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Reduced order LQG controllers for linear time varying plants

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Abstract: The problem of reduced order LQG optimization is addressed in a finite horizon, linear time-varying (LTV) system setting. First-order necessary conditions for local optimality in the parameter space is provided in terms of four coupled matrix differential equations. This result provides a transparent generalization of the optimal projection equations of Hyland and Bernstein for the optimal, steady-state compensation of linear time-invariant (LTI) plants

Keywords: Reduced-order control; optimal projection equations; time-varying plants.

1. Introduction

The solution of the Linear Quadratic Gaussian (LQG) optimal control problem has been known for some three decades, in both the infinite horizon LTI and finite horizon LTV settings [13]. The optimal dynamic compensator couples an optimal state-feedback with an optimal state estimator. In particular, that compensator is of the same order as the plant. In a series of articles [3,4,7] the problem of finding the optimal dynamic compensator of a specified *reduced* order has been addressed in LTI infinite horizon systems. The results provide necessary conditions for optimality which involve four mutually coupled matrix equations (two algebraic Riccati and two algebraic Lyapunov equations). The coupling is due to a projection matrix which appears in all four equations, and whose rank is precisely equal to the order of the compensator and which determines the optimal compensator gains. Intuitively, the reason for the coupling is the fact that the choice of the subspace of the state-space to which the compensator's state belongs is directly related to all aspects of the optimization process. Consequently, it represents a break down of the separation between the operations of state estimation and state estimate feedback; i.e. the certainty equivalence principle is no longer valid. The proof is based on expressing the closed-loop cost as a function of the design parameters (= the compensator's coefficients), and the utilization of a basic Lagrange multipliers argument for optimization over the parameters space. Thus, this approach provides a constrained optimal control methodology in which we do not seek to optimize a performance measure per se, but rather a performance measure within a class of fixed structure controllers.

In this paper we extend the results of [7] to the finite horizon – LTV case. The parameter space is now that of matrix valued functions, and a new feature of the problem is the dependence on data on the statistics of the initial state. Not surprisingly, the solution is based on mutually coupled two differential

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Riccati and two differential Lyapunov equations. Here again, the coupling is due to a (time-varying) projection matrix whose rank along the finite interval is equal to the order of the compensator. As in the LTI case, when the order reduction constraint is removed, the well known decades old solution of the LQG problem reimmerges. Several potential benefits may result by developing the finite horizon, fixed-strucure, reduced-order dynamic compensation results. Specifically, the finite-time differential Riccati equations may provide an alternative approach to homotopy methods [6,12] for solving the steady state optimal projection equations via numerical integration. Furthermore, the theory developed herein can be easily modified to address the reduced order synthesis for infinite time linear periodic systems. This extension will provide the framework for designing reduced order controllers for higher harmonic control of helicopter vibrations.

2. Reduced-order LQG time varying control problem

Our setting is the familiar form of the LQG problem with the standard 'nicety' structural assumptions. We consider an LTV linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w_1(t), \tag{2.1a}$$

$$y(t) = C(t)x(t) + w_2(t), \tag{2.1b}$$

with continuous coefficients [1], defined over a finite interval $[t_0, t_1]$, where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^l$. Here the initial state is a random variable with $E(x(t_0)) = x_0$, $E(x(t_0)x(t_0)^{\mathrm{T}}) = Q_0$; the disturbances w_i, $i = 1, 2$, are Gaussian zero-mean white noise signals with $V_i(t)\delta(t-s) = E(w_i(t)w_i(s)^{\mathrm{T}})$, where both V_i are continuous and V_2 is assumed to be uniformly positive definite; finally, the initial state and the two noise signals are assumed mutually independent: $E(w_1(t)w_2(s)^{\mathrm{T}}) = 0 = E(w_i(t)x(t_0)^{\mathrm{T}})$.

Our objective is to find a linear controller of the form

$$\dot{x}_c(t) = A_c(t)x_c(t) + B_c(t)y(t), \quad x_c(t_0) = x_{c0}, \tag{2.2a}$$

$$u(t) = C_c(t)x_c(t), \tag{2.2b}$$

which minimizes a quadratic cost functional

$$J(A_c, B_c, C_c, x_{c0}) = E\left\{\int_{t_0}^{t_1}\left[x(t)^{\mathrm{T}}R_1(t)x(t) + u(t)^{\mathrm{T}}R_2(t)u(t)\right]\,\mathrm{d}t + x(t_1)^{\mathrm{T}}R_3x(t_1)\right\} \tag{2.3}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, R_1 and R_2 are continuous, R_1 and R_3 are positive semidefinite and R_2 is uniformly positive definite. *The particular feature of the problem discussed in this paper is that the order of the compensator*, n_c, *is predetermined to be lower than that of the plant*: $n_c < n$.

