambda_{\max}(P)$, r is such that Eq. (25) holds, M is given by Eq. (26), and $A_i, i = 1, \dots, N$, is defined in Eq. (15), then Eq. (14) is globally asymptotically mean-square synchronized.

Proof. Consider the functional $V : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}$ given by $V(t, \psi) = V_1(\psi(0)) + V_2(t, \psi)$, where $V_1(\psi(0)) = \psi^T(0)P\psi(0)$ and $V_2(t, \psi) = \sum_{i=1}^N \int_{-\delta_i(t)}^0 \psi^T(\theta)Q_i\psi(\theta) d\theta$. It follows from Eq. (17) and Lemma 3.1 that the infinitesimal operator $\mathcal{L}V(t, S_t)$ associated with the stochastic process (14) is given by

$$\mathcal{L}V(t, S_t) = \mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t), \quad t \geq 0, \tag{30}$$

where

$$\mathcal{L}V_1(S(t)) = 2S^T(t)P(-LS(t) + Bf(\hat{S}(t))) + \text{tr}[\sigma(S(t))P\sigma(S(t))], \tag{31}$$

$$\mathcal{L}V_2(t, S_t) = \sum_{i=1}^N [S^T(t)Q_i S(t) - (1 - \dot{\delta}_i(t))S^T(t - \delta_i(t))Q_i S(t - \delta_i(t))], \tag{32}$$

and $\hat{S}(t) \in \mathbb{R}^n$ is defined by Eq. (15). Next, since $\mathbb{E}[V(t, S_t)] = V(0, S_0) + \mathbb{E}[\int_0^t \mathcal{L}V(u, S_u) du]$, it follows that $\mathbb{E}[dV(t, S_t)] = \mathbb{E}[\mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t)] dt$.

To complete the proof, we show that $\mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t) \leq 0, t \geq 0$, and $\mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t) \equiv 0$ implies $MS(t) \equiv 0$. To see this, note that Eq. (25) implies

$$\text{tr}[\sigma(S)P\sigma(S)] \leq k_1 \text{tr}[\sigma(S)\sigma(S)] \leq k_1 r S^T M^T M S, \quad S \in \mathbb{R}^n. \tag{33}$$

Furthermore, note that for every diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that $\Lambda \geq 0$, it follows that for $f_i(\cdot), i = 1, \dots, n$, given by Eq. (11) or Eq. (12)

$$f^T(\hat{S}(t))\Lambda f(\hat{S}(t)) \leq \hat{S}^T(t)\Lambda \hat{S}(t), \quad t \geq 0. \tag{34}$$

Now, using Eqs. (27), (33), and (34), it follows from Eq. (31) that

$$\begin{aligned} \mathcal{L}V_1(S(t)) &\leq 2S^T(t)P(-LS(t) + Bf(\hat{S}(t))) + k_1 r S^T(t)M^T M S(t) \\ &\leq -2S^T(t)PLS(t) + S^T(t)RS(t) + f^T(\hat{S}(t))\Lambda \hat{f}(S(t)) + k_1 r S^T(t)M^T M S(t) \\ &\leq S^T(t)(-2PL + k_1 r M^T M + R)S(t) + \hat{S}^T(t)\Lambda \hat{S}(t), \quad t \geq 0. \end{aligned} \tag{35}$$

Hence, since $\dot{\delta}_i(t) \leq h_1 < 1, t > 0$, it follows from Eqs. (32) and (35) that

$$\begin{aligned} \mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t) &\leq S^T(t) \left(-2PL + k_1 r M^T M + R + \sum_{i=1}^N Q_i \right) S(t) \\ &\quad + \sum_{i=1}^N \sum_{j=1}^N S^T(t - \delta_i(t)) A_i^T \Lambda A_j S(t - \delta_j(t)) - \sum_{i=1}^N (1 - h_1) S^T(t - \delta_i(t)) Q_i S(t - \delta_i(t)) \\ &= \eta^T(t) \Omega_1 \eta(t) + S^T(t) \Omega_2 S(t) \leq S^T(t) \Omega_2 S(t), \quad t \geq 0, \end{aligned}$$

where $\eta(t) \triangleq [S^T(t - \delta_1(t)), \dots, S^T(t - \delta_N(t))]^T$, and Ω_1 and Ω_2 are defined by Eqs. (28) and (29).

Finally, if $\Omega_1 \leq 0$ and $\Omega_2 < 0$, it follows that $\mathbb{E}[dV(t, S_t)] = \mathbb{E}[\mathcal{L}V_1(S(t)) + \mathcal{L}V_2(t, S_t)] dt \leq 0, t \geq 0$, and $\mathbb{E}[V(t, S_t)] \leq \mathbb{E}[V(0, S_0)]$. Note that since P is positive-definite and $\mathbb{E}[V(t, S_t)]$ is a non-increasing function of time, it follows that $\mathbb{E}[\|S(t)\|^2]$ is bounded for all $t \geq 0$. Since $\mathcal{L}[S^T(t)\Omega_2 S(t)] = 2S^T(t)\Omega_2[-LS(t) + Bf(\hat{S}(t))] + \text{tr}[\sigma(S(t))\Omega_2\sigma(S(t))], t \geq 0$, and $\mathbb{E}[\|S(t)\|^2], t \geq 0$, is bounded, it follows that $\mathbb{E}[\mathcal{L}[S^T(t)\Omega_2 S(t)]], t \geq 0$, is bounded. Since $\mathbb{E}[d[S^T(t)\Omega_2 S(t)]] = \mathbb{E}[\mathcal{L}[S^T(t)\Omega_2 S(t)]] dt, t \geq 0$, and $\mathbb{E}[\mathcal{L}[S^T(t)\Omega_2 S(t)]]$ is bounded, it follows that $\mathbb{E}[S^T(t)\Omega_2 S(t)]$ is uniformly continuous in t . Note that since $\mathbb{E}[V(t, S_t)] \geq 0, t \geq 0$, and $\mathbb{E}[S^T(t)\Omega_2 S(t)]$ is uniformly continuous in t , it follows from Barbalat's lemma [17, p. 221] that $\mathbb{E}[S^T(t)\Omega_2 S(t)] \rightarrow 0$ as $t \rightarrow \infty$. Since $\Omega_2 < 0$, it follows that $\mathbb{E}[\|S(t)\|^2] \rightarrow 0$ as $t \rightarrow \infty$. Thus, $\mathbb{E}[\|MS(t)\|^2] \leq \|M\|^2 \mathbb{E}[\|S(t)\|^2] \rightarrow 0$ as $t \rightarrow \infty$. Hence, $\mathbb{E}[\|MS_t\|^2] \rightarrow 0$ as $t \rightarrow \infty$, that is, Eq. (14) is globally asymptotically mean-square synchronized.

Alternately, if $\Omega_1 \leq 0, \ker(\Omega_2) = \text{span}(\mathbf{1}_n)$, and $\Omega_2 \leq 0$, then a similar argument shows that $\mathbb{E}[S^T(t)\Omega_2 S(t)] \rightarrow 0$ as $t \rightarrow \infty$, which, since $\ker(\Omega_2) = \text{span}(\mathbf{1}_n)$, implies that Eq. (14) is globally asymptotically mean-square synchronized. \square

The next theorem establishes a sufficient condition for global exponential mean-square synchronization of the network system Eq. (14).