We shall allow only '*minimal realizations*' of the compensator. That is, we shall restrict our attention to compensators of the form (2.2) that are both completely controllable and completely observable over any positive interval. As the coefficient triplets (A_c, B_c, C_c) will be embedded with the $C[t_0, t_1] \times C[t_0, t_1] \times C[t_0, t_1]$ of continuous matrix functions, it is noted that the family of admissible compensators forms an open set. That will enable us to invoke variational arguments in our characterization of optimal solutions.

For a given compensator (A_c, B_c, C_c), the closed-loop system can be written as

$$\dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{w}(t) \tag{2.4}$$

[1] Our assumption that the system coefficients, and later on, the compensator coefficients, are continuous, is made for simplicity. Other function spaces, such as L_∞ are also possible.

with the cost functional

$$J(A_c, B_c, C_c) = E\left\{\int_{t_0}^{t_1} \tilde{x}(t)^T \tilde{R} \tilde{x}(t)\, dt + \tilde{x}(t_1)^T \tilde{R}_0 \tilde{x}(t_1)\right\}, \tag{2.5}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} w_1 \\ B_c w_2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}, \quad \tilde{R}_0 = \begin{bmatrix} R_3 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.6}$$

and

$$\begin{gathered} E\left(\tilde{w}(t)\tilde{w}(s)^T\right) = \delta(t-s)\tilde{V}(t) = \delta(t-s)\begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix}(t), \\ E(\tilde{x}(t_0)) = \tilde{x}_0 = \begin{bmatrix} x_0 \\ x_{c0} \end{bmatrix}, \quad E\left(\tilde{x}(t_0)\tilde{x}(t_0)^T\right) = \tilde{Q}_0 = \begin{bmatrix} Q_0 & x_0 x_{c0}^T \\ x_{c0} x_0^T & x_{c0} x_{c0}^T \end{bmatrix}. \end{gathered} \tag{2.7}$$

Let $\tilde{\Phi}(\cdot,\cdot)$ be the transition matrix generated by the augmented matrix $\tilde{A}$. The following is a standard observation (see e.g. Kwakernaak and Sivan [9], pp. 85–112).

Proposition 2.1. *Given a compensator* (2.2), *the second moment matrix* $\tilde{Q}(t) = E(\tilde{x}(t)\tilde{x}(t)^T)$ *satisfies the differential Lyapunov equation*

$$\dot{\tilde{Q}}(t) = \tilde{A}(t)\tilde{Q}(t) + \tilde{Q}(t)\tilde{A}^T(t) + \tilde{V}(t), \quad \tilde{Q}(t_0) = \tilde{Q}_0, \tag{2.8}$$

or equivalently,

$$\tilde{Q}(t) = \tilde{\Phi}(t, t_0)\tilde{Q}_0\tilde{\Phi}(t, t_0)^T + \int_{t_0}^{t} \tilde{\Phi}(t, s)\tilde{V}(s)\tilde{\Phi}(t, s)^T\, ds. \tag{2.9}$$

Let us define also $\tilde{P}(t)$ *by the dual Lyapunov equation*

$$\dot{\tilde{P}}(t) = -\tilde{P}(t)\tilde{A}(t) - \tilde{A}(t)^T\tilde{P}(t) - \tilde{R}(t), \quad \tilde{P}(t_1) = \tilde{R}_0. \tag{2.10}$$

Then

$$J(A_c, B_c, C_c, x_{c0}) = \mathrm{tr}\left\{\tilde{R}_0\tilde{Q}(t_1) + \int_{t_0}^{t_1} \tilde{R}(t)\tilde{Q}(t)\, dt\right\} = \mathrm{tr}\left\{\tilde{P}(t_0)\tilde{Q}_0 + \int_{t_0}^{t_1} \tilde{P}(t)\tilde{V}(t)\, dt\right\}. \tag{2.11}$$

In what follows it may be convenient at times to think of the two matrix equations in Proposition 2.1, translated into equivalent vector forms. For this result we use standard Kronecker algebra notation. (For precise definitions and details see e.g. Brewer [5].) Given a matrix $M \in \mathbb{R}^{l \times m}$ we denote by vec $M \in \mathbb{R}^{lm}$ the vector obtained by stretching the columns of M. '$\otimes$' will stand for the Kronecker matrix product and define $M \oplus N = M \otimes I + I \otimes N$. Subject to these conventions, straightforward manipulation yield the following vector forms for (2.8) and (2.10):

$$\frac{d}{dt}\mathrm{vec}\ \tilde{Q}(t) = \left[\tilde{A}(t) \oplus \tilde{A}(t)\right] \mathrm{vec}\ \tilde{Q}(t) + \mathrm{vec}\ \tilde{V}(t), \quad \mathrm{vec}\ \tilde{Q}(t_0) = \mathrm{vec}\ \tilde{Q}_0, \tag{2.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{vec}\tilde{P}(t) = -\left[\tilde{A}(t)^{\mathrm{T}} \oplus \tilde{A}(t)^{\mathrm{T}}\right]\operatorname{vec}\tilde{P}(t) - \operatorname{vec}\tilde{R}(t), \quad \operatorname{vec}\tilde{R}(t_1) = \operatorname{vec}\tilde{R}_0. \tag{2.13}$$