Theorem 4.2. Consider the biological neural network given by Eq. (14) with $f_i(\cdot), i = 1, 2, \dots, n$, given by either Eq. (11) or Eq. (12), and assume that $\dot{\delta}_i(t) \leq h_1 < 1$, and $h_2 \geq \delta_i(t) \geq 0, t \geq 0, i = 1, 2, \dots, N$, hold. If there exist a positive-definite matrix $P \in \mathbb{R}^{n \times n}$, nonnegative-definite matrices $Q_i \in \mathbb{R}^{n \times n}, i = 1, 2, \dots, N$, and $R \in \mathbb{R}^{n \times n}$, a nonnegative-definite diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$, and a scalar $\varepsilon > 0$ such that Eq. (27) holds

$$\Omega_3 \triangleq \begin{bmatrix} -(1-h_1)e^{-2\varepsilon h_2}Q_1 + A_1^T\Lambda A_1 & A_1^T\Lambda A_2 & \dots & A_1^T\Lambda A_N \\ A_2^T\Lambda A_1 & -(1-h_1)e^{-2\varepsilon h_2}Q_2 + A_2^T\Lambda A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{N-1}^T\Lambda A_N \\ A_N^T\Lambda A_1 & \dots & \dots & -(1-h_1)e^{-2\varepsilon h_2}Q_N + A_N^T\Lambda A_N \end{bmatrix} \leq 0, \tag{36}$$

and either $\Omega_4 < 0$ or both $\Omega_4 \leq 0$ and $\ker(\Omega_4) = \text{span}(\mathbf{1}_n)$ hold, where

$$\Omega_4 \triangleq -PL - LP + k_1 r M^T M + R + \sum_{i=1}^N Q_i + 2\varepsilon P, \tag{37}$$

$k_1 \triangleq \lambda_{\max}(P)$, r is such that Eq. (25) holds, M is given by Eq. (26), and $A_i, i = 1, \dots, N$, is defined in Eq. (15), then Eq. (14) is globally exponentially mean-square synchronized.

Proof. The proof is similar to the proof of Theorem 4.1 using the functional $V : [0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}$ given by

$$V(t, \psi) = e^{2\varepsilon t} \psi^T(0)P\psi(0) + \sum_{i=1}^N \int_{-\delta_i(t)}^0 e^{2\varepsilon(t+\theta)} \psi^T(\theta)Q_i\psi(\theta) d\theta$$

and, hence, is omitted. \square

The following corollary to Theorem 4.2 is immediate.

Corollary 4.1. Consider the biological neural network given by Eq. (14) with $f_i(\cdot), i = 1, 2, \dots, n$, given by either Eq. (11) or Eq. (12), and assume that $\dot{\delta}_i(t) \leq h_1 < 1$, and $h_2 \geq \delta_i(t) \geq 0, t \geq 0, i = 1, 2, \dots, N$, hold. If there exist a positive-definite matrix $P \in \mathbb{R}^{n \times n}$, nonnegative-definite matrices $Q_i \in \mathbb{R}^{n \times n}, i = 1, 2, \dots, N$, and $R \in \mathbb{R}^{n \times n}$, and a nonnegative-definite diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ such that Eq. (27) holds, and $\Omega_1 < 0$ and $\Omega_2 < 0$, where Ω_1 and Ω_2 are given by Eqs. (28) and (29) with $k_1 \triangleq \lambda_{\max}(P)$, r is such that Eq. (25) holds, M is given by Eq. (26), $A_i, i = 1, \dots, N$, is defined in Eq. (15), then Eq. (14) is globally exponentially mean-square synchronized.

Proof. The result is a direct consequence of Theorem 4.2 by noting that if $\Omega_1 < 0$ and $\Omega_2 < 0$ hold, then there exists a sufficiently small $\varepsilon > 0$ such that $\Omega_3 \leq 0$ and $\Omega_4 < 0$ hold, where Ω_3 and Ω_4 are given by Eqs. (36) and (37). \square

Remark 4.1. Note that Theorem 4.1 does not require that the time delays be bounded, whereas Theorem 4.2 and Corollary 4.1 hold for the case where the time delays are bounded.

Remark 4.2. It is important to note that if $f_i(\cdot), i = 1, 2, \dots, n$, in Eq. (14) is replaced by Eq. (10), then the results of Theorems 4.1 and 4.2 as well as Corollary 4.1 still hold.

5. Synchronization of stochastic biological neural network systems with limiting delay systems

Theorems 4.1 and 4.2, as well as Corollary 4.1, require that the system time delays be differentiable with respect to time t , which can be a limiting assumption in practice. In this section, we relax the differentiability assumption on $\delta_i(\cdot)$, $i = 1, \dots, N$. Here, we assume that Eq. (14) converges to a constant time-delay system and show that, under certain conditions, if the constant time-delay system is globally uniformly asymptotically mean-square synchronized, then the original system with time-varying delay is globally asymptotically mean-square synchronized.

To proceed, we rewrite the biological neural network (14) with bounded time delays as

$$\begin{aligned} dS(t) &= (-LS(t) + Bg(S(t - \delta_1(t)), \dots, S(t - \delta_N(t)))) dt + \sigma(S(t)) dw(t), \\ S(\theta) &= \phi(\theta), \quad -r_p \leq \theta \leq 0, \quad t > 0, \quad r_p > 0, \end{aligned} \tag{38}$$

where $g(X_1, X_2, \dots, X_N) \triangleq [f_1(\bar{X}_1), f_2(\bar{X}_2), \dots, f_n(\bar{X}_n)]^T$, $\bar{X} \triangleq \sum_{i=1}^N A_i X_i$, $\bar{X}_i \in \mathbb{R}^n$, $i \in \{1, 2, \dots, N\}$, $\bar{X} = [\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n]^T$, A_i is defined in Eq. (15), and $f_i(\cdot)$ is given by either Eq. (11) or Eq. (12). Clearly, $g(X_1, X_2, \dots, X_N)$ is globally Lipschitz continuous in $[X_1^T, X_2^T, \dots, X_N^T]^T$ on \mathbb{R}^{nN} , and hence, there exists $m > 0$ such that

$$\|g(X_1, X_2, \dots, X_N) - g(Y_1, Y_2, \dots, Y_N)\| \leq m \| [X_1^T, X_2^T, \dots, X_N^T]^T - [Y_1^T, Y_2^T, \dots, Y_N^T]^T \|,$$

that is,

$$\begin{aligned} &\|g(X_1, X_2, \dots, X_N) - g(Y_1, Y_2, \dots, Y_N)\|^2 \\ &\leq m^2 (\|X_1 - Y_1\|^2 + \|X_2 - Y_2\|^2 + \dots + \|X_N - Y_N\|^2), \end{aligned} \tag{39}$$

where $X_i, Y_i \in \mathbb{R}^n$, $i \in \{1, 2, \dots, N\}$. Next, define $\mathcal{X}(t) \triangleq g(S(t - \delta_1(t)), \dots, S(t - \delta_N(t))) - g(S(t - d_1), \dots, S(t - d_N))$, where $0 \leq \delta_i(t) \leq r_p$, $r_p > 0$, and $d_i \triangleq \lim_{t \rightarrow \infty} \delta_i(t)$, $i = 1, 2, \dots, N$, so that Eq. (38) can be rewritten as

$$\begin{aligned} dS(t) &= (-LS(t) + Bg(S(t - d_1), \dots, S(t - d_N)) + B\mathcal{X}(t)) dt + \sigma(S(t)) dw(t), \\ S(\theta) &= \phi(\theta), \quad -r_p \leq \theta \leq 0, \quad t > 0, \quad r_p > 0. \end{aligned} \tag{40}$$