In this case

$$\begin{aligned} J(A_c, B_c, C_c) &= \left(\operatorname{vec}\tilde{R}_0\right)^{\mathrm{T}} \operatorname{vec}\tilde{Q}(t_1) + \int_{t_0}^{t_1}\left(\operatorname{vec}\tilde{R}(t)\right)^{\mathrm{T}}\tilde{Q}(t)\,\mathrm{d}t \\ &= \left(\operatorname{vec}\tilde{P}(t_0)\right)^{\mathrm{T}} \operatorname{vec}\tilde{Q}_0 + \int_{t_0}^{t_1}\left(\operatorname{vec}\tilde{P}(t)\right)^{\mathrm{T}}\tilde{V}(t)\,\mathrm{d}t. \end{aligned} \tag{2.14}$$

3. Necessary conditions for optimality

In this section we obtain necessary conditions for optimality that characterize solutions to the reduced-order LQG time-varying control problem. Our main observation is summarized as follows.

Theorem 3.1. *A necessary condition for a choice (A_c, B_c, C_c, x_{c0}) of the compensator's parameters to be locally optimal (relative to the $C[t_0, t_1] \times C[t_0, t_1] \times C[t_0, t_1] \times \mathbb{R}^{n_c}$ topology and with respect to the cost functional J) is that there will exist matrix functions $\Gamma \in C([t_0, t_1), \mathbb{R}^{n_c \times n})$, $G \in C((t_0, t_1], \mathbb{R}^{n_c \times n})$, P, Q, $\hat{P}$ and $\hat{Q} \in C([t_0, t_1], \mathbb{R}^{n \times n})$ with the following properties*:

$$\Gamma(t)G^{\mathrm{T}}(t) = I_{n_c}, \tag{3.1a}$$

$$\tau(t) := G^{\mathrm{T}}(t)\Gamma(t) \;\Rightarrow\; \tau(t)^2 = \tau(t), \tag{3.1b}$$

$$\operatorname{rank}\hat{P}(t) = \operatorname{rank}\hat{Q}(t) = n_c, \tag{3.1c}$$

$$\exists M, M^{-1} \in C\left((t_0, t_1), \mathbb{R}^{n_c \times n_c}\right), \quad \hat{Q}(t)\hat{P}(t) = G^{\mathrm{T}}(t)M(t)\Gamma(t), \tag{3.1d}$$

and such that the differential matrix equations

$$-\dot{P} = PA + A^{\mathrm{T}}P + R_1 - PBR_2^{-1}B^{\mathrm{T}}P + \tau_\perp^{\mathrm{T}}PBR_2^{-1}B^{\mathrm{T}}P\tau_\perp, \tag{3.2a}$$

$$\dot{Q} = AQ + QA^{\mathrm{T}} + V_1 - QC^{\mathrm{T}}V_2^{-1}CQ + \tau_\perp QC^{\mathrm{T}}V_2^{-1}CQ\tau_\perp^{\mathrm{T}}, \tag{3.2b}$$

$$-\dot{\hat{P}} = \hat{P}(A - QC^{\mathrm{T}}V_2^{-1}C) + (A - QC^{\mathrm{T}}V_2^{-1}C)^{\mathrm{T}}\hat{P} + PBR_2^{-1}B^{\mathrm{T}}P - \tau_\perp^{\mathrm{T}}PBR_2^{-1}B^{\mathrm{T}}P\tau_\perp, \tag{3.2c}$$

$$\dot{\hat{Q}} = (A - BR_2^{-1}B^{\mathrm{T}}P)\hat{Q} + \hat{Q}(A - BR_2^{-1}B^{\mathrm{T}}P)^{\mathrm{T}} + QC^{\mathrm{T}}V_2^{-1}CQ - \tau_\perp QC^{\mathrm{T}}V_2^{-1}CQ\tau_\perp^{\mathrm{T}}, \tag{3.2d}$$

(where $\tau_\perp := I_n - \tau$) are satisfied. The parameters of the locally optimal compensator are then given by

$$A_c = \Gamma(A - QC^{\mathrm{T}}V_2^{-1}C - BR_2^{-1}B^{\mathrm{T}}P)G^{\mathrm{T}} + \dot{\Gamma}G^{\mathrm{T}}, \tag{3.3a}$$

$$B_c = \Gamma QC^{\mathrm{T}}V_2^{-1}, \tag{3.3b}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}PG^{\mathrm{T}}, \tag{3.3c}$$

$$x_{c0} = \Gamma(t_0)x_0. \tag{3.3d}$$

Finally, all the definitions above are associated with the eventual solutions of (2.8) *and* (2.10) *as follows: Denote the blocks of* $\tilde{P}$ *and* $\tilde{Q}$ *(relative to the state space direct sum decomposition* $\mathbb{R}^{n+n_c} = \mathbb{R}^n \times \mathbb{R}^{n_c}$*) by*

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix} \quad \text{and} \quad \tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}. \tag{3.4}$$

Then

$$\Gamma(t) = -P_2(t)^{-1}P_{12}(t)^{\mathrm{T}}, \tag{3.5a}$$

$$G(t) = Q_2(t)^{-1}Q_{12}(t)^{\mathrm{T}}, \tag{3.5b}$$

$$\tau(t) = G(t)^{\mathrm{T}}\Gamma(t), \tag{3.5c}$$

$$P(t) = (P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}})(t), \tag{3.5d}$$

$$Q(t) = (Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}})(t), \tag{3.5e}$$

$$\hat{P}(t) = (P_{12}P_2^{-1}P_{12}^{\mathrm{T}})(t) = -\Gamma(t)P_{12}^{\mathrm{T}}(t), \tag{3.5f}$$

$$\hat{Q}(t) = (Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}})(t) = G^{\mathrm{T}}(t)Q_{12}^{\mathrm{T}}(t). \tag{3.5g}$$

Remark 3.1. With the exception of one case, our theorem does not provide a simple expression for the terminal values of P and $\hat{P}$ and the initial values of Q and $\hat{Q}$, in terms of the specified parameters. The exception is the case where $R_3 = 0$ and $Q_0 = 0$ (then, in particular, $x_0 = 0$). Then $P(t_1) = \hat{P}(t_1) = 0$ and $Q(t_0) = \hat{Q}(t_0) = 0$, respectively. Details are brought in the proof.

Remark 3.2. In the case that $n = n_c$ (3.1a) implies invertibility of Γ and G and the equalities $\Gamma^{-1} = G^{\mathrm{T}}$, $\tau = I$ and $\tau_\perp = 0$. Thus equations (3.2a, b) reduce to the familiar form of the Riccati equations associated with LQG optimization. Without any coupling, solvability of (3.2c) and (3.2d) add no further constraints. A change of state variable from x_c to $G^{\mathrm{T}}x_c$ then brings the compensator (3.3) to the usual form of an LQG-optimal controller. In particular, in that case one can easily establish the continuity of Γ at t_1 and of G at t_0: Indeed, let Φ_c be the transition matrix generated by A_c, as defined in (3.3a) and let Ψ_c be that generated by the more standard $A - QC^{\mathrm{T}}V_2^{-1}C - BR_2^{-1}B^{\mathrm{T}}P$. Then the change of state from x_c to $G^{\mathrm{T}}x_c$ implies $G^{\mathrm{T}}(t)\Phi_c(t, s) = \Psi_c(t, s)G^{\mathrm{T}}(s)$ and $\Phi_c(t, s)\Gamma(s) = \Gamma(t)\Psi_c(t, s)$. In particular, $G^{\mathrm{T}}(t_0) = \Psi_c(t_0, t)G^{\mathrm{T}}(t)\Phi_c(t, t_0)$ and $\Gamma(t_1) = \Phi_c(t_1, t)\Gamma(t)\Psi_c(t, t_1)$ are both bounded.

The proof of Theorem 1 is based on application of well known (see e.g. Luenberger [11] or Lee and Markus [10]) non-linear optimal control/Lagrange multiplier rules to our problem. In point of fact, in the context of our discussion, we shall focus on the 'Matrix Minimum Principle' version of those results, as summarized by Athans in [1]. For the reader's convenience we first recall some basic definitions and facts.

Definition 3.1. Let X and Y be Banach spaces and $T: X \to Y$ a continuous, possibly non linear mapping. We say that T is *Fréchet differentiable* at a point $x \in X$ if there exists a bounded linear operator $T'(x): X \to Y$ (= the *Fréchet derivative* of T at x) such that

$$\forall h \in X, \quad \lim_{\|h\| \to 0} \frac{\| T(x+h) - T(x) - T'(x)h \|}{\| h \|} = 0. \tag{3.6}$$

The point x is called a *regular point* if $T'(x)$ is surjective.

The Basic Optimization Theorem. *Consider continuously* Fréchet differentiable *functions $f : X \to Y$ and $g : X \to \mathbb{R}$ and the optimization problem*

$$\min_{x : f(x) = 0} g(x). \tag{3.7}$$

Then a necessary condition for a regular point of x^* of f to be a local solution is the implication

$$\forall \xi \in X, \quad f'(x^*)\xi = 0 \ \Rightarrow \ g'(x^*)\xi = 0. \tag{3.8}$$

Proof of Theorem 3.1. In order to facilitate the presentation we shall break the proof into four steps.

Step 1. The optimal control problem.