Definition 5.1 (Hui [18]). If for every initial condition $S(\theta) = \phi(\theta) \in \mathcal{C}([-r_p, 0], \mathbb{R}^n)$ of Eq. (40) there exists an unbounded sequence $\{t_n\}_{n=1}^\infty$, with $t_n \rightarrow +\infty$ as $n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} \mathbb{E}[\|\mathcal{X}(t + t_n)\|] = 0$ uniformly in t on every compact subset of $[0, \infty)$, then the system

$$\begin{aligned} dZ(t) &= (-LZ(t) + Bg(Z(t - d_1), \dots, Z(t - d_N))) dt + \sigma(Z(t)) dw(t), \\ Z(t_0 + \theta) &= \psi(t_0 + \theta), \quad -r_p \leq \theta \leq 0, \quad t > t_0 \geq 0, \quad \psi \in \mathcal{C}([-r_p, t_0], \mathbb{R}^n), \end{aligned} \tag{41}$$

is called a *limiting delay system* of Eq. (40).

Definition 5.2. The limiting delay system (41) is said to be *uniformly mean-square synchronized* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$, independent of t_0 , such that $\mathbb{E}[\|MZ_t\|^2] < \varepsilon$ for all $\mathbb{E}[\|\psi\|^2] < \delta$, where $Z_t = Z(t + \theta)$, $\theta \in [-r_p, 0]$, $t \geq t_0$, and M is given by Eq. (26). In addition, the limiting delay system (41) is said to be *globally uniformly asymptotically mean-square synchronized* if it is uniformly mean-square synchronized and for every $c > 0$ and $\eta > 0$, there exists $T = T(\eta, c)$ such that $\mathbb{E}[\|MZ_t\|^2] < \eta$ for all $t > t_0 + T$ and $\mathbb{E}[\|\psi\|^2] < c$.

Lemma 5.1. Consider the biological neural network (14) or, equivalently, Eq. (40), and assume that $f_i(\cdot)$ given by Eq. (11) or Eq. (12) is such that $0 \leq f_i(x) \leq f_{\max}$ for all $x \in \mathbb{R}$ and $i = 1, \dots, n$.

If there exist $r > 0$, positive-definite matrices $P, Q \in \mathbb{R}^{n \times n}$ such that Eq. (25) holds, and

$$\begin{bmatrix} P & -B \\ -B & Q \end{bmatrix} \geq 0, \tag{42}$$

$$\Omega \triangleq -2L + P + rM^T M < 0, \tag{43}$$

then $\mathbb{E}[\|S_t\|^2]$ is bounded for all $t > 0$.

Proof. Consider the function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ given by $V(S) = S^T S$, $S \in \mathbb{R}^n$. It follows from Eqs. (17) and (25) that the infinitesimal operator $\mathcal{L}V(S(t))$ associated with the stochastic process (14) (or Eq. (40)) is given by

$$\begin{aligned} \mathcal{L}V(S(t)) &= 2S^T(t)(-LS(t) + Bf(\hat{S}(t))) + \text{tr}[\sigma^2(S(t))] \\ &\leq -2S^T(t)LS(t) + 2S^T(t)Bf(\hat{S}(t)) + rS^T(t)M^T MS(t), \quad t \geq 0, \end{aligned}$$

which, using Eq. (42), implies that

$$\begin{aligned} \mathcal{L}V(S(t)) &\leq -2S^T(t)LS(t) + S^T(t)PS(t) + f^T(\hat{S}(t))Qf(\hat{S}(t)) + rS^T(t)M^T MS(t) \\ &= S^T(t)(-2L + P + rM^T M)S(t) + f^T(\hat{S}(t))Qf(\hat{S}(t)) \\ &\leq S^T(t)\Omega S(t) + \lambda_{\max}(Q)f^T(\hat{S}(t))f(\hat{S}(t)), \quad t \geq 0, \end{aligned}$$

where Ω is defined by Eq. (43).

Next, since $0 \leq f_i(x) \leq f_{\max}$ for all $x \in \mathbb{R}$ and $i = 1, \dots, n$, it follows that

$$\mathcal{L}V(S(t)) \leq S^T(t)\Omega S(t) + n\lambda_{\max}(Q)f_{\max}^2, \quad t \geq 0. \tag{44}$$

Now, by Eq. (43), $S^T(t)\Omega S(t) \leq \lambda_{\max}(\Omega)S^T(t)S(t) \leq 0$, $t \geq 0$, and hence,

$$\mathcal{L}V(S(t)) \leq \lambda_{\max}(\Omega)S^T(t)S(t) + n\lambda_{\max}(Q)f_{\max}^2, \quad t \geq 0. \tag{45}$$

Let $m \triangleq -n\lambda_{\max}(Q)f_{\max}^2/\lambda_{\max}(\Omega)$. Next, we show that if $\|\phi\|^2 = c < m$, then $\mathbb{E}[V(S(t))] \leq m$ for all $t > 0$. To see this, assume, *ad absurdum*, that $\mathbb{E}[V(S(t))] > m$ for some $t > 0$, which holds since $\mathbb{E}[V(S(t))]$ is continuous in t , and note that there exists some $\tau \in (0, t)$ such that $\mathbb{E}[V(S(\tau))] = m$, and $\mathbb{E}[dV(S(\tau))] > 0$. Now, note that by Eq. (45), $\mathbb{E}[dV(S(\tau))] = \mathbb{E}[\mathcal{L}V(S(\tau))] dt \leq 0$ for some $\tau > 0$ such that $\mathbb{E}[V(S(\tau))] = m$, which contradicts $\mathbb{E}[dV(S(\tau))] > 0$. Hence, $\mathbb{E}[V(S(t))] \leq m$ for all $t > 0$. Alternately, if $\|\phi\|^2 = c \geq m$, then using a similar argument it can be shown that $\mathbb{E}[V(S(t))] \leq c$ for all $t \geq 0$. Hence, $\mathbb{E}[V(S(t))]$ is bounded for all $t > 0$, that is, $\mathbb{E}[\|S_t\|^2]$ is bounded for all $t > 0$.

The following result gives a sufficient condition for stochastic synchronization of the biological neural network Eq. (14) using the limiting delay system (41).

Theorem 5.1. Consider the biological neural network given by Eq. (40) with bounded time-varying delays and its corresponding limiting delay system (41) with constant delays such that $0 \leq \delta_i(t) \leq r_p$ and $0 \leq d_i \leq r_p$, $i = 1, 2, \dots, N$, and assume that $f_i(\cdot)$ is given by Eq. (11) or Eq. (12) and is such that $0 \leq f_i(x) \leq f_{\max}$ for all $x \in \mathbb{R}$ and $i = 1, \dots, n$. If there exist $r > 0$ and positive-definite matrices $P, Q \in \mathbb{R}^{n \times n}$ such that Eqs. (25), (42), and (43) hold, $\sigma_i(S)$ is globally Lipschitz continuous in S on \mathbb{R}^n , and the limiting delay system Eq. (41) is globally uniformly asymptotically mean-square synchronized, then Eq. (40) is globally asymptotically mean-square synchronized.

Proof. To prove that Eq. (40) with bounded time-varying delays is globally asymptotically mean-square synchronized, we show that for every $\varepsilon > 0$ and $\delta > 0$, there exists $T = T(\varepsilon, \delta) > 0$ such that $\mathbb{E}[\|MS(t)\|^2] < \varepsilon$ for all $t > T$, $\|\phi\|^2 < \delta$, where M is given by Eq. (26). To see this, note that

$$\begin{aligned} \mathbb{E}[\|MS(t)\|^2] &= \mathbb{E}[\|MZ(t) + MS(t) - MZ(t)\|^2] \\ &\leq 2\mathbb{E}[\|MZ(t)\|^2] + 2\mathbb{E}[\|MS(t) - MZ(t)\|^2] \\ &\leq 2\mathbb{E}[\|MZ(t)\|^2] + 2\|M\|^2\mathbb{E}[\|S(t) - Z(t)\|^2], \quad t > 0, \end{aligned} \tag{46}$$

where $Z(\cdot)$ is the solution to Eq. (41) with initial condition $Z(t_0 + \theta) = S(t_0 + \theta)$, $\theta \in [-r_p, 0]$, $t_0 \in (0, t)$.

Define $\mathcal{H}(X_t) \triangleq g(X(t-d_1), X(t-d_2), \dots, X(t-d_N))$, where $X_t \in \mathcal{C}([-r_p, \infty), \mathbb{R}^n)$, and note that the infinitesimal operator $\mathcal{L}[\|S(t) - Z(t)\|^2]$ is given by

$$\begin{aligned} \mathcal{L}[\|S(t) - Z(t)\|^2] &= 2(S(t) - Z(t))^T((-LS(t) + B\mathcal{H}(S_t)) \\ &\quad - (-LZ(t) + B\mathcal{H}(Z_t)) + B\mathcal{X}(t)) + \sum_{i=1}^n [\sigma_i(S(t)) - \sigma_i(Z(t))]^2. \end{aligned}$$

Next, since $S(t_0 + \theta) = Z(t_0 + \theta)$, $\theta \in [-r_p, 0]$, it follows that

$$\begin{aligned} \mathbb{E}[\|S(t) - Z(t)\|^2] &= \mathbb{E}[\|S(t_0) - Z(t_0)\|^2] + \mathbb{E}\left[\int_{t_0}^t \mathcal{L}[\|S(u) - Z(u)\|^2] du\right] \\ &= \mathbb{E}\left[\int_{t_0}^t \mathcal{L}[\|S(u) - Z(u)\|^2] du\right], \end{aligned}$$

and hence,

$$\begin{aligned} \mathbb{E}[\|S(t) - Z(t)\|^2] &= \mathbb{E}\left[\int_{t_0}^t 2(S(u) - Z(u))^T((-LS(u) + B\mathcal{H}(S_u)) \right. \\ &\quad \left. - (-LZ(u) + B\mathcal{H}(Z_u)) + B\mathcal{X}(u)) + \sum_{i=1}^n (\sigma_i(S(u)) - \sigma_i(Z(u)))^2 du\right] \\ &\leq \mathbb{E}\left[\int_{t_0}^t 2|(S(u) - Z(u))^T((-LS(u) + B\mathcal{H}(S_u)) - (-LZ(u) + B\mathcal{H}(Z_u)))| du\right] \\ &\quad + \mathbb{E}\left[\int_{t_0}^t 2|(S(u) - Z(u))^T B\mathcal{X}(u)| du\right] + \mathbb{E}\left[\int_{t_0}^t \sum_{i=1}^n (\sigma_i(S(u)) - \sigma_i(Z(u)))^2 du\right], \end{aligned} \tag{47}$$

$t \geq t_0 \geq 0$.

Now, since, for $t_0 \leq u \leq t$,

$$\begin{aligned} &2|(S(u) - Z(u))^T((-LS(u) + B\mathcal{H}(S_u)) - (-LZ(u) + B\mathcal{H}(Z_u)))| \\ &\leq \|S(u) - Z(u)\|^2 + \|(-LS(u) + B\mathcal{H}(S_u)) - (-LZ(u) + B\mathcal{H}(Z_u))\|^2, \end{aligned} \tag{48}$$

it follows from the definition of $\mathcal{H}(S_u)$ and Eq. (39) that there exists $L_1 \geq 2\|L\|^2 + 2m^2\|B\|^2$ such that

$$\begin{aligned} &\|(-LS(u) + B\mathcal{H}(S_u)) - (-LZ(u) + B\mathcal{H}(Z_u))\|^2 \\ &\leq 2\|LS(u) - LZ(u)\|^2 + 2\|B\mathcal{H}(S_u) - B\mathcal{H}(Z_u)\|^2 \\ &\leq 2\|L\|^2\|S(u) - Z(u)\|^2 + 2\|B\|^2\|\mathcal{H}(S_u) - \mathcal{H}(Z_u)\|^2 \\ &\leq L_1\|(S(u) - Z(u))\|^2 + L_1\|(S(u-d_1) - Z(u-d_1))\|^2 \\ &\quad + \dots + L_1\|(S(u-d_N) - Z(u-d_N))\|^2, \quad t_0 \leq u \leq t, \end{aligned} \tag{49}$$

and hence, it follows from Eqs. (48) and (49) that, for $t_0 \leq u \leq t$

$$\begin{aligned}
 & 2|(S(u) - Z(u))^T [(-LS(u) + B\mathcal{H}(S_u)) - (-LZ(u) + B\mathcal{H}(Z_u))] \\
 & \leq \|(S(u) - Z(u))\|^2 + L_1 \|(S(u) - Z(u))\|^2 \\
 & \quad + L_1 \|(S(u - d_1) - Z(u - d_1))\|^2 + \dots + L_1 \|(S(u - d_N) - Z(u - d_N))\|^2.
 \end{aligned} \tag{50}$$