We bring the optimization problem of Theorem 3.1 to the general form of the Basic Optimization Theorem as follows: The optimization parameters will be $(A_c, B_c, C_c, \tilde{Q}, x_{c0})$. We set

$$f(A_c, B_c, C_c, \tilde{Q}, x_{c0}) = \tilde{Q}(t) - \tilde{Q}_0 - \int_{t_0}^{t} \left[\tilde{A}(s)\tilde{Q}(s) + \tilde{Q}(s)\tilde{A}(s)^{\mathrm{T}} + \tilde{V}(s) \right] \mathrm{d}s, \tag{3.9a}$$

$$g(A_c, B_c, C_c, \tilde{Q}, x_{c0}) = \mathrm{tr}\left\{ \tilde{R}_0\tilde{Q}(t_1) + \int_{t_0}^{t_1} \tilde{R}(t)\tilde{Q}(t)\, \mathrm{d}t \right\}. \tag{3.9b}$$

The parameter space will be $X = (C[t_0, t_1])^4 \times \mathbb{R}^{n_c}$ and $C[t_0, t_1]$ will substitute for Y. This defines an unconstrained optimal control problem with $\tilde{Q}$ being the 'state' and the rest of the optimization parameters determining the 'control', as well as that part of the 'initial state' $\tilde{Q}(t_0)$ that can be freely assigned via the choice of x_{c0}.

Given any perturbation $(F_c, G_c\ H_c, \tilde{\Xi}, \xi_{c0})$ of $(A_c, B_c, C_c, \tilde{Q}, x_{c0})$, there holds [2]

$$\mathrm{vec}\left(\frac{\partial f(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial A_c} F_c \right)(t) = -\int_{t_0}^{t} \left[\begin{bmatrix} 0 & 0 \\ 0 & F_c \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & F_c \end{bmatrix} \right](s)\ \mathrm{vec}\ \tilde{Q}(s)\ \mathrm{d}s,$$

$$\begin{aligned} \mathrm{vec}&\left(\frac{\partial f(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial B_c} G_c \right)(t) \\ &= -\int_{t_0}^{t} \left[\begin{bmatrix} 0 & 0 \\ G_c C & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ G_c G & 0 \end{bmatrix} \right](s)\ \mathrm{vec}\ \tilde{Q}(s)\ \mathrm{d}s \\ &\quad - \int_{t_0}^{t} \mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & B_c V_2 G_c^{\mathrm{T}} \end{bmatrix}(s)\ \mathrm{d}s - \int_{t_0}^{t} \mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & G_c V_2 B_c^{\mathrm{T}} \end{bmatrix}(s)\ \mathrm{d}s, \end{aligned}$$

$$\mathrm{vec}\left(\frac{\partial f(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial C_c} H_c \right)(t) = -\int_{t_0}^{t} \left[\begin{bmatrix} 0 & BH_c \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & BH_c \\ 0 & 0 \end{bmatrix} \right](s)\ \mathrm{vec}\ \tilde{Q}(s)\ \mathrm{d}s,$$

$$\mathrm{vec}\left(\frac{\partial f(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial x_{c0}} \xi_c \right)(t) = -\mathrm{vec} \begin{bmatrix} 0 & x_0\xi_c^{\mathrm{T}} \\ \xi_c x_0^{\mathrm{T}} & x_{c0}\xi_c^{\mathrm{T}} + \xi_c x_{c0}^{\mathrm{T}} \end{bmatrix},$$

$$\mathrm{vec}\left(\frac{\partial f(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial \tilde{Q}} \tilde{\Xi} \right)(t) = \mathrm{vec}\ \tilde{\Xi}(t) - \int_{t_0}^{t} \left[\tilde{A}(s) \oplus \tilde{A}(s) \right]\ \mathrm{vec}\ \tilde{\Xi}(s)\ \mathrm{d}s, \tag{3.10}$$

[2] In what follows we use '∂' in reference to partial Fréchet derivative, consistent with the definition above.

and

$$\frac{\partial g(A_c, B_c, C_c\ \tilde{Q}, x_{c0})}{\partial A_c} F_c = 0, \qquad \frac{\partial g(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial B_c} G_c = 0,$$

$$\frac{\partial g(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial C_c} H_c = \int_{t_0}^{t_1} \left[\mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & C_c^{\mathrm{T}} R_2 H_c \end{bmatrix} (s) \right]^{\mathrm{T}} \mathrm{vec}\ \tilde{Q}(s)\ \mathrm{d}s$$

$$+ \int_{t_0}^{t_1} \left[\mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & H_c^{\mathrm{T}} R_2 C_c \end{bmatrix} (s) \right]^{\mathrm{T}} \mathrm{vec}\ \tilde{Q}(s)\ \mathrm{d}s,$$

$$\frac{\partial g(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial x_{c0}} \xi_c = 0,$$

$$\frac{\partial g(A_c, B_c, C_c, \tilde{Q}, x_{c0})}{\partial \tilde{Q}} \tilde{\Xi} = \left[\mathrm{vec}\ \tilde{R}_0\right]^{\mathrm{T}} \mathrm{vec}\ \tilde{\Xi}(t_1) + \int_{t_0}^{t_1} \left[\mathrm{vec}\ \tilde{R}(s)\right]^{\mathrm{T}} \mathrm{vec}\ \tilde{\Xi}(s)\ \mathrm{d}s. \tag{3.11}$$