Since $\sigma_i(S)$ is globally Lipschitz continuous in S on \mathbb{R}^n , it follows that

$$\sum_{i=1}^n [\sigma_i(S(u)) - \sigma_i(Z(u))]^2 \leq L_2 \|(S(u) - Z(u))\|^2, \quad t_0 \leq u \leq t, \tag{51}$$

where $\sqrt{L_2/n}$ is the Lipschitz constant for all $\sigma_i(S)$, $i = 1, \dots, n$. Now, it follows from Eqs. (47), (50), and (51) that

$$\begin{aligned}
 \mathbb{E}[\|S(t) - Z(t)\|^2] & \leq (1 + L_1) \mathbb{E} \left[\int_{t_0}^t \|(S(u) - Z(u))\|^2 du \right] \\
 & \quad + L_1 \mathbb{E} \left[\int_{t_0}^t \sum_{i=1}^N \|(S(u - d_i) - Z(u - d_i))\|^2 du \right] \\
 & \quad + 2 \mathbb{E} \left[\int_{t_0}^t |(S(u) - Z(u))^T B\mathcal{X}(u)| du \right] \\
 & \quad + L_2 \mathbb{E} \left[\int_{t_0}^t \|(S(u) - Z(u))\|^2 du \right], \quad t \geq t_0 \geq 0.
 \end{aligned} \tag{52}$$

Next, since $\mathbb{E}[2|(S(u) - Z(u))^T B\mathcal{X}(u)|] \leq \mathbb{E}[\|B(S(u) - Z(u))\|^2] + \mathbb{E}[\|\mathcal{X}(u)\|^2]$ for $t_0 \leq u \leq t$, it follows from Definition 5.1 that for every $\varepsilon_1 > 0$, there exists $T_1 = T_1(\varepsilon_1) > 0$ such that $\mathbb{E}[\|\mathcal{X}(t)\|^2] < \varepsilon_1$ for $t > T_1$. Now, choose $t_0 > T_1$ and note that

$$\mathbb{E}[2|(S(u) - Z(u))^T B\mathcal{X}(u)|] < \|B\|^2 \mathbb{E}[\|(S(u) - Z(u))\|^2] + \varepsilon_1, \quad T_1 \leq t_0 \leq u \leq t. \tag{53}$$

Furthermore, since $S(t_0 + \theta) = Z(t_0 + \theta)$, $\theta \in [-r_p, 0]$, it follows that

$$\begin{aligned}
 \mathbb{E} \left[\int_{t_0}^t \|S(u - d_i) - Z(u - d_i)\|^2 du \right] & = \mathbb{E} \left[\int_{t_0 - d_i}^{t - d_i} \mathbb{E}[\|S(u) - Z(u)\|^2] du \right] \\
 & = \mathbb{E} \left[\int_{t_0}^{t - d_i} \|S(u) - Z(u)\|^2 du \right] \\
 & \leq \mathbb{E} \left[\int_{t_0}^t \|S(u) - Z(u)\|^2 du \right], \quad t > t_0 > 0,
 \end{aligned} \tag{54}$$

and hence, it follows from Eqs. (52)–(54) that

$$\begin{aligned}
 \mathbb{E}[\|S(t) - Z(t)\|^2] & < (1 + \|B\|^2 + L_1(N + 1) + L_2) \mathbb{E} \left[\int_{t_0}^t \|(S(u) - Z(u))\|^2 du \right] + \varepsilon_1(t - t_0) \\
 & = (1 + \|B\|^2 + L_1(N + 1) + L_2) \int_{t_0}^t \mathbb{E}[\|(S(u) - Z(u))\|^2] du + \varepsilon_1(t - t_0), \quad t > t_0 > T_1.
 \end{aligned}$$

Now, using the Gronwall–Bellman lemma [17, p. 125], it follows that

$$\mathbb{E}[\|S(t) - Z(t)\|^2] < \varepsilon_1(t - t_0) \exp[(1 + \|B\|^2 + (1 + N)L_1 + L_2)(t - t_0)], \quad t > t_0 > T_1. \tag{55}$$

Next, it follows from Lemma 5.1 that there exists $c = c(\delta) > 0$ for every $\delta > 0$ such that $\mathbb{E}[\|S_{t_0}\|^2] < c$, $t_0 > 0$, for $\|\phi\|^2 < \delta$. Note that if Eq. (41) is globally uniformly asymptotically

mean-square synchronized, then it follows that for every $\varepsilon_2 > 0$, there exists $T_2 = T_2(\varepsilon_2, c) > 0$ such that $\mathbb{E}[\|MZ(t)\|^2] < \varepsilon_2$ for $t - t_0 > T_2$ and $Z(t_0 + \theta) = S(t_0 + \theta)$, $\theta \in [-r_p, 0]$, $\mathbb{E}[\|S_{t_0}\|^2] < c$. Hence, it follows from Eqs. (46) and (55) that

$$\mathbb{E}[\|MS(t)\|^2] < 2\varepsilon_2 + 2\varepsilon_1 \|M\|^2(t - t_0) \exp[(1 + \|B\|^2 + (1 + N)L_1 + L_2)(t - t_0)], t_0 > T_1, \quad t > T_2 + t_0.$$

Finally, define

$$e(\varepsilon_1, \varepsilon_2, t - t_0) \triangleq 2\varepsilon_2 + 2\varepsilon_1 \|M\|^2(t - t_0) \exp[(1 + \|B\|^2 + (1 + N)L_1 + L_2)(t - t_0)], \quad (56)$$

where $t_0 > T_1$, $t - t_0 > T_2$, so that

$$\mathbb{E}[\|MS(t)\|^2] < e(\varepsilon_1, \varepsilon_2, t - t_0), \quad t_0 > T_1, \quad t - t_0 > T_2. \quad (57)$$

In this case, it follows that for every $\varepsilon > 0$ and $\delta > 0$, there exists $T = T(\varepsilon, \delta) > 0$ such that $\mathbb{E}[\|MS(t)\|^2] < \varepsilon$, $t > T$, $\|\phi\|^2 < \delta$. To see this, note that since ε_1 and ε_2 are arbitrary, to construct $\varepsilon > 0$ such that previous inequality holds we first choose $\varepsilon_2 \in (0, \varepsilon/2)$ and $T_2 = T_2(\varepsilon_2, c)$, such that $\mathbb{E}[\|S_t\|^2] < c$, $t > 0$, for $\|\phi\|^2 < \delta$, where $c = c(\delta)$. Next, choose some $\Delta > 0$ and obtain ε_1 by solving $e(\varepsilon_1, \varepsilon_2, T_2 + \Delta) = \varepsilon$. Note that $t_0 > T_1$. Now, choose a value for t_0 to obtain $T = T(\varepsilon, \delta)$ with $T \triangleq t_0 + T_2 + \Delta$. Since $t_0 > T_1$ and $T - t_0 > T_2$, it follows from Eq. (57) that $\mathbb{E}[\|MS(T)\|^2] < e(\varepsilon_1, \varepsilon_2, T - t_0) = e(\varepsilon_1, \varepsilon_2, T_2 + \Delta) = \varepsilon$. For $t > T$, let $t'_0 \triangleq t_0 + t - T$. Since $t'_0 > T_1$ and $t - t'_0 = T - t_0 > T_2$, it follows from Eq. (57) that $\mathbb{E}[\|MS(t)\|^2] < e(\varepsilon_1, \varepsilon_2, t - t'_0) = e(\varepsilon_1, \varepsilon_2, T_2 + \Delta) = \varepsilon$. Hence, $\mathbb{E}[\|MS(t)\|^2] < \varepsilon$ for all $t > T$, which implies that Eq. (40) is globally asymptotically mean-square synchronized. \square

Remark 5.1. Note that Lemma 5.1 and Theorem 5.1 still hold in the case where $f_i(\cdot)$, $i = 1, 2, \dots, n$, in Eq. (14) (or Eq. (40)) is replaced by

$$f_i(x) = \begin{cases} [x]_+, & x \leq f_{\max}, \\ f_{\max}, & x > f_{\max}, \end{cases} \quad (58)$$

where $[+]$ is defined by $[x]_+ = x$ if $x \geq 0$, and $[x]_+ = 0$ otherwise.

Remark 5.2. It is important to note that Lyapunov–Krasovskii-based approaches for analyzing stability and synthesizing controllers for stochastic nonlinear time delay systems with state and disturbance-dependent noise have been addressed in the literature (see [19,20], and the numerous references therein). Specifically, Ref. [19] considers a state-feedback \mathcal{H}_∞ control problem for stochastic nonlinear system with state and disturbance-dependent noise, and time-varying state delays. Alternatively, Ref. [20] addresses partially known nonlinear systems with input and state time-varying delays approximated by fuzzy-based linear state space subsystems. An interesting feature of the results of [20] is that the authors do not assume differentiability or boundedness of the system time-delays. This would suggest that the boundedness assumption invoked in this section may be relaxed resulting in a stronger version of Theorem 5.1.

6. Illustrative numerical examples

In this section, we present two numerical examples to demonstrate the concepts presented in the paper.

Example 6.1. Consider the stochastic network system characterized by

$$dS_1(t) = (-S_1(t) + 0.2f_1(0.3S_2(t - \delta_1(t)) - 0.5S_3(t - \delta_2(t)))) dt + 0.1(S_1(t) - S_2(t)) dw_1(t), \quad S_1(\theta) = 2 + \sin \theta, \tag{59}$$

$$dS_2(t) = (-1.1S_2(t) + 0.3f_2(0.4S_1(t - \delta_3(t)) - 0.3S_3(t - \delta_4(t)))) dt + 0.1(S_2(t) - S_3(t)) dw_2(t), \quad S_2(\theta) = -3 + \cos \theta, \tag{60}$$

$$dS_3(t) = (-1.4S_3(t) + 0.5f_3(0.4S_1(t - \delta_5(t)) + 0.3S_2(t - \delta_6(t)))) dt + 0.1(S_3(t) - S_1(t)) dw_3(t), \quad S_3(\theta) = 1 - \theta, \tag{61}$$

where $\theta \in [-1, 0]$, $\delta_1(t) = 1 + 0.1 \sin t$, $\delta_2(t) = 1 + 0.1t$, $\delta_3(t) = 0.5$, $\delta_4(t) = 0.1t$, $\delta_5(t) = 0.3$, $\delta_6(t) = 0.4$, $t \geq 0$, $f_i(\cdot), i = 1, 2, 3$, is defined by either Eq. (11) or Eq. (12), and $dw_i, i = 1, 2, 3$, is the standard Gaussian white noise process.

Using the MATLAB LMI Toolbox[®], it can be shown that

$$P = \begin{bmatrix} 205 & -2.69 & 0.13 \\ -2.69 & 169 & -1.96 \\ 0.13 & -1.96 & 122 \end{bmatrix}, \quad R = \begin{bmatrix} 101 & -1.85 & 0.12 \\ -1.85 & 107 & -2.00 \\ 0.12 & -2.00 & 118 \end{bmatrix}, \quad A = \begin{bmatrix} 81.2 & 0 & 0 \\ 0 & 114 & 0 \\ 0 & 0 & 144 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 34.3 & 0.37 & 0.02 \\ 0.37 & 49.7 & 0.49 \\ 0.02 & 0.49 & 25.8 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 34.3 & 0.33 & 0.02 \\ 0.33 & 30.7 & 0.52 \\ 0.02 & 0.52 & 54.0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 66.3 & 0.37 & 0.03 \\ 0.37 & 30.7 & 0.46 \\ 0.03 & 0.46 & 25.8 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 34.3 & 0.33 & 0.02 \\ 0.3 & 30.7 & 0.52 \\ 0.02 & 0.52 & 45.9 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 76.9 & 0.38 & 0.03 \\ 0.38 & 30.7 & 0.46 \\ 0.03 & 0.46 & 25.8 \end{bmatrix}, \quad Q_6 = \begin{bmatrix} 34.3 & 0.39 & 0.02 \\ 0.39 & 60.8 & 0.52 \\ 0.02 & 0.52 & 25.8 \end{bmatrix},$$

satisfy Eqs. (27)–(29), with $r = 0.03$ and $A_i, i = 1, 2, \dots, 6$, defined as in Eq. (15), and hence, the conditions of Theorem 4.1 are satisfied. Next, define the synchronization error $e(t) \triangleq [(S_1(t) - S_2(t))^2 + (S_2(t) - S_3(t))^2]^{1/2}$. The trajectories of the state variables and the synchronization error with respect to time are shown in Figs. 1 and 2, respectively. Note that even though some of the

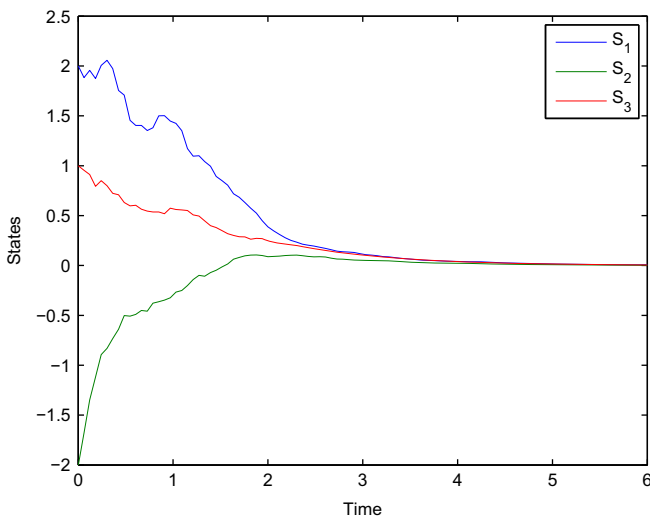


Fig. 1. State trajectories versus time for Example 6.1.