One immediately observes that the linear operator $f'(A_c, B_c, C_c, \tilde{Q}, x_{c0})$ is surjective for any choice of $(A_c, B_c, C_c, \tilde{Q}, x_{c0})$ and that the condition

$$f'(A_c, B_c, C_c, \tilde{Q}, x_{c0}) \begin{bmatrix} F_c \\ G_c \\ H_c \\ \tilde{\Xi} \\ \xi_c \end{bmatrix} = 0 \tag{3.12}$$

translates into a differential equation in $\tilde{\Xi}$ and, eventually, into a representation of this function in terms of the other four components of the perturbation vector. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{vec}\ \tilde{\Xi}(t) = \left[\tilde{A}(t) \oplus \tilde{A}(t)\right] \mathrm{vec}\ \tilde{\Xi}(t) + \left[\begin{bmatrix} 0 & BH_c \\ G_c C & F_c \end{bmatrix} \oplus \begin{bmatrix} 0 & BH_c \\ G_c C & F_c \end{bmatrix} \mathrm{vec}\ \tilde{Q} \right] (t)$$

$$+ \mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & B_c V_2 G_c^{\mathrm{T}} \end{bmatrix} (t) + \mathrm{vec} \begin{bmatrix} 0 & 0 \\ 0 & G_c V_2 B_c^{\mathrm{T}} \end{bmatrix} (t), \tag{3.13a}$$

$$\tilde{\Xi}(t_0) = \begin{bmatrix} 0 & x_0 \xi_c^{\mathrm{T}} \\ \xi_c x_0^{\mathrm{T}} & x_{c0} \xi_c^{\mathrm{T}} + \xi_c x_{c0}^{\mathrm{T}} \end{bmatrix}. \tag{3.13b}$$

Alternatively, the matrix differential equation is

$$\dot{\tilde{\Xi}}(t) = \tilde{A}(t) \tilde{\Xi}(t) + \tilde{\Xi}(t) \tilde{A}(t)^{\mathrm{T}} + \begin{bmatrix} 0 & BH_c \\ G_c C & F_c \end{bmatrix} (t) \tilde{Q}(t) + \tilde{Q}(t) \begin{bmatrix} 0 & BH_c \\ G_c C & F_c \end{bmatrix}^{\mathrm{T}} (t)$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & B_c V_2 G_c^{\mathrm{T}} \end{bmatrix} (t) + \begin{bmatrix} 0 & 0 \\ 0 & G_c V_2 B_c^{\mathrm{T}} \end{bmatrix} (t). \tag{3.14}$$

The Optimization Theorem then says that $(A_c, B_c, C_c, \tilde{Q}, x_{c0})$ is a local solution of (3.7) only if there holds

$$g'\left(A_c, B_c, C_c, \tilde{Q}, x_{c0}\right)\begin{bmatrix} F_c \\ G_c \\ H_c \\ \tilde{\Xi} \\ \xi_c \end{bmatrix} = 0, \tag{3.15}$$

for all choices of (F_c, G_c, H_c, ξ_c) and with $\tilde{\Xi}$, as defined in (3.13).

Let $\tilde{P}$ be the (positive semidefinite), backwards solution of (2.10). If we write (3.15) explicitly and substitute the variation of parameters representations formulae for $\tilde{\Xi}$ and then for $\tilde{P}$, it becomes a matter of straightforward computation to see that the optimization condition translates into the following:

$$\operatorname{tr} \tilde{P}(t_0)\begin{bmatrix} 0 & x_0\xi_c^{\mathrm{T}} \\ \xi_c x_0^{\mathrm{T}} & x_{c0}\xi_c^{\mathrm{T}} + \xi_c x_{c0}^{\mathrm{T}} \end{bmatrix} = 0, \tag{3.16a}$$

$$\begin{aligned} \operatorname{tr}\int_{t_0}^{t_1}\Bigg[\tilde{P}\left[\begin{bmatrix} 0 & BH_c \\ G_cC & F_c \end{bmatrix}\tilde{Q} + \tilde{Q}\begin{bmatrix} 0 & BH_c \\ G_cC & F_c \end{bmatrix}^{\mathrm{T}}\right] + \tilde{P}\begin{bmatrix} 0 & 0 \\ 0 & B_cV_2G_c^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & G_cV_2B_c^{\mathrm{T}} \end{bmatrix}\tilde{P} \\ + \tilde{Q}\begin{bmatrix} 0 & 0 \\ 0 & C_c^{\mathrm{T}}R_2H_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & H_c^{\mathrm{T}}R_2C_c \end{bmatrix}\tilde{Q}\Bigg](t)\,\mathrm{d}t \\ = 2\operatorname{tr}\int_{t_0}^{t_1}\left[\tilde{P}\tilde{Q}\begin{bmatrix} 0 & BH_c \\ G_cC & F_c \end{bmatrix} + \tilde{P}\begin{bmatrix} 0 & 0 \\ 0 & B_cV_2G_c^{\mathrm{T}} \end{bmatrix} + \tilde{Q}\begin{bmatrix} 0 & 0 \\ 0 & C_c^{\mathrm{T}}R_2H_c \end{bmatrix}\right](t)\,\mathrm{d}t = 0, \end{aligned} \tag{3.16b}$$