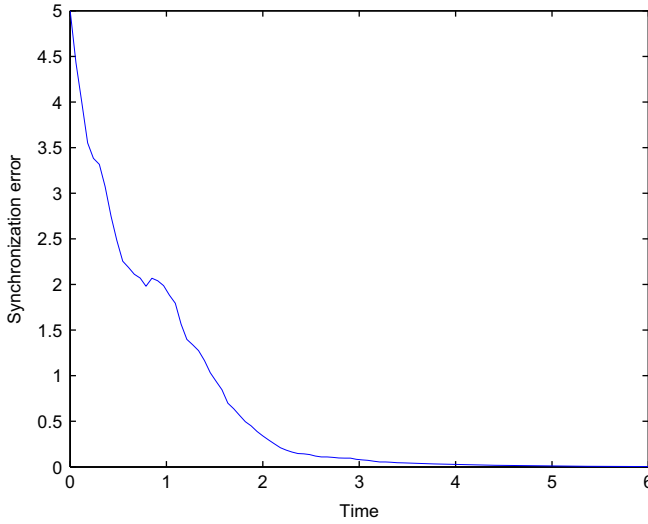


Fig. 2. Synchronization error versus time for Example 6.1.

delays in this example are not bounded, that is, $\delta_i(t) \rightarrow \infty$ as $t \rightarrow \infty$ for $i \in \{2, 4\}$, the system is globally mean-square asymptotically synchronized.

Example 6.2. Consider the stochastic network given by

$$dS_1(t) = (-S_1(t) + f_1(1.5S_1(t - \delta_1(t)) - 0.5S_2(t - \delta_2(t)))) dt + 0.3(S_1(t) - S_2(t)) dw_1(t), \quad (62)$$

$$dS_2(t) = (-S_2(t) + f_2(S_1(t - \delta_3(t)))) dt + 0.3(S_2(t) - S_1(t)) dw_2(t), \quad (63)$$

where $S_1(\theta) = \phi_1(\theta)$, $S_2(\theta) = \phi_2(\theta)$, $\theta \in [-1, 0]$, $\delta_1(t) = \exp(-|t - 3|)$, $\delta_2(t) = 1/(t + 1)$, and $\delta_3(t) = 1 - |\cos(\pi/(1 + t))|$, $f_i(\cdot)$, $i = 1, 2$, is given by Eq. (58) and satisfies $0 \leq f_i(x) \leq f_{\max}$ for all $x \in \mathbb{R}$ and $i = 1, 2$, and dw_i , $i = 1, 2$, is the standard Gaussian white noise process. The network of Eqs. (62) and (63) is the stochastic and time delayed mean field model of a simple biological neural network given by (7.48) and (7.49) of [12, Chapter 7.5] in which all of the excitatory neurons are described by a single firing rate, and all of the inhibitory neurons are described by a second rate.

Since some time delays are not differentiable and $\lim_{t \rightarrow \infty} \delta_1(t) = \lim_{t \rightarrow \infty} \delta_2(t) = \lim_{t \rightarrow \infty} \delta_3(t) = 0$, we consider the limiting delay system given by

$$dZ_1(t) = (-Z_1(t) + f_1(1.5Z_1(t) - 0.5Z_2(t))) dt + 0.3(Z_1(t) - Z_2(t)) dw_1(t), \quad (64)$$

$$dZ_2(t) = (-Z_2(t) + f_2(Z_1(t))) dt + 0.3(Z_2(t) - Z_1(t)) dw_2(t), \quad (65)$$

where $t \geq t_0$, $Z_1(t_0 + \theta) = \phi_1(t_0 + \theta)$, and $Z_2(t_0 + \theta) = \phi_2(t_0 + \theta)$, $\theta \in [-1, 0]$. In addition, consider the function $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $V(Z) = \frac{1}{2}(Z_1 - Z_2)^2$. It follows that the infinitesimal operator $\mathcal{L}V(Z(t))$ associated with the stochastic process (64) and (65) is given by

$$\begin{aligned} \mathcal{L}V(Z(t)) &= -(Z_1(t) - Z_2(t))^2 - (Z_1(t) - Z_2(t))(f_2(Z_1(t)) \\ &\quad - f_1(1.5Z_1(t) - 0.5Z_2(t))) + 0.09(Z_1(t) - Z_2(t))^2 \\ &= -(Z_1(t) - Z_2(t))(f_2(Z_1(t)) - f_1(1.5Z_1(t) - 0.5Z_2(t))) - 0.91(Z_1(t) - Z_2(t))^2, \quad t \geq t_0. \end{aligned}$$

Now, since $-(Z_1(t) - Z_2(t))(f_2(Z_1(t)) - f_1(1.5Z_1(t) - 0.5Z_2(t))) \leq 0.5(Z_1(t) - Z_2(t))^2 + 0.5(f_2(Z_1(t)) - f_1(1.5Z_1(t) - 0.5Z_2(t)))^2$ for all $t \geq t_0$, and $(f_2(x) - f_1(y))^2 \leq (x - y)^2$ for all $x, y \in \mathbb{R}$ and $f_i(\cdot), i = 1, 2$, given by Eq. (58), it follows that

$$\begin{aligned} \mathcal{L}V(Z(t)) &\leq -0.91(Z_1(t) - Z_2(t))^2 + 0.5(Z_1(t) - Z_2(t))^2 + 0.5(0.5Z_1(t) - 0.5Z_2(t))^2 \\ &= -0.285(Z_1(t) - Z_2(t))^2 \leq 0, \quad t \geq t_0, \end{aligned}$$

hence, $\mathbb{E}[dV(Z(t))] = \mathbb{E}[\mathcal{L}V(Z(t))] dt \leq -\mathbb{E}[0.57 V(Z(t))] dt \leq 0, t \geq t_0$, and $\mathcal{L}V(Z(t)) = 0$ if and only if $Z_1(t) = Z_2(t)$. Now, it follows that $\mathbb{E}[V(Z(t))]$ is a non-increasing function of time and $\mathbb{E}[\frac{1}{2}(Z_1(t) - Z_2(t))^2] \rightarrow 0$ as $t \rightarrow \infty$. Since Eqs. (64) and (65) are autonomous, the limiting delay system given by Eqs. (64) and (65) is globally uniformly asymptotically mean-square synchronized.

Next, it can be shown that with $r=0.18$ and

$$P = Q = B = L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

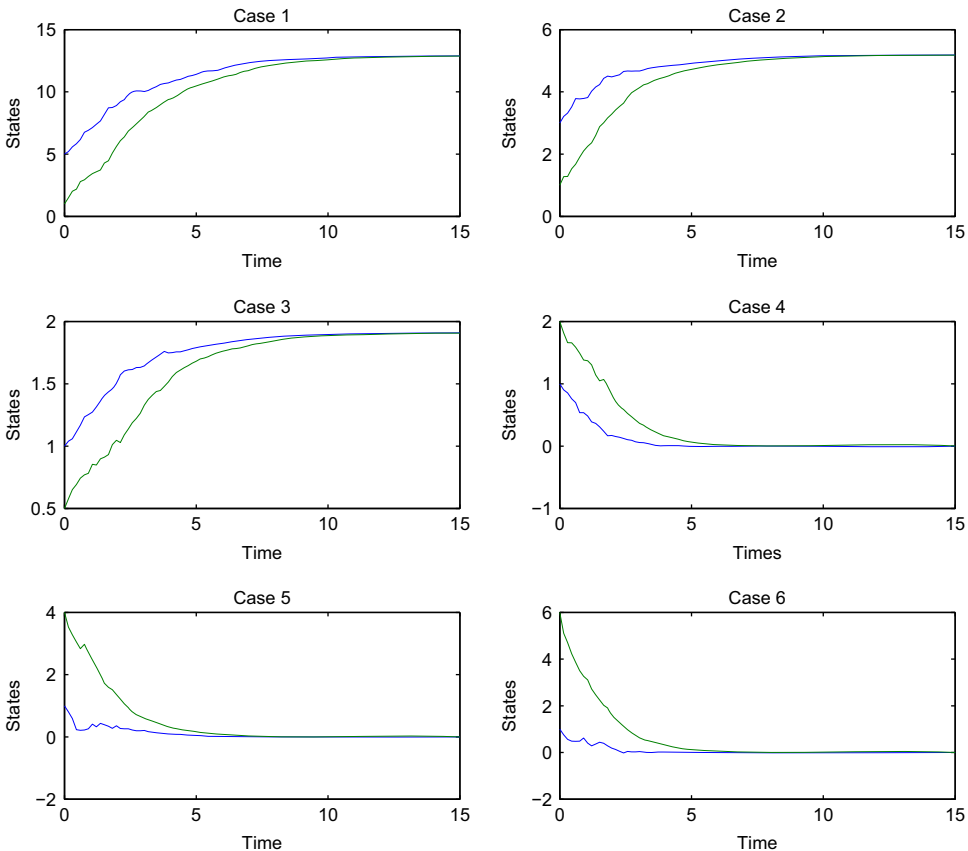


Fig. 3. State trajectories versus time for Example 6.2 with different initial conditions.

Eqs. (25), (42) and (43) hold. Hence, it follows from Theorem 5.1 that Eqs. (62) and (63) are globally asymptotically mean-square synchronized. To illustrate this, we consider 6 different cases for our simulation. Namely,

Case 1: $\phi_1(\theta) = 5, \phi_2(\theta) = 1$, where $\theta \in [-1, 0]$.

Case 2: $\phi_1(\theta) = 3, \phi_2(\theta) = 1$, where $\theta \in [-1, 0]$.

Case 3: $\phi_1(\theta) = 1, \phi_2(\theta) = 0.5$, where $\theta \in [-1, 0]$.

Case 4: $\phi_1(\theta) = 1, \phi_2(\theta) = 2$, where $\theta \in [-1, 0]$.

Case 5: $\phi_1(\theta) = 1, \phi_2(\theta) = 4$, where $\theta \in [-1, 0]$.

Case 6: $\phi_1(\theta) = 1, \phi_2(\theta) = 6$, where $\theta \in [-1, 0]$.

Figs. 3 and 4 show the state trajectories of Eqs. (62) and (63) and the synchronization error $|S_1(t) - S_2(t)|$ versus time, respectively. The figures show that the system synchronizes for all the six cases considered.

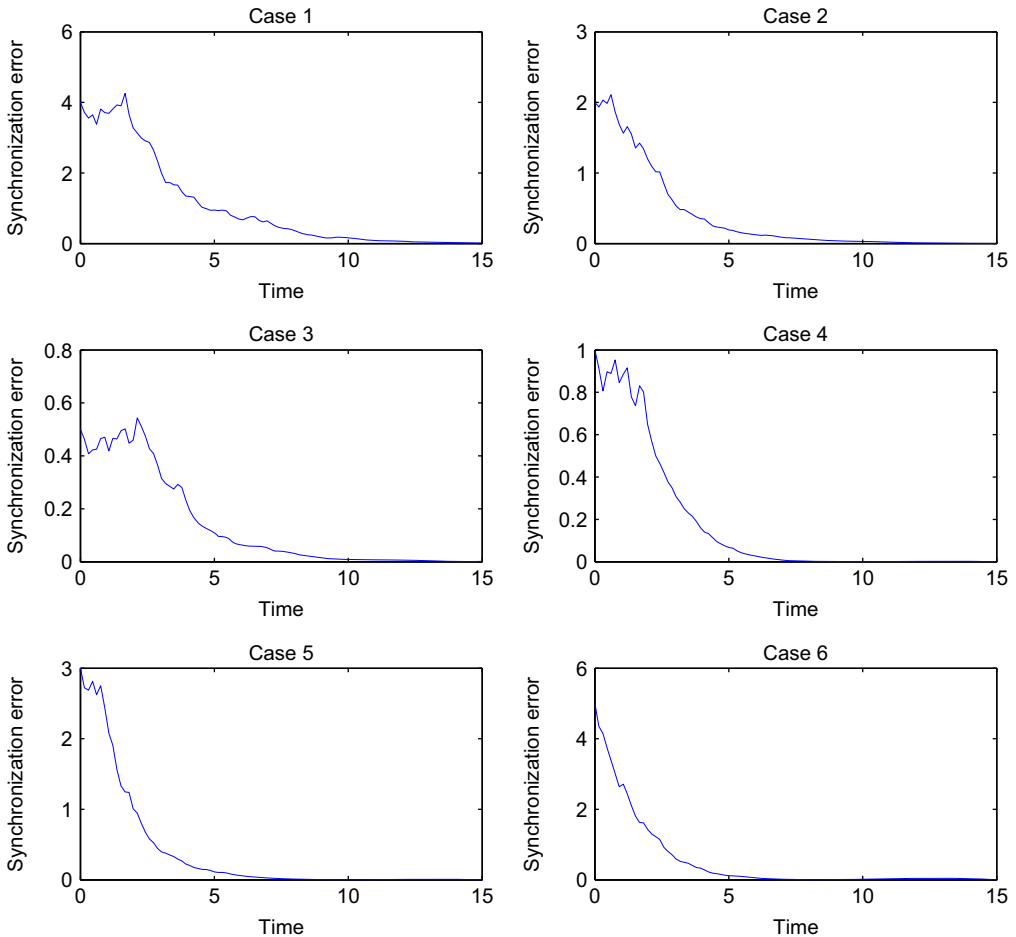


Fig. 4. Synchronization error $|S_1(t) - S_2(t)|$ versus time for Example 6.2 with different initial conditions.

7. Conclusion

There is a substantial literature about oscillators in the brain and even speculation about the effects of anesthetic agents on these oscillators. In this paper, we developed a stochastic synaptic drive firing rate model for an excitatory and inhibitory cortical neuronal network in the face of system time delays and noisy inputs, and provided constructive sufficient conditions for global asymptotic mean-square synchronization for this model. Although the conservatism of our results is problem dependant, it is desirable to better understand the nature of the conservatism in order to utilize our analysis results more effectively. In addition, the issue of necessity remains to be addressed. That is, if the system is globally asymptotically mean-square synchronized with a corresponding Lyapunov–Krasovskii-type functional, then is such a functional necessarily given by one of the theorems in this paper? In future research, we will explore these questions along with the utility of this model to explain the underlying mechanism of action for general anesthesia and consciousness by examining synchronization mechanisms of inhibitory neurons at the onset of anesthesia.

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