for all choices of (F_c, G_c, H_c, ξ_c). Consistently with the block structure given in (3.4), these equalities eventually become:

$$P_2(t_0)x_{c0} + P_{12}^{\mathrm{T}}(t_0)x_0 = 0, \tag{3.17a}$$

$$C(Q_1P_{12} + Q_{12}P_2) + V_2B_c^{\mathrm{T}}P_2 = 0, \tag{3.17b}$$

$$(Q_{12}^{\mathrm{T}}P_1 + Q_2P_{12}^{\mathrm{T}})B + Q_2C_c^{\mathrm{T}}R_2 = 0, \tag{3.17c}$$

$$Q_{12}^{\mathrm{T}}P_{12} + Q_2P_2 = 0. \tag{3.17d}$$

Step 2. The optimal B_c, C_c and x_{c0}.

Claim. The matrices $P_2(t)$, $t_0 \le t < t_1$, and $Q_2(t)$, $t_0 < t \le t_1$, are positive definite matrices.

Proof of the Claim. Consider the augmented system

$$\dot{x} = Ax + \eta, \tag{3.18a}$$

$$\dot{x}_c = A_cx_c + B_cV_2^{1/2}\eta_c. \tag{3.18b}$$

By assumption, the compensator's state equation, (2.2a), is completely controllable over any interval of positive length. In view of the fact that V_2 is uniformly positive definite, the same applies to (3.18b). This property cannot be destroyed by a state feedback that couples (3.18a) with (3.18b). In particular, the system

$$\dot{\tilde{x}} = \tilde{A}(t)\tilde{x} + \tilde{B}(t)\tilde{v} \quad \text{where} \quad \tilde{B} = \begin{bmatrix} V_1^{1/2} & 0 \\ 0 & B_cV_2^{1/2} \end{bmatrix} \tag{3.19}$$

enjoys the property that any designated x_c-component of the terminal state can be reached over any time interval of positive length, by an appropriate choice of the input $\tilde{v}$.

Let $\tilde{\mathscr{G}}(t)$ be the controllability Gramian over the interval (t_0, t), with block partitioning consistent with (3.4). Following from the fact that $\tilde{\mathscr{G}}(t)$ is positive semidefinite we know that so is the block $\mathscr{G}_2(t)$ and have the implication

$$\forall \xi_c, \quad \xi_c^{\mathrm{T}} \mathscr{G}_2(t) = 0 \ \Rightarrow \ \xi_c^{\mathrm{T}} \mathscr{G}_{12}^{\mathrm{T}}(t) = 0. \tag{3.20}$$

Yet the space of possible x_c-components of states of (3.19) at the time t is spanned as the image of $[\mathscr{G}_{12}^{\mathrm{T}}(t) \ \mathscr{G}_2(t)]$, and as argued above, it is the entire $\mathbb{R}^{n_c}$. So $\mathscr{G}_2(t)$ must be non-singular, hence positive definite.

All that remains to be noticed in order to conclude that $Q_2(t)$ is positive definite is the fact that $\tilde{\mathscr{G}}(t)$ is exactly the integral term in the expression (2.9) for $\tilde{Q}(t)$.

The positivity of $P_2(t)$ is established similarly, invoking the uniform positivity of R_2 and the assumed observability in (2.2). Hence the Claim.

Following from the Claim, the definitions in (3.5) are all well made. With these definitions it is now easy to see that the conditions (3.1a,b) in the Theorem are a consequence of (3.17d) and that (3.17a,b,c) imply that the formulae (3.3b,c,d) for the optimal B_c, C_c and x_{c0} are satisfied, respectively.

In addition, we make note of the following points: (1) $\hat{P}$ and $\hat{Q}$ are obviously positive semidefinite. (2) From standard results on block structured positive matrices, since $\tilde{P}$ and $\tilde{Q}$ are positive semidefinite and uniformly bounded, the same must apply to P and Q. This further implies that $\hat{P}$ and $\hat{Q}$ are bounded by P_1 and Q_1. In particular, if $P_1(t_1) = 0$ and $Q_1(t_0) = 0$ then $P(t_1) = \hat{P}(t_1) = 0$ and $Q(t_0) = \hat{Q}(t_0) = 0$. (3) At this point we can see that the definitions (3.5a,b) and the established equality (3.1a) imply that P_{12} and Q_{12}; hence $\hat{P}$ and $\hat{Q}$ are of rank n_c along (t_0, t_1) and that (3.1d) will hold with $M = Q_{12}^{\mathrm{T}} P_{12} = -Q_2 P_2$.

In the ensuing developments we shall refer to this next explicit block structure of (2.8).

$$\dot{Q}_1 = AQ_1 + Q_1 A^{\mathrm{T}} + V_1 - BR_2^{-1} B^{\mathrm{T}} P G^{\mathrm{T}} Q_{12}^{\mathrm{T}} - Q_{12} G P B R_2^{-1} B^{\mathrm{T}}, \tag{3.21a}$$

$$\dot{Q}_{12} = AQ_{12} + Q_{12} A_c - BR_2^{-1} B^{\mathrm{T}} Q_{12} + Q_1 C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}}, \tag{3.21b}$$

$$\dot{Q}_2 = A_c Q_2 + Q_2 A_c^{\mathrm{T}} + \Gamma Q C^{\mathrm{T}} V_2^{-1} C Q_{12} + Q_{12}^{\mathrm{T}} C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}} + \Gamma Q C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}}. \tag{3.21c}$$

Step 3. The optimal A_c and the Riccati and Lyapunov equations (3.2).
Appealing to (3.21b,c) one obtains

$$\begin{aligned} \Gamma \dot{Q}_{12} - \dot{Q}_2 = {} & \Gamma A Q_{12} + \Gamma Q_{12} A_c^{\mathrm{T}} - \Gamma B R_2^{-1} B^{\mathrm{T}} P G^{\mathrm{T}} + \Gamma Q_1 C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}} - A_c Q_2 - Q_2 A_c^{\mathrm{T}} \\ & - \Gamma Q C^{\mathrm{T}} V_2^{-1} C Q_{12} - Q_{12}^{\mathrm{T}} C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}} - \Gamma Q C^{\mathrm{T}} V_2^{-1} C Q \Gamma^{\mathrm{T}}. \end{aligned} \tag{3.22}$$

By (3.5) and (3.17) there also hold

$$\Gamma Q_{12} - Q_2 = 0, \quad \Gamma Q_1 - Q_{12}^{\mathrm{T}} - \Gamma Q = 0, \qquad \Gamma \dot{G}^{\mathrm{T}} = \left(\Gamma \dot{Q}_{12} - \dot{Q}_2\right) Q_2^{-1}. \tag{3.23}$$

Combining (3.22) and (3.23), and the obvious fact $\dot{\Gamma} G^{\mathrm{T}} + \Gamma \dot{G}^{\mathrm{T}} = 0$, we finally get

$$A_c = \Gamma\left(A - BR_2^{-1} B^{\mathrm{T}} P - Q C^{\mathrm{T}} V_2^{-1} C\right) G^{\mathrm{T}} - \Gamma \dot{G}^{\mathrm{T}} = \Gamma\left(A - BR_2^{-1} B^{\mathrm{T}} P - Q C^{\mathrm{T}} V_2^{-1} C\right) G^{\mathrm{T}} + \dot{\Gamma} G^{\mathrm{T}}, \tag{3.24}$$

as claimed.

Equations (3.2a,b) are obtained in a similar way, by direct computation: One invokes the explicit definitions in (3.5d–h), the differential equations (3.21) and their counterparts for the blocks of (3.14). Equations (3.2c,d) are then an easy consequence of (3.2a,b) and the relations

$$\hat{P} = P_1 - P \quad \text{and} \quad \hat{Q} = Q_1 - Q. \tag{3.25}$$

Step 4. Continuity of P, $\hat{P}$, Q and $\hat{Q}$ at the boundary points.

The problem that may arise here is due to the fact that, per our proof, the terms involving τ in (3.2) are not necessarily bounded at the initial and/or the terminal time. We shall be content with establishing continuity of P at t_1. The arguments pertinent to the solutions of (3.2b–d) are completely similar.

Since $P_2(t_0)$ is invertible, $P(t_0)$ (and $\hat{P}(t_0)$) is well defined. Approaching $t = t_1$ integration of (3.2a) yields

$$P(t) - P(t_0) + \int_{t_0}^{t} (PA + A^{\mathrm{T}}P + R_1)(s)\ \mathrm{d}s = -\int_{t_0}^{t} (\tau_\perp^{\mathrm{T}} PBR_2^{-1}B^{\mathrm{T}}P\tau_\perp)(s)\ \mathrm{d}s. \tag{3.26}$$

We have already established that all the terms on the left hand side of (3.26) are uniformly bounded, while the term on the right hand side is a monotonously decreasing, negative semidefinite matrix. Boundedness and monotonicity imply the existence of a limit in the right – hence on the left hand side. In particular, P is continuous at $t = t_1$.

This completes the proof of the theorem. □

4. Conclusion

Optimality conditions have been obtained for the problem of designing reduced-order dynamic compensators for LTV systems. The resulting first-order necessary conditions in parameter space provide a generalization to the optimal projection equations for steady-state, fixed-order dynamic compensation of LTI systems. As in the steady-state LTI case, the optimality conditions presented herein present new challenges. Specifically, since the differential coupled Riccati/Lyapunov equations are nonstandard, their analysis requires extensions of existing frameworks as well as development of new techniques. Furthermore, numerical challenges also arise since the optimal gains call for a solution of coupled two point boundary value problems. These remain areas of future research. Finally, the results of the paper can readily be extended to the reduced order time-varying Kalman filter problem [2], optimal model reduction for time-varying plants [8], and time-varying discrete-time systems [4].

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